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ON ASYMPTOTIC STABILITY FOR LINEAR DELAY EQUATIONS

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1. Introduction. For a linear scalar delay equation

$$\dot{x}(t) + ax(t) + bx(t-r) = 0,$$

the stability of the zero solution can be determined by whether all roots of the characteristic equation

$$\lambda + a + be^{-\lambda r} = 0 \tag{1.1}$$

lie in the left half plane. And it is well known [1] that all roots of (1.1) lie in the left half plane if either

$$a > -b \geq -\frac{1}{r}$$

or

$$rb = \frac{\theta}{\sin\theta} \quad \text{and} \quad a > -b\cos\theta \quad \text{for some } \theta \in (0,\pi).$$

In this paper, we extend this result to a more general equation of the form

$$\lambda + a + \int_0^r d\eta(s)e^{-\lambda s} = 0, \tag{1.2}$$

where η is a function of bounded variation on $[0,r]$ and $\int_0^{0^+} d\eta(s) = 0$, and then apply it to discuss the stability of some classes of delay equations, including a partial delay-differential equation studied by Green and Stech [2].

2. A main theorem. In this section we shall establish a theorem concerning the location of the roots of (1.2).

Lemma 2.1. *Let $\theta \in (0,\pi)$, then $\theta\cos\theta/\sin\theta < 1$.*

Proof: Since

$$\frac{d}{d\theta}(\cos\theta\sin\theta - \theta) = -2\sin^2\theta < 0, \qquad \theta \in (0,\pi),$$

and $\cos\theta\sin\theta - \theta|_{\theta=0} = 0$, it follows that $\cos\theta\sin\theta - \theta < 0$, for $\theta \in (0,\pi)$. Hence

$$\frac{d}{d\theta}\left(\frac{\theta\cos\theta}{\sin\theta}\right) = \frac{\cos\theta\sin\theta - \theta}{\sin^2\theta} < 0, \qquad \theta \in (0,\pi).$$

Note that $\theta\cos\theta\sin\theta \to 1$ as $\theta \to 0^+$, therefore, $\frac{\theta\cos\theta}{\sin\theta} < 1$, $\theta \in (0,\pi)$.

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Lemma 2.2. *For any $\theta \in (0, \pi)$, introduce*

$$D_\theta = \{x + iy : x \in \mathbb{R},\ y \geq -\tfrac{\sin\theta - \theta\cos\theta}{\theta\sin\theta}x\},$$

a half plane of the complex plane $\mathbb{C}$. Then for any $z_i \in D_\theta$, $\alpha_i \geq 0$, $i = 1, 2$, we have $\alpha_1 z_1 + \alpha_2 z_2 \in D_\theta$. Furthermore,

$$z_1 + z_2 = 0 \quad \text{if and only if} \quad z_1,\ z_2 \in D_\theta \text{ and } z_1 = -z_2.$$

Proof: The proof is trivial since D_θ is a half plane and also a cone which contains $0 \in \partial D_\theta$.

Lemma 2.3. *For each $\theta \in (0, \pi)$, let $W_\theta : [0, \infty) \to \mathbb{C}$ be given by*

$$\begin{aligned} W_\theta(v) &= -\theta\cos\theta/\sin\theta + iv + \theta e^{-iv}/\sin\theta \\ &= -\theta(\cos\theta - \cos v)/\sin\theta + i(v - \theta\sin v/\sin\theta), \end{aligned}$$

then $W_\theta([0, \infty)) \in D_\theta$.

Proof: If we let $x(v) = \operatorname{Re} W_\theta(v)$, $y(v) = \operatorname{Im} W_\theta(v)$ for $v \in (0, \pi) \cup (\pi, 2\pi)$, we have

$$\frac{dy}{dx} = \frac{\dot{y}(v)}{\dot{x}(v)} = \frac{-\sin\theta + \theta\cos v}{\theta\sin v},$$

so

$$\frac{d^2y}{dx^2} = \frac{d}{dv}\Big(\frac{\dot{y}(v)}{\dot{x}(v)}\Big)\frac{dv}{dx} = \frac{\sin^2\theta}{\theta^2\sin^3 v}\Big(\frac{\theta}{\sin\theta} - \cos v\Big).$$

Since $\theta/\sin\theta > 1$, it follows that

$$\frac{d^2y}{dx^2} > 0, \quad v \in (0, \pi); \qquad \frac{d^2y}{dx^2} < 0, \quad v \in (\pi, 2\pi).$$

This implies that $W_\theta(v)$ is convex downward for $v \in (0, \pi)$ and convex for $v \in (\pi, 2\pi)$. Moreover note that $x(\theta) = y(\theta) = 0$ and

$$\frac{dy}{dx}\Big|_{(x(\theta), y(\theta))} = -\frac{\sin\theta - \theta\cos\theta}{\theta\sin\theta}.$$

That is, the tangent line of $W_\theta(v)$ at $W_\theta(\theta) = 0$ concides with the boundary of D_θ. This implies that $W_\theta(v) \in D_\theta$, $v \in [0, \pi]$ since $W_\theta(v)$ is convex downward. Furthermore, we have

$$W_\theta(2\pi) = \theta(1 - \cos\theta)/\sin\theta + i2\pi \in D_\theta,$$

again the convexity of $W_\theta(v)$ for $v \in (\pi, 2\pi]$ yields that $W_\theta(v) \in D_\theta$, $v \in (\pi, 2\pi]$. Suppose $v > 2\pi$, then there is an integer k and $v_0 \in [0, 2\pi)$ such that $v = v_0 + 2k\pi$. Since $i2k\pi \in D_\theta$, it follows from Lemma 2.2 that

$$W_\theta(v) = i2k\pi + W_\theta(v_0) \in D_\theta.$$

This completes the proof of the lemma.

Lemma 2.4. *Suppose that η is increasing and there is $\theta \in (0,\pi)$ such that*

$$\int_0^r s\, d\eta(s) = \frac{\theta}{\sin\theta}, \qquad a > -\cos\theta \int_0^r d\eta(s).$$

Define

$$g_\theta(u) = u + a + \cos\theta \int_0^r e^{-us} d\eta(s),$$

then $g_\theta(u) > 0$ for all $u \geq 0$.

Proof: It is obvious that $g_\theta(0) > 0$. Moreover by using Lemma 2.1 we obtain

$$\frac{dg_\theta(u)}{du} = 1 - \cos\theta \int_0^r se^{-us}\, d\eta(s) \geq 1 - \cos\theta \int_0^r s\, d\eta(s) = 1 - \frac{\theta\cos\theta}{\sin\theta} > 0.$$

Hence $g_\theta(u) > 0$ for all $u \geq 0$.

By means of the previous lemmas it is now easy to prove our main

Theorem 2.5. *Under the assumptions of Lemma 2.4, let*

$$\Delta(\lambda) = \lambda + a + \int_0^r e^{-\lambda s} d\eta(s),$$

then

$$\Delta(u + iv) \in D_\theta \setminus \{0\}, \qquad \textit{for all } u \geq 0,\ v \geq 0.$$

Proof: An easy calculation shows that

$$\begin{aligned}\Delta(u+iv) &= u + a + \cos\theta \int_0^r e^{-us} d\eta(s) + iv\left[1 - \frac{\sin\theta}{\theta}\int_0^r se^{-us} d\eta(s)\right] \\ &\quad + \frac{\sin\theta}{\theta}\int_0^r e^{-us}\left[-\frac{\theta\cos\theta}{\sin\theta} + ivs + \frac{\theta}{\sin\theta}e^{-ivs}\right] d\eta(s) \qquad (2.1) \\ &= g_\theta(u) + z_\theta + \frac{\sin\theta}{\theta}\int_0^r e^{-us} W_\theta(vs)\, d\eta(s),\end{aligned}$$

where

$$z_\theta = iv\left[1 - \frac{\sin\theta}{\theta}\int_0^r se^{-us} d\eta(s)\right].$$

Since $W_\theta(vs) \in D_\theta$ and η is increasing, it is obvious that

$$\frac{\sin\theta}{\theta}\int_0^r e^{-us} W_\theta(vs)\, d\eta(s) \in D_\theta.$$

Moreover,

$$1 - \frac{\sin\theta}{\theta}\int_0^r se^{-us}\, d\eta(s) \geq 1 - \frac{\sin\theta}{\theta}\int_0^r s\, d\eta(s) = 0,$$

so $z_\theta \in D_\theta$. It follows from Lemma 2.3 and 2.4 that $g_\theta(u) + z_\theta \in D_\theta \setminus \partial D_\theta$. Therefore, as a consequence of Lemma 2.2 and (2.1) we have

$$\Delta(u + iv) \in D_\theta \setminus \partial D_\theta.$$

3. On stability of delay equations. We now turn to discuss the stability of some classes of delay equations by using Theorem 2.5. As a first application consider the delay equation

$$\dot{x}(t) = -ax(t) - \int_0^r x(t-s)\, d\eta(s), \qquad (3.1)$$

where η satisfies the assumption of Section 1.

Theorem 3.1. *Suppose that η is monotone. If either*

$$\int_0^r s\,d\eta(s) \le 1, \qquad a > -\int_0^r d\eta(s)$$

or

$$1 < \int_0^r s\,d\eta(s) = \frac{\theta}{\sin\theta}, \quad a > -\cos\theta \int_0^r d\eta(s)$$

for some $\theta \in (0, \pi)$, then the zero solution of equation (3.1) *is asymptotically stable.*

Proof: It is enough to show that all eigenvalues of the characteristic equation

$$\Delta(\lambda) = \lambda + a + \int_0^r e^{-\lambda s} d\eta(s) = 0$$

have negative real parts. This is equivalent to proving that

$$\Delta(u+iv) \ne 0, \quad \text{for all } u \ge 0,\ v \ge 0.$$

It is trivial if $\int_0^r d\eta(s) = 0$ (this implies that $\int_0^r s d\eta(s) = 0$), so we suppose that $R^* = \int_0^r s d\eta(s) \ne 0$.

First suppose $R^* \le 1$, and $a > -\int_0^r d\eta(s)$. then

$$\begin{aligned}\Delta(\lambda) &= a + \int_0^r d\eta(s) + \frac{1}{R^*}\int_0^r \lambda s d\eta(s) - \int_0^r d\eta(s) + \int_0^r e^{-\lambda s} d\eta(s)\\ &= a + \int_0^r d\eta(s) + \int_0^r (\frac{\lambda}{R^*} - 1 + e^{-\lambda s}) d\eta(s).\end{aligned}$$

If $R^* > 0$, then $\int_0^r d\eta(s) > 0$. For any $u \ge 0$, $v > 0$,

$$\operatorname{Im}\Delta(u+iv) = \int_0^r (\frac{vs}{R^*} - e^{-us}\sin(vs)) d\eta(s) \ge \int_0^r (\frac{vs}{R^*} - |\sin vs|) d\eta(s) > 0, \tag{3.2}$$

for $vs/R^* - |\sin vs| > 0$, $s \in (0, r]$. And if $v = 0$, we have

$$\Delta(u) = a + \int_0^r d\eta(s) + \int_0^r (\frac{us}{R^*} - 1 + e^{-us}) d\eta(s).$$

Since

$$\frac{us}{R^*} - 1 + e^{-us} \ge us - 1 + e^{-us} \ge 0, \quad s \ge 0,$$

and $a + \int_0^r d\eta(s) > 0$, therefore

$$\Delta(u) > 0 \quad \text{for all } u \ge 0. \tag{3.3}$$

If $R^* < 0$, then $\int_0^r d\eta(s) < 0$. For any $u \ge 0$, we have

$$\frac{us}{R^*} - 1 + e^{-us}\cos vs \le -1 + e^{-us} \le 0, \quad s \ge 0, \ v \in \mathbb{R}.$$

So

$$\int_0^r \left(\frac{us}{R^*} - 1 + e^{-us}\cos vs\right) d\eta(s) \geq 0.$$

Hence

$$\operatorname{Re}\Delta(u+iv) = a + \int_0^r d\eta(s) + \int_0^r \left(\frac{us}{R^*} - 1 + e^{-us}\cos vs\right) d\eta(s) > 0, \quad u \geq 0,\ v \in \mathbb{R}. \tag{3.4}$$

(3.2)–(3.4) conclude our first assertion.

Now suppose

$$1 < \int_0^r s\,d\eta(s) = \frac{\theta}{\sin\theta}, \qquad a > -\cos\theta \int_0^r d\eta(s).$$

(Note that $f(\theta) = \theta/\sin\theta$, $\theta \in (0,\pi)$ is a strictly increasing function and $\lim_{\theta\to 0^+} f(\theta) = 1$, $\lim_{\theta\to\pi} f(\theta) = +\infty$. Hence for any $R^* > 1$, there is a unique $\theta \in (0,\pi)$ such that $R^* = \theta/\sin\theta$.) Applying Theorem 2.5 we find that $\Delta(u+iv) \neq 0$, for $u \geq 0$, $v \geq 0$. Thus the proof is completed.

As an immediate consequence of Theorem 3.1 we have

Corollary 3.2. *For the equation*

$$\dot{x}(t) = -ax(t) - \sum_{i=1}^n a_i x(t - r_i), \quad r_i > 0,\ i = 1,\cdots,n, \tag{3.5}$$

if $a_i \geq 0$, $i = 1,\cdots,n$ *and there is* $\theta \in (0,\pi)$ *such that*

$$\sum_{i=1}^n a_i r_i = \frac{\theta}{\sin\theta}, \quad a > -\cos\theta \sum_{i=1}^n a_i,$$

then the zero solution of (3.5) *is asymptotically stable.*

Next we consider a population model with diffusion effect:

$$\frac{\partial N(x,t)}{\partial t} = K\frac{\partial^2 N(x,t)}{\partial x^2} + rN(x,t)\left[1 - \int_0^T N(x,t-s)\,d\eta(s)\right] \tag{3.6}$$

with the boundary and initial conditions

$$\begin{aligned} &N(0,t) = N(\pi,t) = 0, \qquad t \geq 0 \\ &N(x,s) = \phi(x,s), \qquad -T \leq s \leq 0,\ 0 \leq x \leq \pi, \end{aligned}$$

here $K > 0, r > 0$ and $T > 0$ are constants and $\eta(s)$ is non decreasing with

$$\int_0^T d\eta(s) = 1.$$

Our interest is to discuss the stability of the positive equilibrium solution $\tilde{N}(x)$ of (3.6), which is determined by

$$K\frac{d^2N(x)}{dx^2} + rN(x)[1 - N(x)] = 0, \quad x \in I = (0,\pi) \tag{3.7}$$

$$N(0) = N(\pi) = 0, \quad N(x) > 0, \;\; x \in (0,\pi). \tag{3.8}$$

Green and Stech in [2] have shown that:

1. If $r/K \le 1$, then the only solution of (3.7)–(3.8) is $N \equiv 0$.
2. If $r/K > 1$, (3.7)–(3.8) have a unique solution
$$\tilde{N}(x) = \tilde{N}(x; r, K) \quad \text{with} \quad 0 < \tilde{N}(x) < 1, \; x \in I.$$
3. Let $M(r,K) = \max \tilde{N}(x; r, K)$, then the equilibrium solution $\tilde{N}(x; r, K)$ is asymptotically stable if
$$rM(r,k)\int_0^T s d\eta(s) < 1.$$

By using Theorem 2.5, we can improve this estimate and obtain

Theorem 3.3. *If $r/K > 1$ and*

$$rM(r,K)\int_0^T s d\eta(s) < \frac{\pi}{2},$$

then the equilibrium solution $\tilde{N}(x; r, K)$ is asymptotically stable.

Before the proof of this theorem, we first establish the following

Lemma 3.4. *Let*

$$C_0^2 = \{y \in C^2(I) \cap C(\bar{I}), \;\; y(0) = y(\pi) = 0\}$$

$$L : C_0^2 \to C_0^2, \quad L = KD^2 + r[1 - \tilde{N}(x)].$$

where $D^2 = d^2/dx^2$ and $\tilde{N}(x) = \tilde{N}(x; r, K)$ is the positive equilibrium. Then all eigenvalues of L are real and non positive.

Proof: Obviously, L is a self-adjoint operator, that is

$$\int_0^\pi (L\phi)\psi dx = \int_0^\pi (L\psi)\phi \, dx \quad \text{for all} \quad \phi, \; \psi \in C_0^2,$$

so all eigenvalues of L are real. Suppose L has some eigenvalue $\lambda > 0$, and let y be the corresponding eigenfunction, we have

$$K\frac{d^2y(x)}{dx^2} + (r[1 - \tilde{N}(x)] - \lambda)y(x) = 0.$$

Note that

$$K\frac{d^2\tilde{N}(x)}{dx^2} + r[1 - \tilde{N}(x)]\tilde{N}(x) = 0$$

and

$$r[1 - \tilde{N}(x)] > r[1 - \tilde{N}(x)] - \lambda.$$

The Sturm comparison theorem [3] implies that $\tilde{N}(x)$ has at least one zero in I, which contradicts the positivity of $\tilde{N}$ on I.

Corollary 3.5. *For all* $\psi \in C_0^2$

$$\int_0^\pi (L\psi)\bar{\psi}dx \leq 0. \tag{3.9}$$

Proof: Since L is self-adjoint, the collection $\{\psi\}$ of all eigenfunctions with $\int_0^\pi \psi^2 dx = 1$ form an orthonormal basis of C_0^2 ([4], p.374). Inequality (3.9) follows from Lemma 3.4 and Parseval's equation.

Now we prove Theorem 3.3. First, one can verify that the linearized equation with respect to equilibrium $\tilde{N}(x)$ is

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + r[1-\tilde{N}]u - r\tilde{N}\int_0^T u(x,t-s)d\eta(s).$$

So the eigenvalue problem is

$$\Delta(\lambda,\psi) \stackrel{\text{def}}{=} \Big[\lambda + r\tilde{N}\int_0^T e^{-\lambda s}d\eta(s)\Big]\psi - L\psi = 0, \quad \lambda \in C,\ \psi \in C_0^2,\ \psi \neq 0. \tag{3.10}$$

We claim that (3.10) does not have eigenvalue λ with $\operatorname{Re}\lambda \geq 0$. To see this, for $\lambda = u + iv$, multiplying $\Delta(\lambda,\psi)$ by $\bar{\psi}$ and integrating over I we obtain

$$\int_0^\pi \Delta(\lambda,\psi)\bar{\psi}dx = \int_0^\pi \Big[u + iv + r\tilde{N}\int_0^T e^{-us}e^{-ivs}d\eta(s)\Big]|\psi|^2dx - \int_0^\pi (L\psi)\bar{\psi}dx. \tag{3.11}$$

Since $\tilde{N}(x) > 0$, $x \in (0,\pi)$ and $\int_0^T d\eta(s) = 1$, it follows from (3.9) that for all $0 \neq \psi \in C_0^2$,

$$\int_0^\pi \Delta(0,\psi)\bar{\psi}dx \geq r\int_0^T d\eta(s)\int_0^\pi \tilde{N}(x)|\psi|^2dx > 0.$$

Hence

$$\Delta(0,\psi) \neq 0, \quad \text{for all } 0 \neq \psi \in C_0^2.$$

If $u \geq 0$, $v \geq 0$ and $u + v > 0$, then (3.11) yields that

$$\begin{aligned}\int_0^\pi \Delta(\lambda,\psi)\bar{\psi}dx &= u\int_0^\pi |\psi|^2dx - \int_0^\pi (L\psi)\bar{\psi}dx \\ &\quad + iv\int_0^\pi \Big[1 - \frac{\tilde{N}(x)2r}{\pi}\int_0^T e^{-us}sd\eta(s)\Big]|\psi(x)|^2dx \\ &\quad + \frac{2r}{\pi}\int_0^T e^{-us}(ivs + \frac{\pi}{2}e^{-ivs})d\eta(s)\int_0^\pi \tilde{N}(x)|\psi(x)|^2dx.\end{aligned}$$

By the assumption of $rM(r,K)\int_0^T sd\eta(s) < \pi/2$ we have

$$1 - \frac{2r}{\pi}\tilde{N}(x)\int_0^T se^{-us}d\eta(s) \geq 1 - \frac{2r}{\pi}M(r,k)\int_0^T sd\eta(s) \stackrel{\text{def}}{=} \sigma > 0.$$

So

$$v\int_0^\pi \Big[1-\frac{2r}{\pi}\tilde{N}(x)\int_0^T se^{-us}d\eta(s)\Big]|\psi|^2dx \geq v\sigma\int_0^\pi |\psi|^2dx.$$

Moreover it follows from (3.9) that

$$u\int_0^\pi |\psi|^2dx - \int_0^\pi (L\psi)\bar{\psi}dx \geq u\int_0^\pi |\psi|^2dx.$$

Thus

$$\begin{aligned} z_1 &\stackrel{\text{def}}{=} u\int_0^\pi |\psi|^2dx - \int_0^\pi (L\psi)\bar{\psi}dx \\ &+iv\int_0^\pi \Big[1-\frac{\tilde{N}(x)2r}{\pi}\int_0^T e^{-us}sd\eta(s)\Big]|\psi(x)|^2dx \in D_{\frac{\pi}{2}}\setminus\partial D_{\frac{\pi}{2}}, \end{aligned}$$

where $D_{\frac{\pi}{2}}$ is defined as in Lemma 2.2. Furthermore, since η is increasing, we have

$$\begin{aligned} z_2 &\stackrel{\text{def}}{=} \frac{2r}{\pi}\int_0^T e^{-us}(ivs+\frac{\pi}{2}e^{-ivs})d\eta(s)\int_0^\pi \tilde{N}(x)|\psi(x)|^2dx \\ &= \frac{2r}{\pi}\int_0^\pi \tilde{N}(x)|\psi|^2dx\int_0^T e^{-us}W_{\frac{\pi}{2}}(vs)d\eta(s) \in D_{\frac{\pi}{2}}, \end{aligned}$$

where $W_{\frac{\pi}{2}}$ is defined as in Lemma 2.2. By using Lemma 2.2 we get

$$\int_0^\pi \Delta(\lambda,\psi)\bar{\psi}\,dx = z_1+z_2 \neq 0.$$

Therefore

$$\Delta(\lambda,\psi)\neq 0, \quad \text{for all } u\geq 0,\ v\geq 0,\ 0\neq\psi\in C_0^2.$$

Finally notice that for $u\geq 0$, $v\leq 0$ and $0\neq\psi\in C_0^2$,

$$\Delta(u+iv,\psi)=\bar{\Delta}(u-iv,\bar{\psi})\neq 0,$$

which completes the proof.

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