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On Asymptotic Structure, the Szlenk Index and UKK Properties in Banach Spaces

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Abstract. Let B be a separable Banach space and let $X = B^*$ be separable. We prove that if B has finite Szlenk index (for all $\varepsilon > 0$) then B can be renormed to have the weak* uniform Kadec-Klee property. Thus if $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ so that if (x_n) is a sequence in the ball of X converging ω^* to x so that $\liminf_{n\to\infty} \|x_n - x\| \geqslant \varepsilon$ then $\|x\| \leqslant 1 - \delta(\varepsilon)$. In addition we show that the norm can be chosen so that $\delta(\varepsilon) \geqslant c\varepsilon^p$ for some $p < \infty$ and $c > 0$.

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0. Introduction

The asymptotic structure of a separable infinite dimensional Banach space X as considered in [21] and [20] is a concept which merges the finite dimensional and infinite dimensional structure of X. One obtains for each integer n a class of normalized bases of length n which can each be found arbitrarily far out and arbitrarily separated in X. Usually knowledge of the asymptotic structure does not translate into global information about X. However sometimes it does and one such occurrence is the focal point of this paper.

Our object of attention is the following problem. Suppose X is a space with separable dual having finite Szlenk index for every $\varepsilon > 0$. This means that if one starts with the ball of X^*, B_{X^*}, and then forms the subset of all ω^* limits of ε-separated sequences from B_{X^*} and then forms the new subset of all ω^* limits of ε-separated sequences from this subset and so on then after finitely many such steps one is left with nothing. The question is can X be renormed so that the dual X^* has the ω^*-UKK (weak* uniform Kadec-Klee) property? This latter property involves a modulus $\delta(\varepsilon) > 0$. It says that given $\varepsilon > 0$ and $(x_n^*) \subseteq B_{X^*}$, a sequence converging ω^* to x^* with $\liminf_{n\to\infty} \|x_n^* - x^*\| \geqslant \varepsilon$ then $\|x^*\| \leqslant 1 - \delta(\varepsilon)$. We show that this problem has an affirmative solution and moreover the modulus $\delta(\varepsilon)$ is of power type.

Our proof leads us into further study of the asymptotic structure of X [20]. It turns out that having finite Szlenk index just says that one does not have arbitrarily long bases $(e_i^n)_1^n$ amongst the asymptotic structure of X^* with $\sup_n \|\sum_1^n e_i^n\| < \infty$, and this in turn yields uniform lower ℓ_p estimates for some $p < \infty$ on bases in the asymptotic structure. We then carry this lower ℓ_p estimate back to X^* (for a different p) in a certain manner and ultimately obtain the theorem.

A corollary of our result is a solution to a question raised by R. Huff [10] which can be stated as: If X is a reflexive separable space with $X = B^*$ and Szlenk index $S(B, \varepsilon) < \omega$ for all $\varepsilon > 0$, can X be renormed so that X^* has the UKK (uniform Kadec-Klee) property. This is defined like the ω^*-UKK property except that one uses weak convergence rather than weak* convergence. The nonreflexive version of Huff's problem remains open (see below).

A number of authors have written on the ω^*-UKK and UKK properties and the renorming problem [24], [18], [2], [4]. Prus [24] observed that a reflexive X with a basis has an equivalent UKK norm iff some blocking of the basis into an FDD (finite dimensional decomposition) admits for some $p < \infty$ uniform lower ℓ_p estimates on all block bases. The analogous result for the ω^*-UKK property is established for spaces with a shrinking basis in [4]. UKK properties in Banach lattices are investigated in [2]. In [18] Lancien shows that if X has finite Szlenk index then X^* admits an equivalent ω^* lower semicontinuous *ecart* satisfying the ω^*-UKK and solves the renorming problem for the spaces $L_p(X)$ and certain $C(K)$ spaces.

Our result also bears some resemblance with the characterization of superreflexive Banach spaces as those that can be renormed to be uniformly convex [6] and moreover in such a manner as to have modulus of uniform convexity of power type [23].

In Section 1 we recall the notion of asymptotic structure of a space X with an FDD (E_n), generalize it, and connect it with certain blockings of the FDD (finite dimensional decomposition). In particular we observe that given $\varepsilon_i \downarrow 0$ there exists a blocking (F_n) of (E_n) so that for all n if $(x_i)_1^n$ is any skipped normalized sequence with respect to $(F_i)_n^\infty$ then $(x_i)_1^n$ is (up to ε_n) an element of the nth-asymptotic structure $\{X, (E_i)\}_n$.

Section 3 concerns connections between our indices and lower ℓ_p estimates. In Section 4 we solve our problem in the case where X has a shrinking FDD. Section 5 handles the general case. Section 6 concerns some dual results and contains further remarks. In the last section we discuss results concerning the ω^*-UKK modulus $\delta(\varepsilon)$.

Our notation is standard. $X, B, Y, Z, \ldots$ will be separable infinite dimensional Banach spaces and $F, G, H, \ldots$ will be used for finite dimensional spaces. B_X is the unit ball of X and S_X is the unit sphere of X. $\langle\cdot\rangle$ denotes linear span and $[\cdot]$ is the closed linear span.

1. Asymptotic Structure

In this section we make a connection between the asymptotic structure of a Banach space X with respect to an FDD (E_i) and finite sequences $(x_i)_1^k$, in a certain skipped blocking of (E_i). For example we show that given k and $\varepsilon > 0$ we can find a blocking (F_j) of (E_i) so that, up to ε, every normalized skipped sequence $(x_i)_1^k$ with respect to (F_j) belongs to $\{X, (E_i)\}_k$.

We begin by recalling the precise meaning of this last creature [20]. Let (E_i) be an FDD for X. We shall assume that (E_n) is monotone. This is not essential but rather is for convenience, here and throughout. Given $k \in \mathbb{N}$ and $C \geqslant 1$ we let $\mathcal{M}_k(C)$ be the set of all normalized C-basic sequences of length k. $\mathcal{M}_k(C)$ is a compact metric space under the metric $\log d_b(\cdot, \cdot)$ where

$$d_b\big((x_i)_1^k, (y_i)_1^k\big) = \inf\Bigg\{AB : \forall\ (a_i)_1^k \subseteq \mathbb{R}\ ,$$
$$A^{-1}\Big\| \sum_1^k a_i y_i\Big\| \leqslant \Big\| \sum_1^k a_i x_i\Big\| \leqslant B\Big\| \sum_1^k a_i y_i\Big\|\Bigg\}\ .$$

Set $\mathcal{M}_k = \mathcal{M}_k(1)$.

DEFINITION 1.1. Let $(e_i)_1^k \in \mathcal{M}_k$. Then $(e_i)_1^k \in \{X, (E_i)\}_k$ if

$$\begin{aligned}&\forall\, \varepsilon > 0\ \forall\, n_1\ \exists\, x_1 \in S_{\langle E_i\rangle_{i\geqslant n_1}}\\ &\qquad \forall\, n_2\ \exists\, x_2 \in S_{\langle E_i\rangle_{i\geqslant n_2}} \cdots\\ &\qquad \forall\, n_k\ \exists\, x_k \in S_{\langle E_i\rangle_{i\geqslant n_k}} \text{ so that } d_b\big((e_i)_1^k, (x_i)_1^k\big) < 1+\varepsilon.\end{aligned}$$

We will now introduce a generalization of asymptotic structures. One advantage of this generalization is a simplification of the proofs.

For this let (M, ρ) be a complete metric space and $L : X \to M$ be uniformly continuous on bounded sets mapping bounded sets into relative compact sets. For $k \in \mathbb{N}$ let $\mathrm{C}([-1,1]^k, M)$ be the space of continuous mappings on $[-1,1]^k$ into M equipped with the topology of uniform convergence, i.e. $\mathrm{C}([-1,1]^k, M)$ is equipped with the metric ρ_k, where

$$\rho_k(f,g) = \sup_{|a_i|\leqslant 1, i=1,..k} \rho(f(a_1,\dots a_k), g(a_1,\dots a_k)),$$
$$\text{for } f, g \in \mathrm{C}([-1,1]^k, M).$$

For $x_1, x_2, \dots x_k \in X$ we denote by $L_{(x_1,x_2,\dots x_k)}$ the map:

$$L_{(x_1,x_2,\dots x_k)} : [-1,1]^k \ni (a_1, a_2, \dots, a_k) \mapsto L\left(\sum_{i=1}^k a_i x_i\right).$$

DEFINITION 1.2. Let $f \in C([-1,1]^k, M)$ Then $f \in L - \{X, (E_i)\}_k$ if

$$\begin{aligned} &\forall\, \varepsilon > 0\ \forall\, n_1\ \exists\, x_1 \in S_{\langle E_i\rangle_{i\geqslant n_1}} \\ &\qquad\qquad \forall\, n_2\ \exists\, x_2 \in S_{\langle E_i\rangle_{i\geqslant n_2}} \cdots \\ &\qquad\qquad \forall\, n_k\ \exists\, x_k \in S_{\langle E_i\rangle_{i\geqslant n_k}} \text{ so that } \rho_k(f, L_{(x_1,x_2,\ldots x_k)}) < \varepsilon. \end{aligned}$$

Note that $\{X, (E_i)\}_k$ of Definition 1.1 coincides with $||\cdot|| - \{X, (E_i)\}_k$ of Definition 1.2.

This notion can also be understood in terms of countably branching trees of length k on S_X. We let T_k be the tree $T_k = \{(n_1, \ldots, n_j) : 1 \leqslant j \leqslant k$ where $\forall\, i,\, n_i \in \mathbb{N}\}$ ordered by $(n_1, \ldots, n_j) \leqslant (m_1, \ldots, m_\ell)$ if $j \leqslant \ell$ and $n_i = m_i$ for $i \leqslant j$. For two elements $(n_1, \ldots n_\ell)$ and $(m_1, \ldots, m_\ell)$ of the same length we also introduce the order $\preceq$ given by $(n_1, \ldots n_\ell) \preceq (m_1, \ldots, m_\ell)$ if $n_i \leqslant m_i$, for all $i = 1, 2 \ldots \ell$. Then $T_k(X)$ is the set of all trees on S_X indexed by T_k. Thus $\mathcal{T} \in T_k(X)$ if $\mathcal{T} = \{x(n_1, \ldots, n_j) : (n_1, \ldots, n_j) \in T_k\} \subseteq S_X$ where the order on $\mathcal{T}$ is that induced by T_k. We call such a $\mathcal{T}$ a *block tree* with respect to (E_i) if for all $\alpha \in T_{k-1} \cup \{\emptyset\}$ $(x(\alpha, n))_{n\in\mathbb{N}}$ is a block with respect to (E_i).

A tree $\mathcal{T} \in T_k(X)$ is *C-basic* if all branches $(x(n_1, \ldots, n_j))_{j=1}^k$ are C-basic sequences.

DEFINITION 1.3. Let $\mathcal{T} = (x(n_1, \ldots, n_j))_{j\leqslant k, n_1, \ldots, n_j \in \mathbb{N}} \in T_k(X)$. We say that T *L-converges* to $f \in C([-1,1]^k, M)$ if

$$\lim_{n_1\to\infty} \lim_{n_2\to\infty} \ldots \lim_{n_k\to\infty} \rho_k(L_{(x(n_1,\ldots n_j), j=1,\ldots k)}, f) = 0.$$

By a subtree of a tree $\mathcal{T} = (x(\alpha))_{\alpha\in T_k} \in T_k(X)$ we mean a family $\mathcal{T}' = (x(\alpha))_{\alpha\in T'}$ with $T' \subset T_k$ having the property that for any $\alpha \in (T' \cap T_{k-1}) \cup \{\emptyset\}$ the set $\{n \in \mathbb{N} : (\alpha, n) \in T'\}$ is infinite. Note that $\mathcal{T}'$ can be order isomorphically reordered (with respect to $\leqslant$ and $\preceq$) in a unique way into a tree $(x'(\alpha) : \alpha \in T_k)$. We will identify $\mathcal{T}'$ with the tree $(x'(\alpha) : \alpha \in T_k)$.

Note that if a tree $\mathcal{T}$ L-converges to $f \in C([-1,1]^k, M)$ and $\varepsilon_k \searrow 0$ then there is a subtree so that for all $n_1, n_2, \ldots n_k \in \mathbb{N}$ it follows that $\rho_k(L_{(x(n_1,\ldots n_j), j=1,\ldots k)}, f) < \varepsilon_{n_1}$.

The following proposition follows easily from the definitions above.

PROPOSITION 1.4. *Let (E_n) be a monotone FDD for X. $f \in L - \{X, (E_i)\}_k$ iff there exists $\mathcal{T} \in T_k(X)$, a block tree with respect to (E_i), which L-converges to f.*

The following proposition follows from an easy compactness argument by induction on all $k \in \mathbb{N}$.

PROPOSITION 1.5. *Every tree in $T_k(X)$ has an L-convergent subtree.*

Proof. From the Theorem of Arzela and Ascoli it follows that the set

$$\mathcal{L}_k = \{L_{(x_1,x_2,\ldots x_k)} : x_1, x_2, \ldots x_k \in B_X\}$$

is relatively compact in $C([-1,1]^k, M)$.

For $k = 1$ the claim follows from the relative compactness of $\mathcal{L}_1$. If the claim is true for $k-1$ and if $x(n_1, n_2, \dots n_j)_{(n_1,n_2,\dots n_j)\in\mathcal{T}_k} \in T_k(X)$, we first fix $n \in \mathbb{N}$ and apply the induction hypothesis to the map

$$L^{(x_n)} : C([-1,1], M), \quad y \mapsto L(\cdot)x_n + y)$$

(M is replaced by the space $C([-1,1], M)$) and get a subtree $(x'(n,\alpha))_{\alpha\in\mathcal{T}_k}$ of $(x(n,\alpha))_{\alpha\in\mathcal{T}_k}$ which $L^{(x_n)}$-converges to some $f_n \in C([-1,1]^{k-1}, C([-1,1], M)) \equiv C([-1,1]^k, M)$. Since the f_n's are in the compact set $\overline{\mathcal{L}_k}^{\rho_k}$ they have a convergent subsequence $(f_n)_{n\in N}$ reordering now the family $(x'(n,\alpha))_{n\in N,\alpha\in\mathcal{T}_k}$ gives the answer to the claim. □

Since we shall be concerned as well with subspaces of X we relativize the above definitions. For an interval $I \subseteq \mathbb{N}$, $P_{\langle E_i\rangle_I}$ is the FDD projection of X onto $\langle E_i\rangle_{i\in I}$.

Unless specified otherwise we let Y be a subspace of X, and $L : Y \to M$ be uniformly continuous mapping bounded subsets of Y to relatively compact sets in M.

DEFINITION 1.6. Let (E_n) be a monotone FDD for X and let $Y \subseteq X$ be a subspace. For $k \in \mathbb{N}$ and $f \in C([-1,1]^k, M)$ we say $f \in L - \{Y, (E_i)\}_k$ if $\forall\, \varepsilon > 0$

$$\forall\, n_1\ \forall\, \varepsilon_1 > 0\ \exists\, y_1 \in S_Y \text{ with } \|P_{\langle E_i\rangle_1^{n_1}} y_1\| < \varepsilon_1$$
$$\forall\, n_2\ \forall\, \varepsilon_2 > 0\ \exists\, y_2 \in S_Y \text{ with } \|P_{\langle E_i\rangle_1^{n_2}} y_2\| < \varepsilon_2$$
$$\dots$$
$$\forall\, n_k\ \forall\, \varepsilon_k > 0\ \exists\, y_k \in S_Y \text{ with } \|P_{\langle E_i\rangle_1^{n_k}} y_k\| < \varepsilon_k$$

such that $\rho_k(L_{(y_1,y_2,\dots y_k)}, f) < \varepsilon$.

DEFINITION 1.7. Let (E_i) be an FDD for X and let $Y \subseteq X$. Let $\mathcal{T} = (y(n_1, \dots, n_j))_{T_k} \in T_k(Y, (E_i))$. $\mathcal{T}$ is an *asymptotic block tree on* Y with respect to (E_i), denoted $\mathcal{T} \in a - T_k(Y)$, if for all $s \in \mathbb{N}$, $\lim_{n\to\infty} \|P_{\langle E_i\rangle_1^s} y(n)\| = 0$ and $\lim_{n\to\infty} \|P_{\langle E_i\rangle_1^s}(y(n_1, \dots, n_j, n))\| = 0$ for all $(n_1, \dots, n_j) \in T_{k-1}$.

Proposition 1.4 becomes

PROPOSITION 1.8. *Let (E_i) be an FDD for X and let $Y \subseteq X$ and $k \in \mathbb{N}$. $f \in L - \{Y, (E_i)\}_k$ iff tree $\mathcal{T} \in a - T_k(Y)$ which L-converges to f.*

Next we relate the asymptotic structure of Y to a certain blocking of the FDD (E_i) for X. Recall that (F_j) is a *blocking* of (E_i) if there exist integers $0 = p_0 < p_1 < \cdots$ so that for all j, $F_j = \langle E_i\rangle_{i=p_{j-1}+1}^{p_j}$. $(H_j)_1^\infty$ is a *skipped blocking* of (F_j) if there exists integers $r_1 \leqslant s_1 < s_1 + 1 < r_2 \leqslant s_2 < s_2 + 1 < r_3 \leqslant s_3 < \cdots$ so that $H_j = \langle F_i\rangle_{i=r_j}^{s_j}$ for all j. $(x_j) \subseteq X$ is a (*skipped*) *block sequence* with respect to (F_j) if there exists a (skipped) blocking (H_j) of (F_j) with $x_j \in H_j$ for all j.

PROPOSITION 1.9. *Assume that L is defined on X. Let (E_n) be a monotone FDD for X. Let $\varepsilon_n \downarrow 0$. Then there exists a blocking (F_j) of (E_n) so that for all k and all skipped normalized sequences $(x_i)_1^k$ with respect to $(F_j)_{j=k}^\infty$,*

$$\rho_k\Big(L_{(x_1,x_2,\dots x_k)}, L - \{X, (E_i)\}_k\Big) < \varepsilon_k \ .$$

Rather than prove this we give the proof of the relativized result. This will require that (E_n) be boundedly complete and hence X is naturally a dual space, the dual of $[(E_n^*)] \subseteq X^*$. When we say here and in the sequel that $Y \subseteq X$ is ω^* closed we mean with respect to the ω^* topology thus generated on X. Proposition 1.7 is proved similarly to 1.8 but the boundedly complete hypothesis is never needed. ω^* convergence of a bounded sequence in X is just coordinatewise convergence with respect to (E_n).

PROPOSITION 1.10. *Let (E_n) be a monotone boundedly complete FDD for X and let $Y \subseteq X$ be ω^* closed. Let $\varepsilon_n \downarrow 0$. There exist $\delta_k \downarrow 0$ and a blocking (F_j) of (E_n) with the following property. Given $k \in \mathbb{N}$ if $(y_i)_1^k \subseteq S_Y$ satisfies $\exists\, k \leqslant m_0 < m_1 < \cdots < m_k$ so that for $1 \leqslant j \leqslant k$,*

$$\|(I - P_{\langle F_i\rangle_{m_{j-1}+1}^{m_j-1}})y_j\| < \delta_k$$

then $\rho_k(L_{(y_1,\dots y_k)}, L - \{Y, (E_i)\}_k) < \varepsilon_k$.

In other words if $(y_i)_1^k$ is almost a normalized skipped block sequence with respect to $(F_j)_k^\infty$ then $L_{(y_1,\dots y_k)}$ is close to being in $L - \{Y, (E_i)\}_k$. Proposition 1.10 follows by iterating the following fixed k result.

PROPOSITION 1.11. *Let (E_n) be a monotone boundedly complete FDD for X and let $Y \subseteq X$ be ω^* closed. Then for all $\varepsilon > 0$ and $k \in \mathbb{N}$ there exist $N_1 \in \mathbb{N}$, a blocking (F_j) of $(E_i)_{N_1}^\infty$ and $\delta > 0$ so that if $(y_i)_1^k \subseteq S_Y$ satisfies there exists $0 = m_0 < m_1 < \cdots < m_k$ with*

$$\Big\|(I - P_{\langle F_j\rangle_{m_{j-1}+1}^{m_j-1}})y_j\Big\| < \delta \ \textit{for} \ \ j \leqslant k$$

then $\rho_k(L_{(y_1,y_2,\dots y_k)}, L - \{Y, (E_i)\}_k) < \varepsilon$.

Proof. We begin by showing how to deduce the proposition from the

Claim. $\exists\, \delta > 0\ \exists\, N_1 \in \mathbb{N}\ \forall\, y_1 \in S_Y$ with $\|P_{\langle E_i\rangle_1^{N_1}} y_1\| < \delta\ \exists\, N_2 \in \mathbb{N}\ \forall\, y_2 \in S_Y$ with $\|P_{\langle E_i\rangle_1^{N_2}} y_2\| < \delta \cdots \exists\, N_k \in \mathbb{N}\ \forall\, y_k \in S_Y$ with $\|P_{\langle E_i\rangle_1^{N_k}} y_k\| < \delta$ one has

$$\rho_k(L_{(y_1,y_2,\dots y_k)}, L - \{Y, (E_i)\}_k) < \varepsilon/2.$$

Indeed assume the claim.

Let $\delta' > 0$ be chosen so that for any two sequences $(y_i)_{i=1}^k$ and $(z_i)_{i=1}^k$ in S_Y for which we assume that $||y_i - z_i|| < \delta'$, $i = 1, \dots k$ and that $\rho_k(L_{(z_1,z_2,\dots z_k)}, L - \{Y, (E_i)\}_k) < \varepsilon/2$ it follows that $\rho_k(L_{(y_1,y_2,\dots y_k)}, L - \{Y, (E_i)\}_k) < \varepsilon$. Without loss of generality we can assume that $\delta < \delta'/3$. Choose $p_1 > N_1$ so that $\{y \in S_Y : \|(I - P_{\langle E_i \rangle_{N_1}^{p_1}})y\| < \delta\} \neq \emptyset$ and let S_1 be a finite δ'-net for this set. This can be done as follows. First choose a finite δ-net $\widetilde{S}$ of $P_{\langle E_i \rangle_{N_1}^{p_1}}(\{y \in S_Y : \|(I - P_{\langle E_i \rangle_{N_1}^{p_1}})y\| < \delta\})$, then choose for each $\tilde{y} \in \widetilde{S}$ a y in $\{y \in S_Y : \|(I - P_{\langle E_i \rangle_{N_1}^{p_1}})y\| < \delta\}$ so that $P_{\langle E_i \rangle_{N_1}^{p_1}}(y) = \tilde{y}$. The set of all such elements y has then the required property.

Choose p_2 sufficiently large to satisfy the claim ("$\exists\ N_2, \dots$") for all $y \in S_1$. Define $F_1 = \langle E_i \rangle_{N_1}^{p_1}$ and $F_2 = \langle E_i \rangle_{p_1+1}^{p_2}$. We choose $S_{1,2}$, a finite δ'-net for

$$\{y \in S_Y : \|(I - P_{\langle F_1, F_2 \rangle})y\| < \delta\} .$$

Choose p_3 sufficiently large to satisfy the claim ("$\exists\ N_2, \dots$") for all $y \in S_{1,2}$. Set $F_3 = \langle E_i \rangle_{p_2+1}^{p_3}$.

Notation. If F_i have been defined for all $i \in I$, some interval in $\mathbb{N}$, we let S_I be a finite δ'-net for

$$\{y \in S_Y : \|(I - P_{\langle F_j \rangle_I})y\| < \delta\} .$$

We shall say intervals $I_1 < \dots < I_j$ of integers are *skipped* if

$$\max I_i + 1 < \min I_{i+1} \quad \text{for} \quad i < j \ .$$

Suppose that $F_j = \langle E_i \rangle_{p_{j-1}+1}^{p_j}$ has been defined. Choose p_{j+1} large enough to satisfy the claim for all skipped intervals $I_1 < \dots < I_\ell$ in $\{1, \dots, j\}$ for any $y_1, \dots, y_\ell$ with $y_i \in S_{I_i}$ (using "$\exists\ N_{\ell+1} \dots$"). Define $F_{j+1} = \langle E_i \rangle_{p_j+1}^{p_{j+1}}$.

Let $(y_i)_1^k$ be as in the statement of Proposition 1.11 with respect to the blocking (F_j) of $(E_i)_{N_1}^\infty$ just constructed. Thus for some sequence $I_1 < \dots < I_k$ of skipped intervals we have

$$\|(I - P_{\langle F_j \rangle_{I_\ell}})y_\ell\| < \delta \quad \text{for} \quad \ell \leqslant k \ .$$

For $\ell \leqslant k$ choose $z_\ell \in S_{I_\ell}$ with $\|z_\ell - y_\ell\| < \delta'$. From our construction using the claim we have

$$\rho_k\Big(L_{(z_1,z_2,\dots z_k)}, L - \{Y, (E_i)\}_k\Big) < \varepsilon/2 \ .$$

The choice of δ' finally implies then that

$$\rho_k\Big(L_{(y_1,y_2,\dots y_k)}, L - \{Y, (E_i)\}_k\Big) < \varepsilon \ .$$

Proof of the claim. If false then

$$\begin{aligned}
\forall\, \delta > 0\ \forall\, N_1\ \exists\, y_1 \in S_Y\ ,\ \ & \|P_{\langle E_i\rangle_1^{N_1}} y_1\| < \delta \\
\forall\, N_2\ \exists\, y_2 \in S_Y\ ,\ \ & \|P_{\langle E_i\rangle_1^{N_2}} y_2\| < \delta \\
\dots & \\
\forall\, N_k\ \exists\, y_k \in S_Y\ ,\ \ & \|P_{\langle E_i\rangle_1^{N_k}} y_k\| < \delta
\end{aligned}$$

yet

$$\rho_k\Big(L_{(y_1,y_2,\dots y_k)}, L-\{Y,(E_i)\}_k\Big) \geqslant \varepsilon/2\ .$$

Fix $\delta > 0$. By the above we can find a tree $\mathcal{T} = \{y(n_1,\dots,n_j) : (n_1,\dots,n_j) \in T_k\} \in T_k(Y)$ so that for all s and all $(n_1,\dots,n_j)$, $j < k$

$$\limsup_{n\to\infty} \Big\| P_{\langle E_i\rangle_1^s} y(n_1,\dots,n_j,n)\Big\| \leqslant \delta$$

and for all $(n_1,\dots,n_k)$

$$\rho_k\Big(L_{((y(n_i))_{i=1}^j : j\leqslant k)}, L-\{Y,(E_i)\}_k\Big) \geqslant \varepsilon/2\ .$$

Since for all s, $\limsup_{n\to\infty} \|P_{\langle E_i\rangle_1^s} y(n)\| \leqslant \delta$ using that Y is ω^* closed we may choose $y(n_i) \xrightarrow{\omega^*} y_0 \in Y$ with $\|y_0\| \leqslant \delta$. We then repeat this argument at the next level to the successors of each $y(n_i)$ and so on. Ultimately thus pruning our tree but leaving behind an isomorphic subtree we see that we may assume without loss of generality each $y(n_1,\dots,n_j) = y_0(n_1,\dots,n_{j-1}) + z(n_1,\dots,n_j)$ where $\|y_0(n_1,\dots,n_{j-1})\| \leqslant \delta$, both $y_0(n_1,\dots,n_{j-1})$ and $z(n_1,\dots,n_{j-1})$ belong to Y and $\omega^*\text{-}\lim_{n\to\infty} z(n_1,\dots,n_{j-1},n) = 0$. Let

$$w(n_1,\dots,n_j) = \frac{z(n_1,\dots,n_j)}{\|z(n_1,\dots,n_j)\|}\ .$$

Then $(w(n_1,\dots,n_j)_{T_k} \in a - T_k(Y,(E_i))$.

By Proposition 1.5 and passing to a subtree we may assume that this tree L-converges to some $f \in L-\ \mathrm{C}([-1,1]^k, M)$. From Proposition 1.8, $f \in L - \{Y,(E_i)\}_k$. From $\|y(n_1,\dots,n_j) - z(n_1,\dots,n_j)\| \leqslant \delta$ we obtain that

$$\|y(n_1,\dots,n_j) - w(n_1,\dots,n_j)\| \leqslant \frac{2\delta}{1-\delta}\ .$$

Thus by a perturbation argument choosing $\delta = \delta(\varepsilon)$ sufficiently small we obtain that

$$\rho_k\Big(L_{((y(n_1,\dots n_j))_{j=1}^k)}, L-\{Y,(E_i)\}_k\Big) \geqslant \varepsilon/2$$

for $n_1 < n_2 < \ldots n_k$ large. This is a contradiction. □

2. Indices

We define the Szlenk index of a separable Banach space X and another index which we call the H-index and make some connections between them. The latter index is defined in terms of the asymptotic structure in the setting where X has an FDD or is a subspace of a space with an FDD.

DEFINITION 2.1. The Szlenk Index

Let B be a separable Banach space and let $X = B^*$. Thus (B_X, ω^*) is a compact metric space. Let $0 < \varepsilon < 1$. Let $S_0(B, \varepsilon) = B_X$. If $S_\alpha(B, \varepsilon)$ has been defined for $\alpha < \omega_1$ we let

$$S_{\alpha+1}(B,\varepsilon) = \Big\{x : \exists\, (x_n) \subseteq S_\alpha(B,\varepsilon) \text{ with } \omega^*\text{-}\lim_{n\to\infty} x_n = x \text{ and } \liminf_{n\to\infty} \|x_n - x\| \geqslant \varepsilon\Big\}\,.$$

If $\alpha < \omega_1$ is a limit ordinal we set

$$S_\alpha(B,\varepsilon) = \bigcap_{\beta<\alpha} S_\beta(B,\varepsilon)\,.$$

Szlenk's original index [26] was defined somewhat differently. However by Rosenthal's ℓ_1 theorem [25] the two indices are equivalent if B contains no isomorph of ℓ_1. Furthermore

$$\sup_{\varepsilon>0}\{\alpha : S_\alpha(B,\varepsilon) \neq \emptyset\} < \omega_1$$

if and only if $X = B^*$ is separable.

We will say that B has *finite Szlenk index* if for all $\varepsilon > 0$ there exists $k \in \mathbb{N}$ with $S_k(B, \varepsilon) = \emptyset$. There is a natural relation between this index and trees on $X = B^*$ (see also [1]).

PROPOSITION 2.2. *Let B be a separable Banach space and $X = B^*$. Let $\varepsilon > 0$, $k \in \mathbb{N}$ and $x_0 \in S_{k+1}(B, \varepsilon)$.*

Then there exists a tree $\{x(n_1, \ldots, n_j) : (n_1, \ldots, n_j) \in T_k\} \subseteq X$ so that

(1) ω^*-$\lim x(n) = x_0$

(2) ω^*-$\lim_{n\to\infty} x(n_1, \ldots, n_j, n) = x(n_1, \ldots, n_j)$ *for all* $(n_1, \ldots, n_j) \in T_{k-1}$

(3) $\liminf \|x(n) - x_0\| \geqslant \varepsilon$

(4) $\liminf_{n\to\infty} \|x(n_1, \ldots, n_j, n) - x(n_1, \ldots, n_j)\| \geqslant \varepsilon$ *for all* $(n_1, \ldots, n_j) \in T_{k-1}$.

By taking the difference tree of the above tree as we did in the proof of Proposition 1.11 in the previous section we obtain the following.

PROPOSITION 2.3. *Let $X = B^*$ be a separable dual, $k \in \mathbb{N}$, $\varepsilon > 0$ and assume $S_{k+1}(B, \varepsilon) \neq \emptyset$.*

Then there exists a tree $(z(n_1, \dots, n_j))_{T_k} \subseteq 2B_X$ with for all $(n_1, \dots, n_j)$, $j < k$,

(1) $\liminf_{n\to\infty} \|z(n_1, \dots, n_j, n)\| \geqslant \varepsilon$

(2) $\omega^*\text{-}\lim_{n\to\infty} z(n_1, \dots, n_j, n) = 0$

(3) For all $(n_1, \dots, n_k)$,

$$\Big\| \sum_{j=1}^{k} z(n_1, \dots, n_j) \Big\| \leqslant 2\ .$$

This leads us to make the following definitions.

DEFINITION 2.4. Let (x_i) be a basic sequence (of possibly finite length). Let $0 < \varepsilon < 1$. The *strong index* of (x_i) is

$SI((x_i), \varepsilon) = \sup\{k : \exists\ (a_i)_1^k$ with $\varepsilon \leqslant |a_i| \leqslant 1$ for $i \leqslant k$ and a normalized block basis $(y_i)_1^k$ of (x_i) so that $\| \sum_1^k a_i y_i \| \leqslant 1\}$.

We then use this to define an index based upon the strong index of the asymptotic structure of a space.

DEFINITION 2.5. Let X have a monotone FDD (E_n) and let $Y \subseteq X$ and $\varepsilon > 0$. $H(Y, (E_i), \varepsilon) = \sup\{SI((e_i)_1^k, \varepsilon) : k \in \mathbb{N}$ and $(e_i)_1^k \in \{Y, (E_i)\}_k\}$.

As noted in [20] it is easy to see that if $(x_i)_1^n$ is a normalized block basis of $(e_i)_1^k \in \{Y, (E_i)\}_k$ then $(x_i)_1^n \in \{Y, (E_i)\}_n$. Thus we have

PROPOSITION 2.6. *Let (E_n) be a monotone FDD for X and let $Y \subseteq X$ and $\varepsilon > 0$. Then $H(Y, (E_i), \varepsilon) = \sup\{k : \exists\ (e_i)_1^k \in \{Y, (E_i)\}_k$ and $(a_i)_1^k \subseteq [\varepsilon, 1]$ with $\| \sum_1^k a_i e_i \| \leqslant 1\}$.*

REMARK 2.7. Our next result yields that the Szlenk index of a space B with separable dual Y is finite iff the H-index of Y with respect to a certain FDD is finite as well. Recall that if $B^* = Y$ is separable then B is a quotient of a space with a shrinking basis [3]. It follows that Y is a subspace of a space X with a boundedly complete basis and moreover the ω^* topology on Y induced by B agrees with the relative ω^* topology on Y obtained by regarding the space X as $[(e_n^*)]^*$ where (e_n) is the boundedly complete basis for X. ω^*-convergence in this setting of a bounded sequence is just coordinatewise convergence. For convenience in calculations we take the basis in question or more generally an FDD to be bimonotone.

PROPOSITION 2.8. *Let* (E_n) *be a bimonotone boundedly complete FDD for* X *and let* $Y = B^*$ *be a* ω^* *closed subspace. Let* $\varepsilon > 0$.

(a) $S_{k+1}(B,\varepsilon) \neq \emptyset \Longrightarrow H(Y,(E_i),\varepsilon/2) \geqslant k$

(b) $H(Y,(E_i),\varepsilon) \geqslant k \Longrightarrow S_k(B,\varepsilon/2) \neq \emptyset$.

Proof. Let $\varepsilon > 0$. Suppose that $S_{k+1}(B,\varepsilon) \neq \emptyset$. Let $\overline{\mathcal{T}} = (z(n_1,\dots,n_j))_{T_k} \subseteq 2B_Y$ be the tree given by Proposition 2.3. Define $w(n_1,\dots,n_j) = \frac{z(n_1,\dots,n_j)}{\|z(n_1,\dots,n_j)\|}$ and let $\mathcal{T} = (w(n_1,\dots,n_j)$ be the corresponding tree. Clearly $\mathcal{T} \in a - T_k(Y)$ and by pruning we may assume all branches are 2-basic. Thus we may assume by Proposition 1.5 that $\mathcal{T}$ converges to some $(e_i)_1^k \in \{Y,(E_i)\}_k$. Note that for all $(n_1,\dots,n_k)$,

$$\left\|\sum_1^k \|z(n_1,\dots,n_j)\| w(n_1,\dots,n_j)\right\| \leqslant 2$$

by (3) of Proposition 2.3.

In other words for arbitrarily large n and $\varepsilon' < \varepsilon$ we can find some branch of $\mathcal{T}$ and coefficients all exceeding ε' so that the norm of the ensuing sum does not exceed 2. It follows that $H(Y,(E_i),\varepsilon/2) \geqslant k$ which proves (a).

Next let $H(Y,(E_i),\varepsilon) \geqslant k$. Thus there exists $(e_i)_1^k \in \{Y,(E_i)\}_k$ and $(a_i)_1^k \subseteq [\varepsilon,1]$ with $\|\sum_1^k a_i e_i\| \leqslant 1$. By Proposition 1.8 there exists $\mathcal{T} = (w(n_1,\dots,n_j))_{T_k} \in a - T_k(Y)$ which converges to $(e_i)_1^k$.

Let $y(n_1,\dots,n_j) = \sum_{i=1}^j a_i w(n_1,\dots,n_i)$. By the convergence of $\mathcal{T}$ to $(e_i)_1^k$ we may assume that $\|y(n_1,\dots,n_j)\| < 2$ for all $(n_1,\dots,n_j) \in T_k$. Moreover for $j < k$,

$$\liminf_{n\to\infty} \|y(n_1,\dots,n_j) - y(n_1,\dots,n_j,n)\| \geqslant \varepsilon$$

and $\omega^*\text{-}\lim_{n\to\infty} y(n_1,\dots,n_j,n) = y(n_1,\dots,n_j)$. It follows that $S(Y,\varepsilon/2) \geqslant k$. Indeed $\frac{y(n_1,\dots,n_j)}{2} \in S_{k-j}(B,\varepsilon/2)$ for $1 \leqslant j \leqslant k$ and so $0 = \omega^*\text{-}\lim \frac{y(n)}{2}$ belongs to $S_k(B,\varepsilon/2)$. □

PROPOSITION 2.9. *Let* (E_i) *be a bimonotone FDD for* X *and let* $Y \subseteq X$. *Let* $0 < \varepsilon < 1$. *Then*

(a) $H(Y,(E_i),\varepsilon') \leqslant H(Y,(E_i),\varepsilon)$ *if* $\varepsilon' \geqslant \varepsilon$.

(b) $H(Y,(E_i),\varepsilon^2) \leqslant [H(Y,(E_i),\varepsilon)+1]^2$

Proof. We need only prove (b). Let $H(Y,(E_i),\varepsilon) = k$. Assume $(e_i)_1^{(k+1)^2} \in \{Y,(E_i)\}_{k+1}$ is such that there exist $(a_i)_1^{(k+1)^2} \subseteq [\varepsilon,1]$ with $\|\sum_1^{(k+1)^2} a_i e_i\| \leqslant 1$. For $1 \leqslant j \leqslant k+1$ define $x_j = \frac{1}{b_j}\sum_{i=(j-1)(k+1)+1}^{j(k+1)} a_i e_i$ to be norm 1.

Since (E_i) is bimonotone we see that $b_i \leqslant 1$ for $i \leqslant k+1$. Also $b_i > \varepsilon$ by the definition of $H(Y,(E_i),\varepsilon) = k$. This uses that x_i is formed as a sum of $k+1$ e_j's with coefficients at least ε^2. Note that $\|\sum_1^{k+1} b_j x_j\| \leqslant 1$. But this contradicts our choice of k. □

DEFINITION 2.10. Let X have a bimonotone FDD (E_i) and let $Y \subseteq X$. We say Y has *finite H-index* if $H(Y, (E_i), \varepsilon) < \omega$ for some (and thus by Proposition 2.9 for all) $0 < \varepsilon < 1$.

In the terminology of [9] a space B with separable dual Y has finite Szlenk index for all $\varepsilon > 0$ iff $\|\cdot\| : (B_Y, \omega^*) \to \mathbb{R}$ is Baire-1/2. See [9] and [16] for more on the general theory of Baire-1/2 functions.

REMARK 2.11. One can define the previous concepts using the weak rather than the ω^* topology. This was done by Huff [10] who attributes the idea to Bourgain. Thus the weak index of X would be given by

$$W_{\alpha+1}(X, \varepsilon) = \{x : \exists\, (x_n) \subseteq W_\alpha(X, \varepsilon)\ ,\ x_n \xrightarrow{\omega} x \text{ and } \liminf_{n\to\infty} \|x_n - x\| \geqslant \varepsilon\}\ .$$

Of course in the reflexive case, $X = B^*$ we get that for all $\varepsilon > 0$, $W(X, \varepsilon) \equiv \sup\{\alpha : W_\alpha(X, \varepsilon) \neq \emptyset\} < \omega$ iff B has finite Szlenk index.

The notion of *weak asymptotic structure* could also be defined in terms of trees. For $Y \subseteq X$, X having an FDD (E_n), a normalized basic sequence $(e_i)_1^k \in w - \{Y, (E_i)\})k$ if there exists $\mathcal{T} = (y(n_1, \dots, n_j))_{T_k} \in a - T_k(Y)$ with respect to (E_i) so that $\mathcal{T}$ converges to $(e_i)_1^k$ and so that for all $(n_1, \dots, n_j) \in T_{k-1}$, ω-$\lim_n y(n_1, \dots, n_j, n) = 0$. Of course the weak asymptotic structure could differ from the asymptotic structure but some of the properties of asymptotic structure do still hold in this setting. We state one such result.

PROPOSITION 2.12. *Let X have an FDD (E_i) and let $Y \subseteq X$. Assume that Y does not contain an isomorph of ℓ_1. Let $(e_i)_1^k \in w - \{Y, (E_i)\}$, and let $(y_i)_1^m$ be a normalized block basis of $(e_i)_1^k$. Then $(y_i)_1^m \in w - \{Y, (E_i)\}_m$.*

This follows easily from the following

LEMMA 2.13. *Let $\mathcal{T}$ be a tree in B_Y which is order isomorphic to T_k. Assume Y does not contain ℓ_1 and that the initial nodes of $\mathcal{T}$ are weakly null and all successors of a given node in $\mathcal{T}$ are weakly null. Then there exists a subtree $\mathcal{T}' = (y(n_1, \dots, n_j))_{T_k}$ of $\mathcal{T}$ which is order isomorphic to $\mathcal{T}$ and satisfies ω-$\lim_n y_m = 0$ whenever $y_m = \sum_{j=1}^k y(n_1^{(m)}, \dots, n_j^{(m)})$ for some $(n_1^{(m)}, \dots, n_k^{(m)}) \in T_k$ with $n_1^{(m)} = m$.*

Proof. This can be deduced from a k-dimensional version of Corollary 3 in [17]. $\square$

3. Lower ℓ_p Estimates

PROPOSITION 3.1. *Let (e_i) be a bimonotone basic sequence with $SI((e_i), 1/2) \equiv n_0 < \infty$. Then there exists $p = p(n_0) \in (1, \infty)$ so that if $(x_i)_1^m$ is any block basis of (e_i) then*

$$\Big\| \sum_1^m x_i \Big\| \geqslant \frac{1}{2}\Big(\sum_1^m \|x_i\|^p\Big)^{1/p}.$$

REMARK 3.2. This lemma is known. It follows from proofs of similar results given in [11] or in [13]. In the latter the result is presented in an unconditional setting for disjoint blocks but the same proof works in our setting. We choose to present our own proof. The idea of the proof is used for a later result.

Proof. The proof of Proposition 2.9 also yields that $SI((e_i), 1/4) \leqslant [SI((e_i), 1/2) + 1]^2$. Let $n = 4n_0 + 1$ and choose $p \in (1, \infty)$ with $2^p = n$. We may assume $\|e_i\| = 1$ for all i. If (x_i) is a block basis of (e_i) then $SI((e_i), 1/2) \geqslant SI((x_i), 1/2)$ so it suffices to prove that for all $(a_i)_1^m \in S_{\ell_p^m}$ that $\| \sum_1^m a_i e_i \| \geqslant 1/2$.

If this were false choose such an $(a_i)_1^m \in S_{\ell_p^m}$ with $\| \sum_1^m a_i e_i \| < 1/2$. Assume m is minimal with this property, (i.e., that such a sequence $(a_i)_1^m$ exists). By the fact that (e_i) is bimonotone, $|a_i| < 1/2$ for $i \leqslant m$. Choose n_1 minimal with $\sum_1^{n_1} |a_i|^p \geqslant (1/2)^p$. Then choose $n_2 > n_1$ minimal so that $\sum_{n_1+1}^{n_2} |a_i|^p \geqslant (1/2)^p$ and so on until obtaining $n_k < m$ with $\sum_{n_k+1}^{m} |a_i|^p \leqslant (1/2)^p$. It follows from the minimality of n_{j+1} that

$$\Big(\sum_{n_j+1}^{n_{j+1}} |a_i|^p\Big)^{1/p} \in \Big[\frac{1}{2}, 2^{1/p}\cdot\frac{1}{2}\Big] \text{ for } 0 \leqslant j < k$$

(taking $n_0 = 0$). Thus $(1 - (\frac{1}{2})^p)^{1/p} \leqslant \frac{1}{2}2^{1/p}k^{1/p}$ which implies that $k > \frac{1}{2}(1 - \frac{1}{n})n > \frac{n}{4}$. Set $x_j = \sum_{n_{j-1}+1}^{n_j} a_i e_i$ for $1 \leqslant j \leqslant k$. By the minimality of m and the fact that $(\sum_{n_{j-1}+1}^{n_j} |a_i|^p)^{1/p} \geqslant \frac{1}{2}$ we have that $\|x_j\| \geqslant \frac{1}{4}$. Thus $SI((e_i), \frac{1}{4}) \geqslant k > \frac{n}{4}$. This contradicts $n > 4SI((e_i), \frac{1}{4})$. □

DEFINITION 3.3. Let (E_n) be an FDD and $p < \infty$. (E_n) is *block p-Besselian* if there exists $c > 0$ so that whenever (x_i) is a block sequence of (E_n),

$$\| \sum x_i \| \geqslant c\Big(\sum \|x_i\|^p\Big)^{1/p}$$

(E_n) is *skipped block p-Besselian* if the above holds for all skipped sequences of (E_n).

DEFINITION 3.4. Let (E_n) be an FDD and let $p < \infty$. (E_n) is *asymptotically block p-Besselian* if there exists $c > 0$ so that whenever $k \in \mathbb{N}$ and $(x_i)_{i=1}^k$ is a block sequence of $(E_n)_{n=k}^\infty$ then $\| \sum_1^k x_i \| \geqslant c(\sum_1^k \|x_i\|^p)^{1/p}$.

(E_n) is *asymptotically skipped* block p-Besselian if the above holds for all skipped sequences $(x_i)_{i=1}^k$ of $(E_n)_{n=k}^\infty$.

PROPOSITION 3.5. *Let (E_n) be an FDD which is asymptotically block p-Besselian for some $p < \infty$. Then (E_n) is block q-Besselian for all $q > p$.*

Proof. We may assume that (E_n) is bimonotone. Suppose that $c > 0$ is such that for all k and all block sequences $(x_i)_1^k$ of $(E_n)_k^\infty$,

$$\Big\| \sum_1^k x_i \Big\| \geqslant c\Big(\sum_1^k \|x_i\|^p \Big)^{1/p} .$$

Let $q > p$. Choose K so large that

$$cK^{-1}\Big(\frac{K^q}{2} - 1\Big)^{1/p} > 1 . \tag{*}$$

Let $n_0 \in \mathbb{N}$ with $n_0 > K^q + 1$.

Claim If $(x_i)_1^s$ is a block sequence of $(E_j)_{n_0}^\infty$ then $\| \sum_1^s x_i \| \geqslant K^{-1}(\sum_1^s \|x_i\|^q)^{1/q}$.

If the claim is true the result follows. Assume the claim is false. Then there exists a normalized block sequence $(e_i)_1^s$ of $(E_j)_{n_0}^\infty$ and scalars $(a_i)_1^s$ with $\sum_1^s |a_i|^q = 1$ and $\|x\| < K^{-1}$ for $x = \sum_1^s a_i e_i$. Furthermore we may assume s is minimal so that such a situation arises. As in the proof of Proposition 3.1 we may write $x = \sum_{i=1}^{N+1} x_i$ where $x_i = \sum_{j=n_{i-1}+1}^{n_i} a_j e_j$ is the shortest vector (after x_{i-1}) with $\|x_i\|_{\ell_q} \geqslant K^{-1}$ for $i \leqslant N$ and $\|x_{N+1}\|_{\ell_q} < K^{-1}$. Note that $\|x_i\|_{\ell_q} \leqslant K^{-1}2^{1/q}$ for $i \leqslant N$ since $|a_j| < \frac{1}{K}$ by the bimonotone property and the fact that $\|x\| \leqslant K^{-1}$. Also

$$1 \geqslant \Big(\sum_{i=1}^N \|x_i\|_{\ell_q}^q \Big)^{1/q} \geqslant K^{-1}N^{1/q}$$

and so $N \leqslant K^q$. Furthermore

$$\sum_{i=1}^{N+1} (K^{-1}2^{1/q})^q \geqslant \sum_{i=1}^{N+1} \|x_i\|_{\ell_q}^q = 1$$

and so $2(N+1)K^{-q} \geqslant 1$ which yields that

$$N^{1/p} > \Big(\frac{K^q}{2} - 1 \Big)^{1/p} .$$

By the minimality of s we have that

$$\|x_i\| \geqslant K^{-1}\|x_i\|_{\ell_q} \geqslant K^{-2} \text{ for } i \leqslant N .$$

Combining these with our hypothesis and $(*)$ we have that

$$\begin{aligned}\|x\| &\geqslant c\Big(\sum_{i=1}^{N}\|x_i\|^p\Big)^{1/p} \geqslant cK^{-2}N^{1/p}\\ &> K^{-1}\bigg[cK^{-1}\Big(\frac{K^q}{2}-1\Big)^{1/p}\bigg] > K^{-1}\end{aligned}$$

which is a contradiction. □

4. Blockings in Spaces of Finite Index

In this section we focus on spaces X having an FDD and finite H-index. We prove that the FDD can be blocked to yield certain lower ℓ_p estimates for some $p < \infty$.

THEOREM 4.1. *Let (E_n) be an FDD for X.*

(a) If X is of finite H-index with respect to (E_n) then there exists $p \in [1, \infty)$ and a blocking (F_j) of (E_n) which is skipped block p-Besselian.

(b) If X is of finite H-index with respect to (E_n) and (E_n) is boundedly complete then there exists a blocking (H_j) of (E_n) and $p \in [1, \infty)$ so that (H_j) is block p-Besselian.

Proof. (a) follows directly from our work thus far. Let (F_n) be the blocking of (E_n) given by Proposition 1.9 for a suitable $\varepsilon_n \downarrow 0$ rapidly. It follows that there exists $n_0 \in \mathbb{N}$ so that if for all k if $(x_i)_1^k$ is a normalized skipped sequence of $(F_n)_k^\infty$ then $SI((x_i)_1^k, 1/2) \leqslant n_0$. Hence by Propositions 3.1 and 3.5 there exist $p < \infty$ so that (F_n) is skipped block p-Besselian.

To prove part (b) we need a trick of W.B. Johnson [14]. We give the proof because we need a generalization in the next section.

LEMMA 4.2. *Let (E_n) be a boundedly complete FDD for X. Let $\varepsilon_n \downarrow 0$. Then there exist integers $0 = n_0 < n_1 < \cdots$ so that if $x = \sum x_j \in S_X$, $x_j \in E_j$ for all j, then for all j there exists $i_j \in (n_{j-1}, n_j]$ so that $\|x_{i_j}\| < \varepsilon_j$.*

Proof. It suffices to show that $\forall\, m\ \forall\, \varepsilon > 0\ \exists\, n > m$ so that if $x = \sum x_i \in S_X$ with $x_i \in E_i$ then there exists $j \in (m, n]$ with $\|x_j\| < \varepsilon$. If not then $\forall\, n\ \exists\, x^n = \sum x_j^n \in S_X$ with $x_j^n \in E_j$ and $\|x_j^n\| \geqslant \varepsilon$ for all $j \in (m, n]$. Choose a subsequence (x^{n_k}) of (x^n) with $x_j^{n_k} \xrightarrow{k\to\infty} x_j \in E_j$ for all j. Thus $\|x_j\| \geqslant \varepsilon$ for $j > m$ and $\sup_\ell \|\sum_1^\ell x_i\| < \infty$. This contradicts that (E_j) is boundedly complete. □

Proof of (b). Let $\varepsilon_n \downarrow 0$ rapidly. Let (F_j) and p be as in (a). Let $0 = n_0 < n_1 < \cdots$ be given by Lemma 4.2 and define $H_j = \langle F_i \rangle_{n_{j-1}+1}^{n_j}$. Let $x = \sum x_i = \sum y_i$ with $x \in S_X$, $x_i \in F_i$ and $y_i \in H_j$ for all i, j. For each j choose $i_j \in (n_{j-1}, n_j]$

with $\|x_{i_j}\| < \varepsilon_j$. Set

$$z_j = \sum_{i=i_{j-1}+1}^{i_j} x_i \qquad (i_0 = 0) .$$

Then (x_j) is a skipped sequence with respect to (F_j) and so $\|\sum z_j\| \geqslant \frac{1}{2}(\sum \|z_j\|^p)^{1/p}$. Furthermore $\|\sum z_j\| \leqslant \|x\| + \sum_j \|x_{i_j}\| < 2$ (for suitably small ε_j's). Also for all j, $\|y_j\| \leqslant \|z_j\| + \|x_{i_j}\| + \|z_{j+1}\|$. Thus

$$\Big(\sum \|y_j\|^p\Big)^{1/p} \leqslant \Big(\sum_j (\|z_j\| + \|z_{j+1}\| + \varepsilon_j)^p\Big)^{1/p} \leqslant 9 ,$$

for suitably small ε_j's. □

COROLLARY 4.3. *Let (E_n) be a boundedly complete FDD for X and assume that X is of finite H-index with respect to (E_n). Then there exist $1 \leqslant p < \infty$, a blocking (H_j) of (E_n) and an equivalent norm $|\cdot|$ on X so that if $(x_j) \subseteq X$ is any block sequence of (H_j) then $|\sum x_j| \geqslant (\sum |x_j|^p)^{1/p}$. In particular X can be renormed to have the ω^*-UKK property.*

Proof. Let (H_j) and p be as in (b). Define for $x \in X$, $|x| = \sup\{(\sum \|x_i\|^p)^{1/p} : x = \sum x_i$ where (x_i) is a block sequence with respect to $(H_j)\}$. □

This result partially solves the problem raised by Huff [10]. If $X = B^*$ is reflexive and B has an FDD and is of finite Szlenk index then X can be renormed to have the UKK. Thus given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ so that if $(x_n) \subseteq B_X$, ω-$\lim_n x_n = x$ and $\|x_n - x_m\| \geqslant \varepsilon$ for $n \neq m$ then $\|x\| \leqslant 1 - \delta(\varepsilon)$. In the next section we remove the assumption that X have an FDD.

5. Blockings and Subspaces of Finite Index

We relativize the results of the previous section to subspaces of X. First we need an extension of Lemma 4.2.

LEMMA 5.1. *Let X have a bimonotone boundedly complete FDD (F_n) and let $Y \subseteq X$ be ω^* closed. $\forall\, \varepsilon > 0\ \forall\, m \in \mathbb{N}\ \exists\, n > m$ such that if $y = \sum_1^\infty y_i \in B_Y$ with $y_i \in F_i$ for all i then $\exists\, k \in (m, n]$ with*

(a) $\|y_k\| < \varepsilon$

(b) $\operatorname{dist}(\sum_{i=1}^{k-1} y_i, Y) < \varepsilon$.

Proof. We proved (a) in Lemma 4.2. In particular we can find $m = n_0 < n_1 < n_2 < \cdots$ so that if $x = \sum_1^\infty x_i \in B_X$, $x_i \in F_i$ for all i, then for all j there exists $k_j \in (n_{j-1}, n_j]$ with $\|x_j\| < \varepsilon$. Thus if (b) fails then for all j there exists $y^{(j)} = \sum_i^j y_i^{(j)} \in B_X$, $y_i^{(j)} \in F_i$ for all i, so that for all $s < j$ there exists $k(j,s) \in (n_{s-1}, n_s]$ with $\|y_{k(j,s)}^{(j)}\| < \varepsilon$ and $\operatorname{dist}(\sum_{i=1}^{k(j,s)-1} y_i^{(j)}, Y) \geqslant \varepsilon$. Passing to a subsequence of $(y^{(j)})$ we may assume that $\lim_{j\to\infty} y_i^{(j)} \equiv x_i \in F_i$ exists for all i and that $k(j,s) \equiv k(s)$ for $s \leqslant j$. By the fact that $\| \sum_1^\ell x_i \| \leqslant 1$ for all ℓ and the boundedly complete property of (E_n) we have $x = \sum_1^\infty x_i \in B_X$. Also $y^{(j)} \xrightarrow{\omega^*} x$ and so $x \in Y$. Thus

$$\operatorname{dist}\left(Y, \sum_{i=1}^{k(s)-1} x_i\right) \overset{s\to\infty}{\longrightarrow} 0\ , \text{ a contradiction.} \qquad (\square)$$

THEOREM 5.2. *Let (E_n) be a bimonotone boundedly complete FDD for X and let Y be a ω^* closed subspace whose predual has finite Szlenk index. There exists a blocking (H_j) of (E_n) and $p = p(H(Y,(E_i)), 1/2) \in [1,\infty)$ so that $|\cdot|$ is an equivalent norm on Y where for $x \in X$,*

$$|x| = \sup\left\{ \left(\sum_1^\infty \|x_i\|^p \right)^{1/p} : \exists \text{ a blocking } (G_i) \text{ of } (H_i) \right.$$
$$\left. \text{with } x_i \in G_i \text{ for all } i \text{ and } x = \sum_1^\infty x_i \right\}$$

Of course $|x|$ could be infinite for some $x \in X$. We are only claiming it is an equivalent norm on Y. Before proving the theorem we give some corollaries.

COROLLARY 5.3. *Let Y be a separable dual space whose predual has finite Szlenk index. Then there exist a Banach space Z with a boundedly complete FDD (H_j) and $p \in [1,\infty)$ so that Y embeds isomorphically (norm and ω^*) into Z and $\| \sum z_j \| \geqslant (\sum \|z_j\|^p)^{1/p}$ for all block bases (z_j) of (H_j).*

Proof. As discussed earlier we may assume that Y is a ω^* closed subspace of a space X having a boundedly complete FDD [3]. We let (H_j) and $|\cdot|$ be as in Theorem 5.2. Define Z to be the completion of $\langle (H_j)\rangle$ under $|\cdot|$. Y embeds into Z by the theorem. $\square$

COROLLARY 5.4. *Let B be a separable Banach space of finite Szlenk index ($S(B,\varepsilon) < \infty$ for all $\varepsilon > 0$). Then B admits an equivalent ω^*-UKK norm.*

Proof. Let $Y = B^*$. By [3] there exists a space W having a shrinking basis and a quotient map $Q : W \to B$. Thus $Q^* : Y \to W^*$ embeds Y as a ω^* closed subspace of W^*. Moreover Q^* is a ω^* isomorphism as well. Q^*Y has finite index with respect to the dual basis of W, a boundedly complete basis for W^*. We then apply Corollary 5.3 obtaining Z, (H_j) and p as in the conclusion of Corollary 5.3. Thus we have renormed Y by $\|\cdot\|$ so as to preserve its ω^* topology as the dual space of X in such a manner that Y has a ω^*-UKK norm. The latter comes from the lower ℓ_p estimate for Z. This then defines an equivalent norm on B by regarding Y as the dual of B. Thus for $x \in B$,

$$\|x\| = \sup\{\langle x, y\rangle : y \in B_Y\}\ .$$

So $(Y, \|\cdot\|) = (B, \|\cdot\|)^*$ and $(B, \|\cdot\|)$ has the ω^*-UKK. □

Proof of Theorem 5.2. It will suffice to produce such a p and a blocking (H_j) of (F_j) so that if (G_j) is any further blocking of (H_j) and $y = \sum y_j \in Y$ with $y_j \in G_j$ for all j then

$$\|y\| \geqslant \frac{1}{13}\Big(\sum \|y_j\|^p\Big)^{1/p} .$$

Let $H(Y, (E_i), 1/4) \equiv n_0 < \infty$. Let $\varepsilon > 0$ be small (specified below). By Proposition 1.9 there exist $\delta > 0$ and a blocking (F_j) of (E_i) so that if $(y_i)_1^{n_0+1} \subseteq S_Y$ satisfies: there exists a skipped blocking $(G_j)_1^{n_0+1}$ of $(F_j)_2^\infty$ so that

$$\|(I - P_{G_j})y_j\| < \delta \text{ for } j \leqslant n_0 + 1$$

then $d_b((y_i)_1^{n_0+1}, \{Y, (E_i)\}_{n_0+1}) < 1 + \varepsilon/2$.

LEMMA 5.5. *There exist $p = p(n_0)$ and $\bar\varepsilon_n \downarrow 0$ so that if (G_j) is any skipped blocking of (F_j) and $(y_j) \subseteq Y$ satisfies $\|(I - P_{G_j})y_j\| \leqslant \bar\varepsilon_j\|y_j\|$ for all j then*

$$\Big\|\sum y_j\Big\| \geqslant \frac{1}{4}\Big(\sum \|y_j\|^p\Big)^{1/p} .$$

Proof. We may assume that each $y_j \neq 0$. By taking $\bar\varepsilon_j$ sufficiently small this will insure that $(y_j/\|y_j\|)$ is $1+\varepsilon$-close to being bimonotone. We claim that

$$SI((y_i), 1/2) \leqslant n_0 + 1 .$$

Indeed if $(z_i)_1^{n_0+2}$ is any normalized block basis of (y_i) then, $d_b((z_i)_2^{n_0+2}, \{Y, (E_i)\}_{n_0+1}) < 1+\varepsilon$ from our initial assumptions on (F_j) and standard perturbation arguments which of course impose restrictions on $(\bar\varepsilon_n)$. Thus $SI((z_i)_2^{n_0+1}, 1/2) \leqslant n_0$ which gives the claim. If (y_i) were bimonotone we would have the desired estimate by Proposition 3.1, with a lower constant of $1/2$. Since (y_i) is only nearly bimonotone the $1/2$ becomes $1/4$ by taking ε sufficiently small. □

Continuing with the proof of 5.2 we let $\bar{\bar\varepsilon}_n \downarrow 0$ rapidly (specified below) and choose, using Lemma 5.1, integers $0 = m_0 < m_1 < \cdots$ so that for all $y = \sum y_i \in B_Y$ with $y_i \in F_i$ for all i, given $j \in \mathbb{N}$ there exists $i_j \in (m_{j-1}, m_j]$ with $\|y_{i_j}\| < \bar{\bar\varepsilon}_j$ and $d(\sum_1^{i_j-1} y_i, Y) < \bar{\bar\varepsilon}_j$. Define $H_j = \langle F_i\rangle_{(m_{j-1},m_j]}$ for $j \in \mathbb{N}$. Let (G_j) be any further blocking of H_j, say $G_j = \langle H_i\rangle_{(k_{i-1},k_i]}$ for some $0 = k_0 < k_1 < \cdots$. Let $y = \sum y_i \in S_Y$ with $y_i \in F_i$ for all i.

For each j choose $i_j \in (m_{j-1}, m_j]$ with $\|y_{i_j}\| < \bar{\bar\varepsilon}_j$ and $d(\sum_1^{i_j-1} y_i, Y) < \bar{\bar\varepsilon}_j$. Take $i_0 = 0$ and $z_j = \sum_{i=i_{j-1}+1}^{i_j-1} y_i$. Then $d(z_1, Y) < \bar{\bar\varepsilon}_1$ and for $j > 1$

$$d(z_j, Y) < \bar{\bar\varepsilon}_j + \bar{\bar\varepsilon}_{j-1} + \bar{\bar\varepsilon}_{j-1} < 3\bar{\bar\varepsilon}_{j-1} .$$

Choose $w_j \in Y$ with $\|z_j - w_j\| < 3\bar{\bar{\varepsilon}}_{j-1}$ for $j > 1$ and $\|z_1 - w_1\| < \bar{\bar{\varepsilon}}_1$.

We claim that $(\sum \|w_j\|^p)^{1/p} \leqslant 5$ and so ($\bar{\bar{\varepsilon}}_j$ sufficiently small) $(\sum \|z_j\|^p)^{1/p} \leqslant 6$. Indeed set $\bar{\varepsilon}_1 = \bar{\varepsilon}_0 = \bar{\varepsilon}_{-1}$ and let $I = \{i : \|z_i\| \geqslant 2\bar{\bar{\varepsilon}}_{i-2}\}$. If $j \notin I$ then $\|w_j\| \leqslant 3\bar{\bar{\varepsilon}}_{j-1} + 2\bar{\bar{\varepsilon}}_{j-2}$. If $j \in I$ then ($\bar{\bar{\varepsilon}}_j$ suitably small)

$$\|(I - P_{\langle F_i\rangle_{(i_{j-1}, i_j)}})w_j\| < \bar{\varepsilon}_j\|w_j\| \,.$$

Thus by Lemma 5.5

$$\left(\sum_I \|w_i\|^p\right)^{1/p} \leqslant 4\left\|\sum_I w_i\right\| \leqslant 4\left(1 + \sum(3\bar{\bar{\varepsilon}}_{j-1} + 2\bar{\bar{\varepsilon}}_{j-2})\right) < 5$$

if $\bar{\bar{\varepsilon}}_j$ are suitably small. The claim follows.

Finally let $y = \sum b_j$ where $b_j \in G_j$. Then $\|b_j\| \leqslant \|z_{j-1} + y_{i_{j-1}} + z_j + y_{i_j}\|$. This yields

$$\begin{aligned}\left(\sum \|b_j\|^p\right)^{1/p} &\leqslant 2\Big(\sum \|z_j\|^p\Big)^{1/p} + 2\Big(\sum \bar{\varepsilon}_j^p\Big)^{1/p}\\ &\leqslant 13\end{aligned}$$

for suitably small $\bar{\varepsilon}_j$'s. □

In the case where Y is reflexive we obtain the following:

THEOREM 5.6. *Let Y be a reflexive space whose predual has finite Szlenk index. Then Y can be renormed to have the UKK property. Moreover the UKK modulus is of power type.*

Indeed by a result of Zippin [27] we can regard $Y \subseteq X$ where X is reflexive and has a basis. The result then follows from our previous results and the following proposition.

PROPOSITION 5.7. *Let Z be the space constructed in Corollary 5.3.*

(a) If X has a basis then Z has a basis.

(b) If X is reflexive then Z is reflexive.

Proof. (a) is clear. To see (b) we first recall that the lower ℓ_p estimate on blocks of (H_j) gave that (H_j) was boundedly complete. It remains to show that (H_j) is shrinking. If not there exists a $|\cdot|$ normalized block basis (x_j) of (H_j) so that for all $(a_i) \subseteq \mathbb{R}^+$ with $\sum a_i = 1$ we have $|\sum a_i x_i| > 1/2$.

Choose $\delta > 0$ so that $\delta^{p-1} < 6^{-p}$. Let $(a_i) \subseteq [0, \delta)$ with $\sum a_i = 1$ and using the definition of the norm $|\cdot|$ choose a blocking (G_j) of (H_j) so that for $x = \sum a_i x_i$, $\frac{1}{2^p} < \sum_j \|P_{G_j}x\|^p$. We assume $P_{G_j}x \neq 0$ for all j. We consider each block G_j and if necessary split it into at most 3 blocks as follows. If $P_{G_j}x_i \neq 0$ for at most one i we do nothing. Otherwise let i be maximal so that $P_{G_j}x_i \neq 0$ and $P_{G_{j+1}}x_i \neq 0$ as well. (If no such x_i exists we do nothing.) We split G_j into two blocks, the first acting on $\langle x_1, \dots, x_{i-1}\rangle$ and the second on x_i. We also make a corresponding split if necessary according to the minimal i so that $P_{G_j} x_i \neq 0$ and $P_{G_{j-1}} x_i \neq 0$.

We let (R_j) be the new blocking. It follows that if $P_{R_j} x_i \neq 0$ for more than one i, then for any such i $P_{R_{j'}} x_i = 0$ for $j \neq j'$. Also for such j, $\|P_{R_j} x\| \leqslant \sum_{I_j} a_i$ where $I_j = \{i : P_{R_j} x_i \neq 0\}$. Due the the splitting of (G_j) our above estimate becomes

$$\frac{1}{2^p} \leqslant 3^p \sum_j \|P_{R_j} x\|^p .$$

Let $J = \{j : P_{R_j} x_i \neq 0$ for more than one $i\}$ then $\sum_{j \notin J} \|P_{R_j} x\|^p \leqslant \sum_{i \notin \cup I_j} a_i^p$ since $|x_i| = 1$ for all i. Now we claim that for some $j \in J$, $\|P_{R_j} x\| \geqslant \delta$. Indeed if not we have

$$\begin{aligned} \frac{1}{6^p} &\leqslant \sum_{j \in J} \|P_{R_j} x\|^p + \sum_{j \notin J} \|P_{R_j} x\|^p \\ &< \delta^{p-1} \sum_{j \in J} \|P_{R_j} x\| + \sum_{i \notin \cup I_j} a_i^p \\ &< \delta^{p-1} \left[\sum_{i \in \cup I_j} a_i + \sum_{i \notin \cup I_j} a_i \right] = \delta^{p-1} . \end{aligned}$$

But this is impossible by our choice of δ.

Hence for such an x, $\|x\| \geqslant \|P_{R_j} x\| \geqslant \delta$. But this contradicts that (x_i) is necessarily weakly null for $\|\cdot\|$. Indeed one can always find $(a_i) \subseteq [0, \delta)$ with $\|\sum a_i x_i\| < \delta$ and $\sum a_i = 1$. □

6. Dual Results and Further Remarks

We next explore dual concepts to those above which will ultimately lead to upper ℓ_q estimates for some $q > 1$.

To say that a basic sequence (x_i) has finite strong index is equivalent to saying that we have uniform lower ℓ_p estimates on all block bases for some $p < \infty$. Thus given K there exists n so that if $(y_i)_1^n$ is a normalized block basis of (x_i) then $\|\sum_1^n y_i\| > K$. In other words (x_i) does not admit (what might be called) $\ell_\infty^n +$ uniformly as block bases.

The dual notion is an $\ell_1^n +$ index.

DEFINITION 6.1. Let (x_i) be a basic sequence and $\varepsilon > 0$.

$$I^+((x_i), \varepsilon) = \sup\Bigg\{k : \exists \text{ a normalized block basis } (y_i)_1^k \text{ of } (x_i) \text{ satisfying} \\ \Big\| \sum_1^k a_i y_i \Big\| \geqslant \varepsilon \sum_1^k a_i \text{ if } (a_i)_1^k \subseteq \mathbb{R}^+ \Bigg\}.$$

It is easy to see that $I^+(x_i) < \infty$ iff there exists $n_0 \in \mathbb{N}$ so that for all normalized block bases $(y_i)_1^{n_0}$ of (x_i) we have $\| \sum_1^{n_0} y_i \| < n_0/2$. Also by James' result that ℓ_1 is not distortable [12] adapted to the ℓ_1^+ situation, $I^+((x_i), \varepsilon) < \infty$ for some $\varepsilon < 1$ iff $I^+((x_i), \varepsilon) < \infty$ for all $\varepsilon < 1$. See [1] for more on the I^+ index.

The analog of Proposition 3.1 is

PROPOSITION 6.2. ([11], [13]) *Let (x_i) be a monotone basis. Suppose that $I^+((x_i), 1/2) = n_0 < \infty$. Then there exists $q = q(n_0) > 1$ so that $\| \sum a_i x_i \| \leqslant 6(\sum |a_i|^q)^{1/q}$ for all $(a_i) \subset \mathbb{R}$.*

The same sort of arguments used to prove Theorems 4.1 and 5.2 yield the following. We shall say that if $Y \subseteq X$ where X has an FDD (E_n) then Y is of *finite asymptotic I^+-index* with respect to (E_n) if for some $0 < \varepsilon < 1$ (hence all $\varepsilon < 1$)

$$\sup\left\{I^+((e_i)_1^k, \varepsilon) : (e_i)_1^k \in \{Y, (E_i)\}_k, k \in \mathbb{N}\right\} < \infty .$$

THEOREM 6.3. *Let (E_n) be an FDD for X*

(a) *If X is of finite asymptotic I^+-index with respect to (E_n) then there exist $q > 1$, $K < \infty$ and a blocking (F_j) of (E_n) so that for all block sequences (x_i) with respect to (F_j), $\| \sum x_i \| \leqslant K(\sum \|x_i\|^q)^{1/q}$.*

(b) *If (E_n) is boundedly complete and $Y \subseteq X$ is ω^* closed of finite asymptotic I^+-index with respect to (E_n) then there exist $q > 1$, a blocking (H_j) of (E_n) and $K < \infty$ so that if $y \in Y$ with $y = \sum y_j$ where (y_j) is a block sequence with respect to (H_j) then $\|y\| \leqslant K(\sum \|y_i\|^q)^{1/q}$.*

In this theorem we do not need to require skipped sequences in (a) because the upper estimate results from the separate estimates applied to $\sum x_{2i}$ and $\sum x_{2i-1}$.

The H-index is a sort of ℓ_∞^+-index. Thus it is natural to ask the following question. Suppose X has infinite H-index with respect to (E_n). Is c_0 block finitely representable in (E_n)? The answer is not necessarily.

EXAMPLE 6.4. *There exists a space X with a bimonotone basis (b_i) so that for all n there exists $(e_i)_1^n \in \{X, (b_i)\}_n$ with $\| \sum_1^n e_i \| = 1$ yet c_0 is not block finitely representable in (b_i).*

T_ω is the countably branching tree of ω levels, i.e.,

$$T_\omega = \{(n_1, \dots, n_j) : j \in \mathbb{N}\ ,\ n_1, \dots, n_j \in \mathbb{N}\}$$

ordered by extension. X will be the completion of $c_{00}(T_\omega) \equiv \{f : T_\omega \to \mathbb{R} : f$ has finite support$\}$ under a suitable norm. The node basis $(e_\alpha)_{\alpha \in T_\omega}$ given by $e_\alpha(\beta) = \delta_{\alpha\beta}$ will be a normalized bimonotone basis for X when linearly ordered in any manner that is compatible with the tree order on T_ω. Thus if $\alpha < \beta$ in T_ω then $e_\alpha < e_\beta$ in the basis order.

In addition we will have the following properties.

(1) There exists a basis (e_i) so that if $(\alpha_i)_1^n$ is any initial segment of a branch in T_ω then $(e_{\alpha_i})_1^n$ is 1-equivalent to $(e_i)_1^n$. Moreover $\| \sum_1^n e_i \| = 1$.

(2) If $(x_i)_1^n$ is any normalized block basis of (e_α) then $\| \sum_1^n \varepsilon_i x_i \| \geqslant n/3$ for some choice of $\varepsilon_i = \pm 1$.

Because of the tree structure (1) yields that $(e_i)_1^n \in \{X, (e_\alpha)\}_n$ for all n. (2) yields that c_0 is not block finitely representable in X.

We shall specify a set $\Gamma \subseteq c_{00}(T_\omega)$ and define for $x \in c_{00}(T_\omega)$,

$$\|x\| = \sup\{\langle f, x\rangle : f \in \Gamma\}\ .$$

$f \in \Gamma$ iff f is finitely supported, $f(\alpha) \in \{0, \pm 1\}$ for all α and on any branch of T_ω, f does not take on successive nonzero values of the same sign. Thus if $\alpha < \beta$ in T_ω and $f(\alpha) = 1$ and $f(\gamma) = 0$ for $\alpha < \gamma < \beta$ then $f(\beta) = -1$ or 0.

All the properties of X are now easily verified except for (2) which requires some effort. Let $(x_i)_1^n$ be a normalized block basis of (e_α). Choose $f_i \in \Gamma$ with $\langle f_i, x_i\rangle = 1$ for $i \leqslant n$. We may suppose that range $f_i =$ range x_i with respect to the linearly ordered basis (e_α); the range of $x \in c_{00}(T_\omega)$ is the smallest interval of α's (in the basis ordering) containing the support of x.

Let I_i be the set of initial nodes with respect to the tree order in supp f_i. We shall partition I_i into 3 sets I_i^s, I_i^o and I_i^d and write $f_i = f_i^s + f_i^o + f_i^d$ where f_i^s is f_i restricted to $\{\beta \in T_\omega : \alpha \leqslant \beta$ for some $\alpha \in I_i^s\}$ and so on. We begin with $i = 2$. Let A_1 be the set of terminal nodes (in the tree order) of supp f_1.

$$I_2^s = \{\beta \in I_2 : \exists\, \alpha \in A_1 \text{ with } \alpha < \beta \text{ and } f_1(\alpha) = f_2(\beta)\}$$
$$I_2^o = \{\beta \in I_2 : \exists\, \alpha \in A_1 \text{ with } \alpha < \beta \text{ and } f_1(\alpha) = -f_2(\beta)\}$$
$$I_2^d = I_2 \setminus (I_2^s \cup I_2^0)\ .$$

The letters s, o, d represent same, opposite and disjoint.

Choose $g \in \{f_2^s, f_2^o, f_2^d\}$ so that $\langle g, x_2\rangle \geqslant 1/3$. If $g = f_2^o$ or f_2^d let $\varepsilon_2 = 1$ and $f(2) = f_1 + g$. If $g = f_2^s$ let $\varepsilon_2 = -1$ and $f(2) = f_1 - g$. It follows that $f(2) \in \Gamma$ and

$$\langle f(2), x_1 + \varepsilon_2 x_2\rangle \geqslant 1 + \frac{1}{3}\ .$$

We continue in this manner using $f(2)$ to partition I_3 into 3 sets and ultimately determine f_3^s, f_3^o, f_3^d and ε_3 etc. The construction yields (2). □

The analogous question for the I^+-index has a similar answer. If (e_i) is the summing basis for $c_0 = X$ then ℓ_1^{n+} belongs to $\{X, (e_i)\}_n$ for all n but ℓ_1 is not block finitely represented in (e_i).

We do not know how to find reflexive examples with these properties.

PROBLEM 6.5. Does there exist a reflexive space with a basis (e_i) having infinite H-index (respectively, infinite I^+-index) yet c_0 (respectively, ℓ_1) is not block finitely represented in (e_i)?

The H-index was defined for a fixed $\varepsilon > 0$. One can vary the ε at each level and obtain a variable H-index. If (E_n) is an FDD for X, $(x_n) \subseteq X$ is bounded and $x \in X$ we write $x_n \rightharpoonup x$ if (x_n) converges to x coordinatewise with respect to (E_n). Let $(\varepsilon_i)_1^n \subseteq (0, 1)$. $H_0(X, (E_i), (\varepsilon_i)_1^n) = B_X$. For $k < n$ let

$$H_{k+1}(X, (E_i), (\varepsilon_i)_1^n) = \{x : \exists\, (x_j) \subseteq H_k(X, (E_i), (\varepsilon_i)_1^n) \text{ with } x_j \rightharpoonup x \text{ and } \liminf_{j\to\infty} \|x_j - x\| \geqslant \varepsilon_{k+1}\}\ .$$

In this notation having finite H-index just says that for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ with $H_n(X, (E_i), (\varepsilon)_1^n) = \emptyset$.

DEFINITION 6.6. X has *summable H-index* with respect to (E_n) if $\exists\ K < \infty$ $\forall\ n\ \forall\ (\varepsilon_i)_1^n \subseteq (0, 1)$

$$H_n(X, (E_i), (\varepsilon_i)_1^n) \neq \emptyset \Longrightarrow \sum_1^n \varepsilon_i \leqslant K\ .$$

Again there is a connection with trees and the asymptotic structure of X.

PROPOSITION 6.7. *Let (E_n) be an FDD for X. The following are equivalent.*

(a) X has summable H-index.

(b) There exists $K < \infty$ so that for all n and for all $(e_i)_1^n \in \{X, (E_i)\}_n$,

$$(e_i)_1^n \textit{ is } K\textit{-equivalent to the unit vector basis of } \ell_1^n$$

(c) There exists a blocking (H_j) of (E_i) which is skipped asymptotic ℓ_1; i.e., for some $K < \infty$ if $(x_i)_1^n$ is a skipped block sequence of $(H_j)_n^\infty$ then

$$\Big\| \sum x_i \Big\| \geqslant K^{-1} \sum \|x_i\|\ .$$

Proof. The equivalence of (b) and (c) follows from Proposition 1.9. The equivalence with (a) comes from the following connection between the variable H-index and trees.

Suppose that $H(X, (E_i), (\varepsilon_i)_0^n) \neq \emptyset$. Then, as in the proof of Proposition 2.8, there exists $\mathcal{T} \in a - T_n(X, (E_i))$ which converges to $(e_i)_1^n \in \{X, (E_i)\}_n$ and satisfies $\| \sum_1^n \varepsilon_i' e_i \| \leqslant 1$ for some $\varepsilon_i/2 \leqslant \varepsilon_i' \leqslant 1$. If (b) holds then $\sum_1^n \varepsilon_i' \leqslant K$.

Finally assume (a) and let $(e_i)_1^n \in \{X, (E_i)\}_n$. Assume the variable index of X is $\leqslant K$. Let $(\varepsilon_i)_1^n \subseteq (0,1)$ with $\sum_1^n \varepsilon_i > K$. Suppose $\|\sum_1^n \varepsilon_i e_i\| \leqslant 1$. Choose $\mathcal{T} \in a - T_n(X,(E_i))$ that converges to $(e_i)_1^n$. It follows that $H(X,(E_i),(\varepsilon_i)_1^n) \neq \emptyset$ which is a contradiction. Thus $\|\sum_1^n \varepsilon_i e_i\| \leqslant 1$ implies $\sum_1^n \varepsilon_i \leqslant K$. Since $(\pm e_i)_1^n \in \{X,(E_i)\}_n$ we have $\|\sum_1^n \pm\varepsilon_i e_i\| \leqslant 1$ implies $\sum_1^n \varepsilon_i \leqslant K$. Thus $(e_i)_1^n$ is K-equivalent to the unit vector basis of ℓ_1^n. □

These results can also be generalized to a ω^* closed subspace of a space X with a boundedly complete FDD. By Proposition 6.7 Tsirelson's space T [7] has summable H-index. There is a (formally) weaker notion than summable index.

DEFINITION 6.8. Let (E_n) be an FDD for X. We say X has *proportional H-index* with respect to (E_n) if there exists $K < \infty$ so that for all $0 < \varepsilon < 1$, $H(X,(E_n),\varepsilon) \leqslant K/\varepsilon$. It is clear that summable index implies proportional index.

PROPOSITION 6.9. *Let (E_n) be a monotone FDD for X and suppose that X has proportional H-index with respect to (E_n). Then X has summable H-index with respect to (E_n).*

Proof. If not then for all $\varepsilon > 0$ there exist $k \in \mathbb{N}$ and a block tree with respect to (E_n), $\mathcal{T} = (x(n_1,\dots,n_j))_{T_k} \in T_k(X)$, which converges to $(e_i)_1^k \in \{X,(E_i)\}_k$ and such that there exists $(a_i)_1^k \subseteq [0,\infty)$ with $\sum_1^k a_i = 1$ and $\|\sum_1^k a_i e_i\| < \varepsilon$. Without loss of generality we may assume that for some $N \in \mathbb{N}$ each $a_i = \frac{n_i}{N}$ for some $n_i \in \mathbb{N}$. Write N as $N = n_i m_i + k_i$, where m_i and k_i are integers with $0 \leqslant k_i < n_i$.

Using Proposition 1.9 and pruning $\mathcal{T}$ we may assume that every collection of $j \leqslant kN$ elements of $\mathcal{T}$, suitably ordered, is essentially in $\{X,(E_n)\}_j$. We form a seminormalized block basis of $\mathcal{T}$ as follows. The order will be $(x_1^1,\dots,x_{n_1}^1, x_1^2,\dots,x_{n_2}^2,\dots x_1^k,\dots,x_{n_k}^k)$. The first k_1 of the x_j^1's will be a sum of m_1+1 $x(n)$'s with weight a_1. The remaining x_j^1's will be a sum of m_1 $x(n)$'s with the same weight a_1. All together the x_j^1's will involve N of the $x(n)$'s. The first k_2 x_j^2's will each be a sum of m_2+1 $x(n,m)$'s with weight a_2 and the remaining x_j^2's will be a sum of m_2 $x(n,m)$'s with the same weight a_2. Moreover, each $x(n,m)$ in the support of one of the x_j^2's will be a successor to one of the $x(n)$'s in the support of the x_j^1's and so on.

It follows since X has proportional index that for some fixed $c > 0$,

$$\Big\| \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_j^i \Big\| \geqslant c \sum_{i=1}^{k} n_i = cN\ .$$

However if $(m_1,\dots,m_k) \in T_k$ is such that $x(m_1,\dots,m_j) \in \operatorname{supp} x_{\ell(j)}^j$ for some $\ell(1),\dots,\ell(k)$ then

$$\Big\| \sum_{j=1}^{k} \frac{n_j}{N} x(\ell(1),\dots,\ell(j)) \Big\| \approx \Big\| \sum_{1}^{k} a_i e_i \Big\| < \varepsilon\ .$$

Since there are N such “columns” in the tree we obtain from the triangle inequality that

$$\Big\| \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_j^i \Big\| < \varepsilon N \ .$$

This yields a contradiction. □

The nonreflexive version of our main theorem remains open when one replaces ω^* convergence by ω convergence.

PROBLEM 6.10. *[10] If $W(X, \varepsilon) < \infty$ for all $\varepsilon > 0$ can X be given an equivalent UKK norm?*

If the answer is no it still may be true in the case where X does not contain ℓ_1.

7. The ω^*-UKK Modulus

Let us redefine the modulus for a ω^*-UKK dual space X as follows. Given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ so that if $(x_n) \subseteq X$, $x \in X$, $\|x + x_n\| \leqslant 1$ and $\|x_n\| \geqslant \varepsilon$ for all n with $\omega^*\text{-}\lim_{n\to\infty} x_n = 0$ then $\|x\| \leqslant 1 - \delta$.

We have proved that if X (or more properly B where $X = B^*$) has finite Szlenk index then there exists an equivalent norm $\| \cdot \|$ on X (and B) and $p < \infty$ so that for x as above

$$\|x\| \leqslant (1 - \varepsilon^p)^{1/p} \sim 1 - \frac{1}{p}\varepsilon^p \text{ for small } \varepsilon \ .$$

So $\delta(\varepsilon) \geqslant c\varepsilon^p$ for some c.

We examine what can be said about X from knowledge of the ω^*-UKK modulus $\delta(\varepsilon)$. We begin with an easy observation.

PROPOSITION 7.1.

(a) *$\ell_1 = c_0^*$ is ω^*-UKK with $\delta(\varepsilon) = \varepsilon$.*

(b) *Let X be ω^*-UKK with $\delta(\varepsilon) \geqslant c\varepsilon$ for some $c > 0$ and all $\varepsilon > 0$. Then every normalized ω^*-null sequence in X admits a subsequence equivalent to the unit vector basis of ℓ_1.*

Proof. (a) is obvious.

(b) The hypothesis yields that if $\omega^*\text{-}\lim x_n = 0$ and $\lambda = \lim \|x + x_n\|$ with $\lim_n \|x_n\| = \varepsilon$ then $\|x\| \leqslant \lambda - c\varepsilon$.

Let (y_n) be normalized ω^*-null in X. Let $\varepsilon_n \downarrow 0$ rapidly. By passing to a subsequence we may assume that for all k and $(a_i)_1^{k+1} \subseteq [-1, 1]$, $\ell > k$,

$$\Big| \Big\| \sum_1^k a_i y_i + a_{k+1} y_\ell \Big\| - \Big\| \sum_1^{k+1} a_i y_i \Big\| \Big| < \varepsilon_{k+1} \ .$$

Let $\sum_1^{k+1} |a_i| = 1$. Since

$$\lim_{\ell\to\infty} \Big\| \sum_1^k a_i y_i + a_{k+1} y_\ell \Big\| \geqslant \Big\| \sum_1^k a_i y_i \Big\| + c|a_{k+1}|$$

it follows that

$$\Big\| \sum_1^{k+1} a_i y_i \Big\| \geqslant \Big\| \sum_1^k a_i y_i \Big\| + c|a_{k+1}| - \varepsilon_{k+1} \ .$$

Iterating the argument we obtain

$$\Big\| \sum_1^{k+1} a_i y_i \Big\| \geqslant c \sum_1^{k+1} |a_i| - \sum_1^{k+1} \varepsilon_i \geqslant \frac{c}{2}$$

if $\sum \varepsilon_i < c/2$. □

Actually more can be said.

REMARK 7.2. (1) It follows from [15] that if $X = B^*$ is as in (b) then B embeds into c_0.

(2) Tsirelson's space T can be renormed for $p > 1$ to have $\delta(\varepsilon) \geqslant c_p \varepsilon^p$ but of course cannot be renormed to have $\delta(\varepsilon) \geqslant c\varepsilon$.

(3) Suppose X is as in (b) and X has a boundedly complete FDD, (E_n). Then (E_n) can be blocked into an ℓ_1 FDD for X. This can be deduced either from (1) or from our arguments. More generally if X is a ω^* closed subspace of a space with a boundedly complete FDD (E_n) then there exists a blocking (H_j) of (E_n) so that setting $|x| = \sum \|x_j\|$ for $x = \sum x_j$, $x_j \in H_j$ then X embeds into $\overline{(\langle H_j \rangle, |\cdot|)}$, a space with an ℓ_1-FDD.

PROPOSITION 7.3. *Let Y be a ω^* closed subspace of X, a space with a boundedly complete FDD, (E_n). Assume Y is ω^*-UKK with $\delta(\varepsilon) \geqslant c\varepsilon^p$ for some $c > 0$, $1 < p < \infty$. Then there exists a blocking (H_j) of (E_n) and a norm $|\cdot|$ on $\langle (H_j) \rangle$ that makes $(\langle H_j \rangle, |\cdot|)$ 1-block p-Besselian and so that $|\cdot| \sim \|\cdot\|$ on Y.*

Proof. We may assume (E_n) is bimonotone. From our previous work it suffices to prove that for some $c' > 0$ if $\|y_n\| \geqslant \varepsilon$, ω^*-lim $y_n = 0$ and $\lim \|y + y_n\| = \lambda$ for y, $(y_n) \subseteq Y$ then $\|y\|^p \leqslant \lambda^p - c'\varepsilon^p$.

We present the argument for $p = 2$ where the calculations are simpler.

From $\delta(\varepsilon) \geqslant c\varepsilon^2$ we have $\|\frac{y}{\lambda}\| \leqslant 1 - c(\frac{\varepsilon}{\lambda})^2$ and so $\|y\| \leqslant \lambda - \frac{c\varepsilon^2}{\lambda}$. Thus

$$\begin{aligned} \|y\|^2 &\leqslant \lambda^2 - 2c\varepsilon^2 + c^2 \left(\frac{\varepsilon}{\lambda}\right)^2 \varepsilon^2 \\ &\leqslant \lambda^2 - c\varepsilon^2 \end{aligned}$$

since $\varepsilon \leqslant \lambda$. □

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