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ON ASYMPTOTIC SOLUTIONS OF BOUNDARY-VALUE PROBLEMS DEFINED ON THIN DOMAINS*

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Abstract. The solution of the Poisson equation subject to Dirichlet conditions is examined asymptotically on thin domains. The evolution of the structure of the solution is followed as the shape of the domain changes. It is found that the "end wall" boundary layers present when the domain is rectangular, shrink and weaken as the endwalls become less sloped and vanish when the domain slope is uniformly bounded. Such structural changes are important in certain viscous flows containing moving contact lines.

1. Introduction. An elliptic equation, defined on a domain $\mathfrak{D}$, has the solution ϕ satisfy say, Dirichlet conditions on $\partial\mathfrak{D}$. If $\mathfrak{D}$ is the rectangle $-l \leqslant x \leqslant l$, $0 \leqslant y \leqslant d$ as shown in Fig. la, then the domain is called thin if

$$\varepsilon = d/l < < 1. \tag{1.1}$$

It is now well known [1–3] that asymptotic approximations to ϕ on rectangular domains can be obtained valid for $\varepsilon \to 0$. There is a leading order "outer" solution ϕ_0 valid away from the ends $x = \pm l$ in which ϕ is approximately the solution in the infinite strip. ϕ_0 frequently comes out to be x-independent. Near the ends $x = \pm l$ there are boundary-layer corrections ϕ_I that frequently satisfy the full original equation defined on a simplified U-shaped domain. The matching of ϕ_0 and ϕ_I yields a uniformly valid solution on the whole rectangle. Work along these lines appears in many areas including elasticity [1–3], and fluid mechanics [4, 5].

The analysis of fluid flow in rivulets [6] has led to the consideration of the Poisson equation defined on a domain $\mathfrak{D}$ such that shown in Fig. 1b. Here the boundary $y = 0$ is a solid plate while the known curve $y = h(x)$ represents an interface of the liquid in $\mathfrak{D}$ with the adjacent passive gas. The solution ϕ represents the unidirectional parallel flow (directed into the page). Workers [6, 7] have analyzed this problem for $\varepsilon \to 0$ and obtained a leading order outer solution ϕ_0 which seemingly satisfies *all* the posed boundary

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conditions. Thus, even though the limit $\varepsilon \to 0$ is formally a singular perturbation, ϕ_0 seems uniformly valid and no boundary-layer correction is required.

In the present work we look into the link between problems as illustrated in Fig. 1a, b for the limit $\varepsilon \to 0$ and determine how the boundary-layer structure evolves when the shape of the domain is changed. Thus, we consider the intermediate domain $\mathfrak{D}$ pictured in Fig. 1c. Here $y = h(x)$ may have vertical tangents at $x = \pm l$ but nowhere else. For definiteness, we consider the simplest boundary-value problem, the Poisson problem with Dirichlet conditions. We follow the evolution of the solution structure of this Poisson problem as $y = h(x)$ varies but the conclusions are applicable for more general systems.

2. Formulation. Consider the boundary-value problem

$$\phi_{xx} + \phi_{yy} = k \tag{2.1}$$

on the domains of Fig. 1b, c with

$$\phi = 0 \quad \text{on } y = 0, \tag{2.2a}$$
$$\phi = 0 \quad \text{on } y = h(x), \tag{2.2b}$$

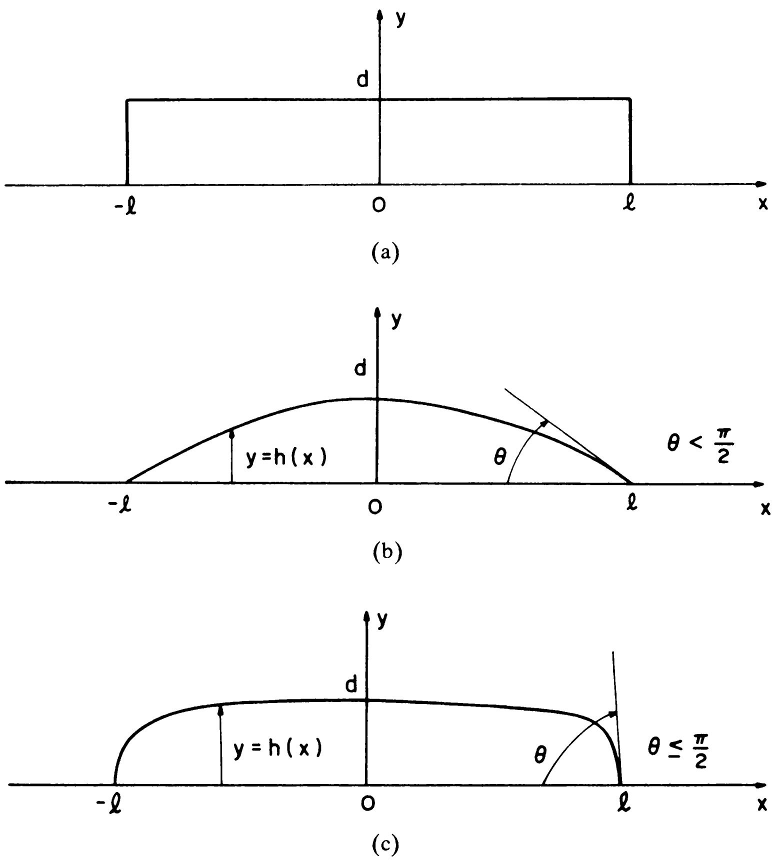

Fig. 1.

where k is constant, and h satisfies

$$\text{(i)} \quad h(\pm l) = 0, \tag{2.3a}$$

$$\text{(ii)} \quad h(0) = d, \tag{2.3b}$$

$$\text{(iii)} \quad |h^{(n)}(x)| < \infty \quad \text{for } -l < x < l. \tag{2.3c}$$

Here subscripts denote partial differentiation. Without loss of generality we restrict h to even functions in x and consider the half-interval $0 \leqslant x \leqslant l$ only.

We scale the variables as follows:

$$X = x/l, \quad Y = y/d, \quad \Phi = \phi/d^2k, \quad H = h/d. \tag{2.4}$$

We than have the problem (2.1)–(2.3) transformed into the new one

$$\Phi_{YY} + \varepsilon^2\Phi_{XX} = 1, \tag{2.5a}$$

$$\Phi(X,0) = \Phi(X, H) = 0 \tag{2.5b}$$

where

$$H(0) = 1 \tag{2.5c}$$

and

$$\varepsilon = d/l. \tag{2.6}$$

The transformed domain is shown in Fig. 2.

The outer solution to system (2.5) is obtained by writing

$$\Phi \sim \Phi_0(X, Y) + \varepsilon^2\Phi_2(X, Y) + O(\varepsilon^4). \tag{2.7}$$

By direct substitution we find that

$$\Phi_0 = \tfrac{1}{2}Y(Y - H) \tag{2.7a}$$

and

$$\Phi_2 = \tfrac{1}{12}Y(Y^2 - H^2)H'' \tag{2.7b}$$

so that

$$\Phi \sim \tfrac{1}{2}Y(Y - H)\{1 + \tfrac{1}{6}(Y + H)H''\varepsilon^2 + O(\varepsilon^4)\}. \tag{2.8}$$

Here primes denote the differentiation.

If H is as pictured in Fig. 1b, e.g. $H = 1 - X^2$, expansion (2.8) can be shown to be uniformly valid for $X \in [0, 1]$. However, if H'' is unbounded at $X = 1$, then non-uniformities can arise. For example if $H = (1 - X^2)^{1/2}$, then the combination $\varepsilon^2 HH''$ of equation

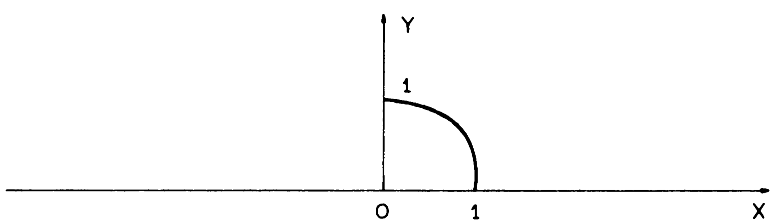

Fig. 2.

(2.8) behaves like $\varepsilon^2(1 - X^2)^{-1}$ which reveals a boundary layer for

$$1 - X = O(\varepsilon^2). \tag{2.9}$$

3. Boundary layer near $X = 1$. We again consider $H(X)$ to be general and examine possible boundary layers near $X = 1$.

We write

$$\xi = (1 - X)/g(\varepsilon) \tag{3.1}$$

where $g(\varepsilon)$ is determined from the order of the non-uniformity of $\varepsilon^2 HH''$ in expansion (2.8). We can then change variables in H and obtain

$$H(X) \sim f(\varepsilon)H_0(\xi) + o(f). \tag{3.2}$$

Given that H vanishes at $X = 1$, we must also stretch the vertical coordinate Y; we write

$$\eta = Y/f(\varepsilon). \tag{3.3}$$

The use of forms (3.1) and (3.3) leads to the rescaled version of equation (2.5a),

$$\Psi_{\eta\eta} + \frac{\varepsilon^2 f^2(\varepsilon)}{g^2(\varepsilon)}\Psi_{\xi\xi} = f^2(\varepsilon) \tag{3.4}$$

where Ψ represents Φ written in the new variables.

In order to describe the solution in the region near $X = 1$ we must retain both $\Psi_{\xi\xi}$ and $\Psi_{\eta\eta}$ in Eq. (3.4) so that

$$\varepsilon f(\varepsilon) = g(\varepsilon), \tag{3.5}$$

which is consistent with our determining $g(\varepsilon)$ from the order of the non-uniformity in expansion (2.8) i.e. such that $\varepsilon^2 HH'' = O(1)$. We also write for the local solution

$$\Psi \sim f^2(\varepsilon)\{\Psi_0 + o(1)\} \tag{3.6}$$

so that Ψ_0 satisfies

$$\Psi_{0_{\xi\xi}} + \Psi_{0_{\eta\eta}} = 1, \tag{3.7a}$$

$$\Psi_0 = 0 \quad \text{on } \eta = 0, \tag{3.7b}$$

$$\Psi_0 = 0 \quad \text{on } \eta = H_0(\xi). \tag{3.7c}$$

$f^2(\varepsilon)\Psi_0$ must match to $\Phi_0 = \frac{1}{2}Y(Y - H) \sim \frac{1}{2}f^2(\varepsilon)[\eta^2 - \eta H_0(\xi)]$. Hence

$$\Psi_0 \sim \tfrac{1}{2}\left[\eta^2 - \eta H_0(\xi)\right] \quad \text{as } \xi \to \infty. \tag{3.7d}$$

Note that condition (3.5) insures that $g(\varepsilon)/f(\varepsilon) = 0(\varepsilon)$ so that as $\varepsilon \to 0$, $\xi \to \infty$ faster than does $\eta \to \infty$.

4. Analytical example. There is one case that can be treated fully. Consider

$$H(X) = (1 - X^2)^{1/2}. \tag{4.1a}$$

From condition (2.9)

$$g(\varepsilon) = \varepsilon^2. \tag{4.1b}$$

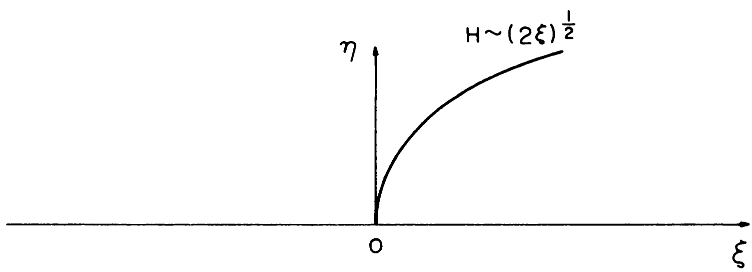

Fig. 3.

From definition (3.2) and forms (4.1a, b) we have

$$f(\varepsilon) = \varepsilon \tag{4.1c}$$

$$H_0(\xi) = (2\xi)^{1/2}. \tag{4.1d}$$

Thus, the problem (3.7) is defined in a parabolic sector shown in Fig. 3. System (3.7) can be solved exactly using parabolic coordinates ξ_1, ξ_2:

$$\xi = \tfrac{1}{2}(\xi_1^2 - \xi_2^2), \qquad \eta = \xi_1 \xi_2. \tag{4.2}$$

We find that

$$\begin{aligned}\psi \sim \varepsilon^2 \Bigg\{ & \frac{1}{2}\eta^2 - \frac{1}{2}\eta\left[\sqrt{\eta^2 + \left(\xi - \frac{1}{2}\right)^2} + \left(\xi - \frac{1}{2}\right)\right]^{1/2} \\ & + \frac{1}{6}\left[\sqrt{\eta^2 + \left(\xi - \frac{1}{2}\right)^2} - \left(\xi - \frac{1}{2}\right)\right]^{1/2}\left[\sqrt{\eta^2 + \left(\xi - \frac{1}{2}\right)^2} - \left(\xi - \frac{1}{2}\right) - 1\right] \\ & + \frac{1}{2}\sum_{k=1}^{\infty}(-1)^k \sin[\![k\pi\left[\sqrt{\eta^2 + \left(\xi - \frac{1}{2}\right)^2} - \left(\xi - \frac{1}{2}\right)\right]^{1/2}]\!] \\ & \times [\![\exp\left(-k\pi\left[\sqrt{\eta^2 + \left(\xi - \frac{1}{2}\right)^2} + \left(\xi - \frac{1}{2}\right)\right]^{1/2}\right)]\!] \Bigg\}.\end{aligned} \tag{4.3}$$

As $\xi \to \infty$, the first two terms match to Φ_0 while the remainder tend to zero. Clearly, one can obtain a uniformly valid solution here by standard methods.

5. General cases. We now turn to the description of the structure of the solutions as $H(X)$ changes. We categorize a family of H as follows:

$$H(X) = (1 - X^2)^a, \qquad 0 \leqslant a \leqslant \infty,$$

where the exponent essentially characterizes the local shape of H near $X = 1$. (i) $a = 0$; This case has the rectangular geometry of Fig. 1a where $H(X) \equiv 1$. We know that there is a boundary layer at the end with $Y = O(1)$ and $1 - X = O(\varepsilon)$. The boundary-layer solution can be obtained explicitly and the composite solution is the exact solution of (2.5) for this case.

(ii) $0 < a < 1$: This case has the geometry shown in Fig. 1c where all derivatives $H^{(n)}(X)$ are unbounded at $X = 1$. There is a boundary layer at the end with thickness

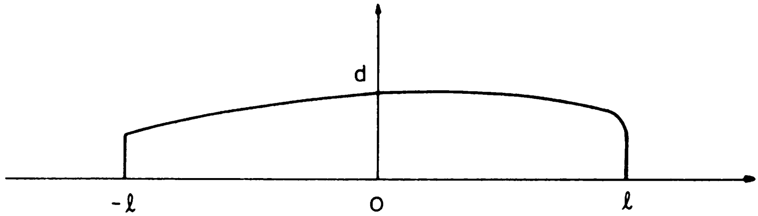

Fig. 4.

$1 - X = O(\varepsilon^{1/(1-a)})$ and height $Y = O(\varepsilon^{a/(1-a)})$ in the sector $0 \leqslant \eta \leqslant (2\xi)^a$. When $a \to 0$ we regain the case (i) in which $Y = 0(1)$. As a increases from zero the boundary layer thickness $\varepsilon^{1/(1-a)}$ decreases in size and ultimately vanishes as $a \to 1$.

(iii) $a = 1$: This case has the geometry shown in Fig. 1b where all derivatives $H^{(n)}X$ are bounded at $\dot{X} = 1$. The boundary layer is absent and the outer solution is uniformly valid for $X \in [0, 1]$.

(iv) $a > 1$: This case has $H(X)$ with $H'(1) = 0$. Again, as in case (iii), the outer solution is uniformly valid (to all orders) and boundary layers are absent.

Clearly, one can build up domain shapes that are combinations of the above cases an example of which is shown in Fig. 4. The cases already discussed can be used to analyze these as well.

6. Discussion. We have examined a class of Poisson equations with Dirichlet conditions. We have seen that asymptotics for thin domains, while formally singular perturbation theories, lead to (regular) uniformly valid results when the domain has corners but no "vertical" tangents. Thus, local angles at the boundary smaller than $\pi/2$ are distinguished from those equal to $\pi/2$. When vertical tangents are present, such theories lead to boundary layers whose thicknesses depend on an index, which we call a and which measures the local shape of the domain. The thickest, strongest boundary layers occur in rectangular domains. These become thinner and weaker as the "endwall" bends downward and they finally vanish when the vertical tangent vanishes.

The present investigation leads to insights on more complicated mathematical systems. For example when a viscous liquid drop on a smooth horizontal plate spontaneously spreads, lubrication theory can be used to analyze the flows [8, 9]. Here one derives a nonlinear, free boundary problem whose solution is also uniformly valid even though the lubrication theory is also formally singular. The reasons for such behavior are clarified by the present analysis on a model problem.

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