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On Asymptotically $\lambda-$ Statistical Equivalent Sequences of Order α in Probability

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Abstract. In this study, we introduce and examine the concepts of asymptotically $\lambda-$statistical equivalent sequences of order α in probability and strong asymptotically $\lambda-$equivalent sequences of order α in probability. We give some relations connected to these concepts.

1. Introduction and Background

The idea of statistical convergence was introduced by Steinhaus [27] and Fast [11] and later reintroduced by Schoenberg [24] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Cinar *et al.* [6], Connor [3], Colak [4], Colak and Bektas [5], Et *et al.* ([8],[9],[10]), Fridy [12], Gadjiev and Orhan [13], Ghosal *et al.* ([7],[14],[15]), Isik et al. ([16],[17],[18]), Mursaleen [20], Sengul and Et [25], Sengul [26] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Čech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability.

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$. The density of a subset E of $\mathbb{N}$ is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ provided the limit exists,}$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

A sequence $x = (x_n)$ of real numbers is said to be statistically convergent to a real number L if for each $\varepsilon > 0$, the set $K = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ has natural density of zero and in this case we write $x_n \xrightarrow{S} L$.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The generalized de la Vallée-Poussin mean is defined by $t_n(x) = \frac{1}{\lambda_n} \sum\limits_{k\in I_n} x_k$, where $I_n = [n - \lambda_n + 1, n]$

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for $n = 1, 2, \ldots$. A sequence $x = (x_k)$ is said to be (V, λ) −summable to a number L if $t_n(x) \to L$ as $n \to \infty$. If $\lambda_n = n$, then (V, λ) −summability is reduced to Cesàro summability. By Λ we denote the class of all non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$.

Marouf [19] introduced definitions for asymptotically equivalent sequences and asymptotic regular matrices. Patterson [21] extended these concepts by presenting an asymptotically statistically equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Later on asymptotically equivalent sequences have been studied in ([1], [2],[22],[23]).

In this paper we introduce and examine the concepts of asymptotically λ−statistical equivalent sequences of order α in probability and strong asymptotically λ−equivalent sequences of order α in probability.

Let X_n $(n \in \mathbb{N})$ be a random variable which is defined on a given event space S with respect to a given class of events Δ and a probability function $P : \Delta \to \mathbb{R}$, then we say that $X_1, X_2, X_3, \ldots, X_n \ldots$ is a sequence of random variables. A sequence of random variables is denoted by $\{X_n\}_{n\in\mathbb{N}}$.

A sequence of random variables $\{X_n\}$ is said to be bounded in probability, if for every $\delta > 0$ there exists $M > 0$ such that

$$P(|X_n| > M) < \delta, \text{ for all } n \in \mathbb{N}$$

that is ;

$$\lim_{n\to\infty} P(|X_n| > M) = 0.$$

Definition 1.1. *Let (S, Δ, P) be a probability space, $\lambda = (\lambda_n)$ be a sequence as above and α be a positive real number such that $0 < \alpha \leq 1$. Two nonnegative sequences of random variables $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are said to be asymptotically λ−statistical equivalent of order α in probability provided that for every $\varepsilon, \delta > 0$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : P\left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon \right) \geq \delta \right\} \right| = 0.$$

In this case we write $X \overset{PS_\lambda^\alpha}{\sim} Y$. In case of $\alpha = 1$ we write $X \overset{PS_\lambda}{\sim} Y$ instead of $X \overset{PS_\lambda^\alpha}{\sim} Y$ and in case of $\lambda_n = n$ for all $n \in \mathbb{N}$ we write $X \overset{PS^\alpha}{\sim} Y$ instead of $X \overset{PS_\lambda^\alpha}{\sim} Y$ and also we write $X \overset{PS}{\sim} Y$ instead of $X \overset{PS_\lambda^\alpha}{\sim} Y$, if $\alpha = 1$ and $\lambda_n = n$ for all $n \in \mathbb{N}$.

Definition 1.2. *Let (S, Δ, P) be a probability space, $\lambda = (\lambda_n)$ be a sequence as above and α be a positive real number such that $0 < \alpha \leq 1$. Two nonnegative sequences of random variables $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are said to be strong asymptotically λ−equivalent of order α in probability provided that for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} P\left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon \right) = 0.$$

In this case we write $X \overset{PN_\lambda^\alpha}{\sim} Y$. In case of $\alpha = 1$ we write $X \overset{PN_\lambda}{\sim} Y$ instead of $X \overset{PN_\lambda^\alpha}{\sim} Y$ and in case of $\lambda_n = n$ for all $n \in \mathbb{N}$ we write $X \overset{PN^\alpha}{\sim} Y$ instead of $X \overset{PN_\lambda^\alpha}{\sim} Y$ and also we write $X \overset{PN}{\sim} Y$ instead of $X \overset{PN_\lambda^\alpha}{\sim} Y$, if $\alpha = 1$ and $\lambda_n = n$ for all $n \in \mathbb{N}$.

2. Main Results

In this section we give the main results of this study.

The proof of the following theorem is obtained by using the standard techniques, therefore we give it unproven.

Theorem 2.1. *Let the sequence $\lambda = (\lambda_n)$ be as above and α, β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then*

i) $X \overset{PS_\lambda^\alpha}{\sim} Y$ implies $X \overset{PS_\lambda^\beta}{\sim} Y$,

ii) $X \overset{PN_\lambda^\alpha}{\sim} Y$ implies $X \overset{PN_\lambda^\beta}{\sim} Y$,

iii) $X \overset{PN_\lambda^\alpha}{\sim} Y$ implies $X \overset{PS_\lambda^\beta}{\sim} Y$.

Theorem 2.2. *Let the sequence $\lambda = (\lambda_n)$ be as above and α be a fixed real number such that $0 < \alpha \leq 1$, if*

$$\lim_{n\to\infty} \inf \frac{\lambda_n^\alpha}{n^\alpha} > 0 \tag{1}$$

then $X \overset{PS^\alpha}{\sim} Y$ implies $X \overset{PS_\lambda^\alpha}{\sim} Y$.

Proof. For given $\varepsilon, \delta > 0$ we have

$$\left\{k \leq n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\} \supset \left\{k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\}.$$

Therefore

$$\begin{aligned} \frac{1}{n^\alpha}\left|\left\{k \leq n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\}\right| &\geq \frac{1}{n^\alpha}\left|\left\{k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\}\right| \\ &= \frac{\lambda_n^\alpha}{n^\alpha}\frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\}\right|. \end{aligned}$$

Taking limit as $n \to \infty$ and using (1), we get $X \overset{PS_\lambda^\alpha}{\sim} Y$. □

Corollary 2.3. *Let the sequence $\lambda = (\lambda_n)$ be as above, if*

$$\lim_{n\to\infty} \inf \frac{\lambda_n}{n} > 0$$

then $X \overset{PS}{\sim} Y$ implies $X \overset{PS_\lambda}{\sim} Y$.

Theorem 2.4. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$. Then each of the following assertions holds true;*

(*i*) *If*

$$\lim_{n\to\infty} \inf \frac{\lambda_n^\alpha}{\mu_n^\beta} > 0 \tag{2}$$

then $X \overset{PS_\mu^\beta}{\sim} Y$ implies $X \overset{PS_\lambda^\alpha}{\sim} Y$.

(*ii*) *If*

$$\lim_{n\to\infty} \frac{\mu_n}{\lambda_n^\beta} = 1 \tag{3}$$

then $X \overset{PS_\lambda^\alpha}{\sim} Y$ implies $X \overset{PS_\mu^\beta}{\sim} Y$.

Proof. (*i*) Suppose that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let (2) be satisfied. For given $\varepsilon, \delta > 0$ we have

$$\left\{k \in J_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\} \supseteq \left\{k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\}$$

where $I_n = [n - \lambda_n + 1, n]$ and $J_n = [n - \mu_n + 1, n]$. Therefore we can write

$$\frac{1}{\mu_n^\beta}\left|\left\{k \in J_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \frac{\lambda_n^\alpha}{\mu_n^\beta}\frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\}\right|$$

and this gives the proof.

(ii) Let $X \overset{PS^{\alpha}_{\lambda}}{\sim} Y$ and (3) be satisfied. Since $I_n \subset J_n$ for $\varepsilon, \delta > 0$ we may write

$$\begin{aligned}
&\frac{1}{\mu_n^{\beta}}\left|\left\{k \in J_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\}\right| \\
= &\frac{1}{\mu_n^{\beta}}\left|\left\{n - \mu_n + 1 < k \leq n - \lambda_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\}\right| \\
&+\frac{1}{\mu_n^{\beta}}\left|\left\{k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\}\right| \\
\leq &\frac{\mu_n - \lambda_n}{\mu_n^{\beta}} + \frac{1}{\lambda_n^{\beta}}\left|\left\{k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\}\right| \\
\leq &\left(\frac{\mu_n - \lambda_n^{\beta}}{\lambda_n^{\beta}}\right) + \frac{1}{\lambda_n^{\alpha}}\left|\left\{k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\}\right| \\
\leq &\left(\frac{\mu_n}{\lambda_n^{\beta}} - 1\right) + \frac{1}{\lambda_n^{\alpha}}\left|\left\{k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta\right\}\right|
\end{aligned}$$

for all $n \in \mathbb{N}$. This implies that the condition $X \overset{PS^{\alpha}_{\lambda}}{\sim} Y$ implies $X \overset{PS^{\beta}_{\mu}}{\sim} Y$. □

From Theorem 2.4 we obtain the following results.

Corollary 2.5. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$.*
If (2) *holds, then*
(i) $X \overset{PS^{\alpha}_{\mu}}{\sim} Y$ *implies* $X \overset{PS^{\alpha}_{\lambda}}{\sim} Y$ *for each* $\alpha \in (0,1]$,
(ii) $X \overset{PS_{\mu}}{\sim} Y$ *implies* $X \overset{PS^{\alpha}_{\lambda}}{\sim} Y$ *for each* $\alpha \in (0,1]$,
(iii) $X \overset{PS_{\mu}}{\sim} Y$ *implies* $X \overset{PS_{\lambda}}{\sim} Y$.
If (3) *holds, then*
(i) $X \overset{PS^{\alpha}_{\lambda}}{\sim} Y$ *implies* $X \overset{PS^{\alpha}_{\mu}}{\sim} Y$ *for each* $\alpha \in (0,1]$,
(ii) $X \overset{PS^{\alpha}_{\lambda}}{\sim} Y$ *implies* $X \overset{PS_{\mu}}{\sim} Y$ *for each* $\alpha \in (0,1]$,
(iii) $X \overset{PS_{\lambda}}{\sim} Y$ *implies* $X \overset{PS_{\mu}}{\sim} Y$.

Theorem 2.6. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then we have*

(i) *Let the condition in* (2) *hold, then* $X \overset{PN^{\beta}_{\mu}}{\sim} Y$ *implies* $X \overset{PN^{\alpha}_{\lambda}}{\sim} Y$,

(ii) *Let the condition in* (3) *hold and $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be two bounded sequences of random variables, then* $X \overset{PN^{\alpha}_{\lambda}}{\sim} Y$ *implies* $X \overset{PN^{\beta}_{\mu}}{\sim} Y$.

Proof. (i) Suppose that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let (2) be satisfied. The proof follows from the following inequality:

$$\frac{1}{\mu_n^{\beta}}\sum_{k\in J_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\frac{1}{\lambda_n^{\alpha}}\sum_{k\in I_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right).$$

(ii) Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be two bounded sequences of random variables, $X \overset{PN^{\alpha}_{\lambda}}{\sim} Y$ and suppose that (3) holds. Since $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are bounded, we can find $M_1, M_2 > 0$ such that $\lim_{n\to\infty} P(|X_n| > M_1) = 0$,

$\lim_{n\to\infty} P(|Y_n| > M_2) = 0$ and $\left|\frac{X_n}{Y_n}\right| \le \frac{M_1}{M_2} < M$ for all $n \in \mathbb{N}$. Since $\lambda_n \le \mu_n$ for all $n \in \mathbb{N}$, we may write

$$\begin{aligned}
\frac{1}{\mu_n^\beta}\sum_{k\in J_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \ge \varepsilon\right) &= \frac{1}{\mu_n^\beta}\sum_{k\in J_n - I_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \ge \varepsilon\right) + \frac{1}{\mu_n^\beta}\sum_{k\in I_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \ge \varepsilon\right) \\
&\le \left(\frac{\mu_n - \lambda_n}{\mu_n^\beta}\right)M + \frac{1}{\mu_n^\beta}\sum_{k\in I_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \ge \varepsilon\right) \\
&\le \left(\frac{\mu_n - \lambda_n^\beta}{\lambda_n^\beta}\right)M + \frac{1}{\lambda_n^\beta}\sum_{k\in I_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \ge \varepsilon\right) \\
&\le \left(\frac{\mu_n}{\lambda_n^\beta} - 1\right)M + \frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \ge \varepsilon\right)
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore $X \overset{PN_\mu^\beta}{\sim} Y$. □

Theorem 2.6 yields the following results.

Corollary 2.7. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \le \mu_n$ for all $n \in \mathbb{N}$.*

If (2) *holds, then*

(*i*) *$X \overset{PN_\mu^\alpha}{\sim} Y$ implies $X \overset{PN_\lambda^\alpha}{\sim} Y$ for each $\alpha \in (0,1]$,*

(*ii*) *$X \overset{PN_\mu}{\sim} Y$ implies $X \overset{PN_\lambda^\alpha}{\sim} Y$ for each $\alpha \in (0,1]$,*

(*iii*) *$X \overset{PN_\mu}{\sim} Y$ implies $X \overset{PN_\lambda}{\sim} Y$.*

Let the condition in (3) *hold and $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be two bounded sequences of random variables, then*

(*i*) *$X \overset{PN_\lambda^\alpha}{\sim} Y$ implies $X \overset{PN_\mu^\alpha}{\sim} Y$ for each $\alpha \in (0,1]$,*

(*ii*) *$X \overset{PN_\lambda^\alpha}{\sim} Y$ implies $X \overset{PN_\mu}{\sim} Y$ for each $\alpha \in (0,1]$,*

(*iii*) *$X \overset{PN_\lambda^\alpha}{\sim} Y$ implies $X \overset{PN_\mu}{\sim} Y$.*

Theorem 2.8. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \le \mu_n$ for all $n \in \mathbb{N}$ and let α and β be fixed real numbers such that $0 < \alpha \le \beta \le 1$.*

(*i*) *Let the condition in* (2) *hold, then $X \overset{PN_\mu^\beta}{\sim} Y$ implies $X \overset{PS_\lambda^\alpha}{\sim} Y$.*

(*ii*) *Let the condition in* (3) *hold and $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be two bounded sequences of random variables then $X \overset{PS_\lambda^\alpha}{\sim} Y$ implies $X \overset{PN_\mu^\beta}{\sim} Y$.*

Proof. (*i*) For any sequences $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ of random variables and $\varepsilon, \delta > 0$, we have

$$\begin{aligned}
&\sum_{k\in J_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \ge \varepsilon\right) \\
&= \sum_{\substack{k\in J_n \\ P\left(\left|\frac{X_k}{Y_k} - L\right| \ge \varepsilon\right) \ge \delta}} P\left(\left|\frac{X_k}{Y_k} - L\right| \ge \varepsilon\right) + \sum_{\substack{k\in J_n \\ P\left(\left|\frac{X_k}{Y_k} - L\right| \ge \varepsilon\right) < \delta}} P\left(\left|\frac{X_k}{Y_k} - L\right| \ge \varepsilon\right)
\end{aligned}$$

$$\geq \sum_{\substack{k\in I_n \\ P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)\geq\delta}} P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)+\sum_{\substack{k\in I_n \\ P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)<\delta}} P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)$$

$$\geq \sum_{\substack{k\in I_n \\ P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)\geq\delta}} P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)$$

$$\geq \left|\left\{k\in I_n : P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)\geq\delta\right\}\right|\delta$$

and so that

$$\begin{aligned}\frac{1}{\mu_n^{\beta}}\sum_{k\in J_n} P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right) &\geq \frac{1}{\mu_n^{\beta}}\left|\left\{k\in I_n : P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)\geq\delta\right\}\right|\delta \\ &\geq \frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\frac{1}{\lambda_n^{\alpha}}\left|\left\{k\in I_n : P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)\geq\delta\right\}\right|\delta.\end{aligned}$$

Therefore $X \overset{PS_{\lambda}^{\alpha}}{\sim} Y$.

(ii) Let the condition in (3) hold, $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be two bounded sequences of random variables and suppose that $X \overset{PS_{\lambda}^{\alpha}}{\sim} Y$. We can select a number $M > 0$ such as in the proof of Theorem 2.6 (ii). Then for every $\varepsilon, \delta > 0$ we may write

$$\begin{aligned}\frac{1}{\mu_n^{\beta}}\sum_{k\in J_n} P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right) &= \frac{1}{\mu_n^{\beta}}\sum_{k\in J_n-I_n} P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)+\frac{1}{\mu_n^{\beta}}\sum_{k\in I_n} P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right) \\ &\leq \left(\frac{\mu_n-\lambda_n}{\mu_n^{\beta}}\right)M+\frac{1}{\mu_n^{\beta}}\sum_{k\in I_n} P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right) \\ &\leq \left(\frac{\mu_n-\lambda_n^{\beta}}{\mu_n^{\beta}}\right)M+\frac{1}{\mu_n^{\beta}}\sum_{k\in I_n} P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right) \\ &= \left(\frac{\mu_n}{\lambda_n^{\beta}}-1\right)M+\frac{1}{\mu_n^{\beta}}\sum_{\substack{k\in I_n \\ P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)\geq\delta}} P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right) \\ &\quad +\frac{1}{\mu_n^{\beta}}\sum_{\substack{k\in I_n \\ P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)<\delta}} P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right) \\ &\leq \left(\frac{\mu_n}{\lambda_n^{\beta}}-1\right)M+\frac{M}{\mu_n^{\beta}}\left|\left\{k\in I_n : P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)\geq\delta\right\}\right|+\frac{\mu_n}{\lambda_n^{\beta}}\delta \\ &\leq \left(\frac{\mu_n}{\lambda_n^{\beta}}-1\right)M+\frac{M}{\lambda_n^{\alpha}}\left|\left\{k\in I_n : P\left(\left|\frac{X_k}{Y_k}-L\right|\geq\varepsilon\right)\geq\delta\right\}\right|+\frac{\mu_n}{\lambda_n^{\beta}}\delta\end{aligned}$$

for all $n \in \mathbb{N}$. Using (3) we obtain that $X \overset{PN_{\mu}^{\beta}}{\sim} Y$. □

The following results easily derive from Theorem 2.8.

Corollary 2.9. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$.*

Let the condition in (2) *hold, then*

(i) $X \overset{PN_\mu^\alpha}{\sim} Y$ *implies* $X \overset{PS_\lambda^\alpha}{\sim} Y$ *for each* $\alpha \in (0,1]$,

(ii) $X \overset{PN_\mu}{\sim} Y$ *implies* $X \overset{PS_\lambda^\alpha}{\sim} Y$ *for each* $\alpha \in (0,1]$,

(iii) $X \overset{PN_\mu}{\sim} Y$ *implies* $X \overset{PS_\lambda}{\sim} Y$.

Let the condition in (3) *hold and* $\{X_n\}_{n\in\mathbb{N}}$ *and* $\{Y_n\}_{n\in\mathbb{N}}$ *be two bounded sequences of random variables, then*

(i) $X \overset{PS_\lambda^\alpha}{\sim} Y$ *implies* $X \overset{PN_\mu^\alpha}{\sim} Y$ *for each* $\alpha \in (0,1]$,

(ii) $X \overset{PS_\lambda^\alpha}{\sim} Y$ *implies* $X \overset{PN_\mu}{\sim} Y$ *for each* $\alpha \in (0,1]$,

(iii) $X \overset{PS_\lambda}{\sim} Y$ *implies* $X \overset{PN_\mu}{\sim} Y$.

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