On Atmospheric Oscillations.

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Introduction.

1. The chief question discussed in this paper (§§ 6—12) is that of the free oscillations of an atmosphere whose temperature varies with the altitude; and in particular the case of a uniform vertical temperature-gradient is studied in some detail. For consistency it is assumed that the expansions and contractions follow the adiabatic law. The problem is treated as a two-dimensional one, the space co-ordinates involved being horizontal and vertical; and the more definite conclusions arrived at relate to the case where the (horizontal) wave-length is somewhat large in comparison with the height of the atmosphere.

The results are most easily interpreted when the temperature-gradient does not fall much below that characteristic of a state of convective equilibrium. The normal modes of oscillation then fall into well-defined types.

In the most important type, the motion of the air-particles is mainly horizontal, and independent of the altitude, and the waves may therefore be described as "longitudinal." The velocity of propagation of progressive waves is found to be equal to $\sqrt{(g\mathrm{H})}$, where H denotes what may be called the "virtual height" of the atmosphere, *i.e.* the height of a "homogeneous atmosphere" corresponding to the temperature of the *lowest stratum*. That the result should come out intermediate in value between the velocity of sound in the lowest stratum, viz., $\sqrt{(\gamma g\mathrm{H})}$, and the zero velocity corresponding to the zero temperature which is postulated in the higher regions was to be anticipated; but that it should be identical in form with that obtained on the hypothesis of an *isothermal* atmosphere whose expansions are subject to Boyle's law,* the effect of the upward decrease of temperature being exactly compensated by the greater elasticity implied in the adiabatic law, is somewhat remarkable.

When the temperature-gradient falls distinctly below the "convective" value, the character of the oscillation is less simple. The wave-velocity is somewhat increased, but must always remain below the value $\sqrt{(\gamma g\mathrm{H})}$, which is the velocity of sound in the lowest stratum.

2. A second type of oscillations depends on the degree of stability of the atmosphere.

* Rayleigh, 'Phil. Mag.' (4), 1890, vol. 29, p. 173; 'Scientific Papers,' vol. 3, p. 335.

The work required to bring unit mass of air from the density ρ_1 to the density ρ_2 under the adiabatic condition $p/\rho^\gamma = p_1/\rho_1{}^\gamma$ is

$$\int_{\rho^2}^{\rho_1} pd\left(\frac{1}{\rho}\right) = \frac{p_1}{(\gamma-1)\rho_1}\left\{\left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1} - 1\right\}. \tag{1}$$

Hence if we imagine two thin strata of equal mass, whose densities are ρ_1, ρ_2, and pressures p_1, p_2, to be interchanged, the work required to effect this will be, per unit mass,

$$\frac{p_1}{(\gamma-1)\rho_1}\left\{\left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1} - 1\right\} + \frac{p_2}{(\gamma-1)\rho_2}\left\{\left(\frac{\rho_1}{\rho_2}\right)^{\gamma-1} - 1\right\} = \frac{\rho_2{}^{\gamma-1} - \rho_1{}^{\gamma-1}}{\gamma-1}\left(\frac{p_1}{\rho_1{}^\gamma} - \frac{p_2}{\rho_2{}^\gamma}\right). \tag{2}$$

If we avail ourselves of the notion of "potential temperature,"* *i.e.* the temperature ϑ which any particular portion of air would assume if brought adiabatically to some standard density D, we have

$$p/\rho^\gamma = \mathrm{R}\vartheta/\mathrm{D}^{\gamma-1}, \tag{3}$$

where R is the constant of the formula

$$p = \mathrm{R}\rho\theta, \tag{4}$$

θ denoting the absolute temperature. Hence (2) becomes

$$\frac{\mathrm{R}(\rho_2{}^{\gamma-1} - \rho_1{}^{\gamma-1})(\vartheta_1 - \vartheta_2)}{(\gamma-1)\mathrm{D}^{\gamma-1}}. \tag{5}$$

Hence if $\rho_2 > \rho_1$, we must for stability have $\vartheta_1 > \vartheta_2$; *i.e.* the potential temperature must increase upwards. Now, if y denote depth below a standard level, we have, in equilibrium,

$$dp/dy = g\rho; \tag{6}$$

and combining this with (3) and (4), we find

$$\frac{1}{\vartheta}\frac{d\vartheta}{dy} = \frac{1}{\theta}\left\{\gamma\frac{d\theta}{dy} - \frac{(\gamma-1)g}{\mathrm{R}}\right\}. \tag{7}$$

In convective equilibrium, where p/ρ^γ, and consequently ϑ, is the same at all altitudes, we have

$$\frac{d\theta}{dy} = \frac{(\gamma-1)g}{\gamma\mathrm{R}}. \tag{8}$$

This equilibrium, though stable for some types of disturbance (§ 8), is in other respects neutral. For complete stability, $d\vartheta/dy$ must be negative, and therefore

$$\frac{d\theta}{dy} < \frac{(\gamma-1)g}{\gamma\mathrm{R}}. \tag{9}$$

When this condition is fulfilled, we have a series of possible modes of

* v. Bezold, 'Berl. Sitzb.,' 1888, vol. 46.

oscillation whose periods, depending as they do on the extent to which the temperature-gradient differs from the convective value (8), are comparatively long. Oscillations of this character, governed by local conditions, must undoubtedly occur in the atmosphere, and may conceivably account for some of the minor fluctuations of the barometer.

There remains a third type of oscillations which, when the wave-length is moderately great, approximate to the character of waves propagated vertically in the atmosphere. These have been discussed in a previous paper by the author.* From a meteorological standpoint they can hardly be of importance.

3. The theory of the "longitudinal" waves is of interest in relation to the large-scale oscillations of the earth's atmosphere as a whole. This subject was treated by Laplace,† and is of some importance in connection with the suggestion put forward by Lord Kelvin‡ as to the origin of the semi-diurnal variation of the barometer. Laplace's investigation was based on the hypotheses of a uniform equilibrium temperature and an isothermal law of expansion, and on the further *assumption* that the vertical motion of the air-particles may be neglected.§ Since the circumstances are then practically those of sound-waves propagated horizontally, his results naturally involve the "Newtonian" velocity of sound, $\sqrt{(g\mathrm{H})}$, where H is the height of the homogeneous atmosphere corresponding to the assumed uniform temperature θ_0, viz., $\mathrm{H} = \mathrm{R}\theta_0/g$.

The hypotheses referred to were, of course, adopted only for mathematical convenience. As representations of actual conditions they are very imperfect; and there is, moreover, great uncertainty as to the most suitable value to be attributed to θ_0. It appeared to the writer that a firmer ground for quantitative conclusions would be gained if it were possible to calculate the wave-velocity (for long waves), even in the two-dimensional problem, on somewhat more natural suppositions as to the constitution of the atmosphere and the law of expansion.

In the actual atmosphere the temperature, as a rule, diminishes upwards, although (as we have seen) it is necessary for stability that the gradient should nowhere exceed the convective value. The special hypothesis of a *uniform* gradient, which is here adopted as a basis of calculation, is itself an artificial one; but in spite of the fact that it implies an upper limit to the atmosphere, it may claim to give, on the whole, a better representation of the

* 'Lond. Math. Soc. Proc.' (2), 1908, vol. 7, p. 122.

† 'Mécanique Céleste,' Livre 4, Chap. 5. See also Rayleigh, *loc. cit.*

‡ 'Roy. Soc. Edin. Proc.,' 1882, vol. 11; 'Math. and Phys. Papers,' vol. 3, p. 341.

§ Some such assumption is necessary to make the problem determinate, in the absence of a prescribed condition to be fulfilled, or approximated to, in the upper regions of the atmosphere.

true conditions than the isothermal view, on which, indeed, the earth's atmosphere is merely a local concentration of a medium diffused through space.

As regards the law of expansion, since permanent inequalities of temperature are postulated in the equilibrium condition, it is proper to ignore the transfer of heat between adjacent portions of the air during the oscillations. In any case, theory shows that the effect of conduction on such long waves as we have here in view may safely be neglected.*

The main conclusion of Laplace was that the free and forced oscillations of an atmosphere covering a globe, whether this be at rest or in uniform rotation, are identical with those of a liquid ocean of uniform depth H; but in view of the nature of his premises, and of the uncertainty as to the temperature to be adopted in estimating the value of H, considerable doubt has been felt as to how far this analogy can be relied upon for quantitative results. The present investigation tends, I think, to show that inferences of this kind will not be very far from the truth, provided the temperature adopted be the mean temperature of the lower strata of the earth's atmosphere, so far as this can be ascertained. The formal adaptation of the theory of longitudinal waves to the case of an atmosphere of relatively small depth covering a globe would follow the same course as in Laplace's investigation.

4. As regards the semi-diurnal variation of the barometer, the passage of Kelvin's paper already referred to runs as follows :—

"The cause of the semi-diurnal variation of barometric pressure cannot be the gravitational tide-generating influence of the sun, because, if it were, there would be a much larger lunar influence of the same kind, while in reality the lunar barometric tide is insensible or nearly so. It seems, therefore, certain that the semi-diurnal variation of the barometer is due to temperature. Now, the *diurnal* term, in the harmonic analysis of the variation of *temperature*, is undoubtedly much larger in all, or nearly all, places than the *semi-diurnal*. It is then very remarkable that the *semi-diurnal term of the barometric effect of* the variation of temperature should be greater, and so much greater as it is, than the diurnal. The explanation probably is to be found by considering the oscillations of the atmosphere, as a whole, in the light of the very formulæ which Laplace gave in his 'Mécanique Céleste' for the ocean, and which he showed to be also applicable to the atmosphere. When thermal influence is substituted for gravitational, in the tide-generating force reckoned for, and when the modes of oscillation

* This follows from the equations (due substantially to Kirchhoff and Rayleigh) given in the author's 'Hydrodynamics,' 3rd edit., § 343. Radiation has a different tendency in this respect.

corresponding respectively to the diurnal and semi-diurnal terms of the thermal influence are investigated, it will probably be found that the period of free oscillation of the former agrees much less nearly with 24 hours than does that of the latter with 12 hours; and that, therefore, with comparatively small magnitudes of the tide-generating force, the resulting tide is greater in the semi-diurnal term than in the diurnal."

The first question which here arises, viz., whether as a matter of fact the earth's atmosphere has a mode of oscillation of the requisite type, with a period of about 12 mean solar hours, can at the present time be examined more closely than was possible at the date (1882) of the above extract. The free oscillations of an ocean of water of uniform depth covering a globe of the size of the earth, rotating with the same angular velocity, have been very fully investigated by Hough* in the course of his classical work on tidal theory. He finds, in particular, that in the case of the most important free oscillation having the same general character as a semi-diurnal tide wave (*i.e.* its most salient spherical harmonic constituent is the sectorial harmonic of the second order), the depth h for which the period is exactly 12 sidereal hours is given by

$$gh/4\omega^2a^2 = 0\cdot10049,$$

where a is the earth's radius, and ω its velocity of rotation. This is evaluated at 29,182 feet. It is to be remarked, however, that throughout the calculation the mutual attraction of the disturbed fluid has been taken into account, whereas in the aërial ocean this influence must be quite insensible. If the disturbance were accurately of the type of a spherical harmonic of the second order the requisite modification would consist merely in multiplying the previous result by the factor

$$\frac{g_2}{g} = 1 - \tfrac{3}{5} \times 0\cdot18093 = 0\cdot89144,$$

where the decimal fraction in the second member is the ratio of the density of the water to the mean density of the globe, as adopted in Hough's computation. This would make

$$gh/4\omega^2a^2 = 0\cdot08958.$$

As the result of a more direct calculation, using Hough's algorithm, together with such of his numerical results as are applicable, I find

$$gh/4\omega^2a^2 = 0\cdot08986,$$

the last figure being somewhat doubtful. If we put $g = 32\cdot200$, $\omega = 2\pi/86164$, $a = 20{,}902{,}000$, this gives

$$h = 25{,}930 \text{ feet}.$$

* 'Phil. Trans.,' A, 1897, vol. 191, p. 139. See pp. 164, 179.

The substitution of a mean solar for a sidereal half-day as the period involves a further slight diminution, which can be estimated pretty closely from another of Hough's results. He finds that for $gh/4\omega^2a^2 = 0{\cdot}1$, the speed ($\sigma$) of the free oscillation in question is given by $\sigma/\omega = 1{\cdot}9968$. Comparing this with the former result, we infer that for a period of 12 mean solar hours ($\sigma/\omega = 1{\cdot}9945$) we must have $gh/4\omega^2a^2 = 0{\cdot}09965$, about. Assuming that when mutual attraction is ignored this figure is to be reduced in the same ratio as the former one, we have, finally,

$$gh/4\,\omega^2a^2 = 0{\cdot}08911,$$

or, with the previous numerical data,

$$h = 25{,}710 \text{ feet}.$$

It must be remembered, of course, that these numerical results can claim no greater accuracy than the theory on which they rest, in which, in particular, the ellipticity of the earth, which is of the order 1/300, is neglected.

On the other hand, the value of H for air at 0° C. is about 26,200 feet, with an increase of about 96 feet for every degree above this temperature. The mean temperature of the air near the earth's surface is usually estimated at 15° C. This would make H = 27,640 feet; but a somewhat lower value for the mean temperature of the lower strata, away from the immediate influence of the ground, would perhaps be more appropriate.

Without pressing too far conclusions based on the hypothesis of an atmosphere uniform over the earth, and approximately in convective equilibrium, we may, I think, at least assert the existence of a free oscillation of the earth's atmosphere, of "semi-diurnal" type, with a period not very different from, but probably somewhat less than, 12 mean solar hours.

At the same time, the reason for rejecting the explanation of the semi-diurnal barometric oscillation as due to a gravitational solar tide seems to call for a little further examination. The amplitude of this variation at places on the equator is given by Kelvin as 0·032 inch. The amplitude given by the "equilibrium" theory of the tides is about 0·00047 inch.* Some numerical results given by Hough in illustration of the kinetic theory of oceanic tides indicate that in order that this amplitude should be increased by dynamical action some seventy-fold, the free period must suffer from the imposed period of 12 solar hours by not more than 2 or 3 minutes. Since the difference between the lunar and solar semi-diurnal periods amounts to 26 minutes, it

* The numerical values given on p. 520 of the author's 'Hydrodynamics' relate to the lunar tide, and are, moreover, by an oversight, stated as "amplitudes," instead of as "ranges."

is quite conceivable that the solar influence might in this way be rendered much more effective than the lunar. The real difficulty, so far as this point is concerned, is the *a priori* improbability of so very close an agreement between the two periods. The most decisive evidence, however, appears to be furnished by the *phase* of the observed semi-diurnal equality, which is accelerated instead of retarded (as it would be by tidal friction) relatively to the sun's transit.*

5. The concluding part of the paper (§§ 13, 14) is an attempt to examine more closely than has hitherto been done the theory of waves on a surface of discontinuity in the atmosphere. That such waves may play a part in meteorological phenomena has been pointed out independently by Helmholtz† and Lord Kelvin,‡ but both writers have confined themselves to analogies drawn from the case of superposed homogeneous liquids. It is to be observed that even on this view the disturbance extends, upwards and downwards from the plane of discontinuity, through a space which is an appreciable fraction of the wave-length; hence, apart altogether from the influence of compressibility, the conditions of the question will be modified when the wave-length is such that the ordinary variation of density within this space becomes sensible. It seemed worth while to investigate the matter; but it must be acknowledged that when there are no currents, the discontinuity being one of temperature and density only, the analogy proves to be adequate, under such conditions as are likely to occur in the atmosphere, for a considerable range of wave-lengths. For very long waves it would break down, the disturbance ceasing to be even approximately concentrated in the neighbourhood of the plane of discontinuity. The discontinuity then becomes, in fact, an unimportant incident in the general upward diminution of density.

When there is a discontinuity of *velocity*, the upper fluid being in steady horizontal motion relative to the lower, the question, when compressibility is taken into account, is more difficult, and I have not been able to arrive at any very simple results. There can be no doubt, however, that the aforesaid analogy is sufficient in this case also for wave-lengths less than a certain limit. In particular, the dynamical instability pointed out by Kelvin§ will remain.

* 'Brit. Ass. Rep.,' 1908, p. 606. The forced tides due to diurnal and semi-diurnal wa es of temperature have been studied by Margules, 'Wien. Sitzb.,' 1890, vol. 99, p. 204.

† 'Berl. Sitzb.,' 1889; 'Wiss. Abh.,' vol. 3, p. 309.

‡ 'Brit. Assoc. Rep.,' 1876; 'Math. and Phys. Papers,' vol. 4, p. 457.

§ 'Math. and Phys. Papers,' vol. 4, p. 76.

Theory of Long Atmospheric Waves.

6. We consider an atmosphere arranged in horizontal layers of uniform density. The motions contemplated are restricted to two dimensions, x, y, of which x is horizontal and y vertical, the positive direction of y being downwards. The equilibrium values of the pressure, density, and temperature are denoted by p_0, ρ_0, ϑ_0; these are functions of y only, and are subject to the hydrostatic condition

$$dp_0/dy = g\rho_0, \tag{10}$$

as well as to the general relation

$$p_0 = \mathrm{R}\rho_0\theta_0. \tag{11}$$

The equations of small motion are, in the usual notation,

$$\rho_0\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, \qquad \rho_0\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y}+g\rho. \tag{12}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0, \tag{13}$$

where

$$\mathrm{D}/\mathrm{D}t = \partial/\partial t + u\partial/\partial x + v\partial/\partial y. \tag{14}$$

The expansions being supposed subject to the adiabatic law, we have also

$$\frac{\mathrm{D}p}{\mathrm{D}t} = c^2\,\frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{15}$$

where

$$c^2 = \gamma p_0/\rho_0 = \gamma\mathrm{R}\theta_0, \tag{16}$$

i.e. c is the velocity of sound corresponding to the equilibrium temperature at the point considered. It is accordingly in general a function of y. If we put

$$p = p_0+\varpi, \qquad \rho = \rho_0+\delta, \tag{17}$$

and continue to neglect small terms of the second order, we have

$$\rho_0\frac{\partial u}{\partial t} = -\frac{\partial\varpi}{\partial x}, \quad \rho_0\frac{\partial v}{\partial t} = -\frac{\partial\varpi}{\partial y}+g\delta. \tag{18}$$

$$\frac{\partial\delta}{\partial t}+v\frac{d\rho_0}{dy} = -\rho_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right). \tag{19}$$

Also, from (15), (13), and (10),

$$\frac{\partial\varpi}{\partial t}+g\rho_0 v = \frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right). \tag{20}$$

Hence, eliminating δ and ϖ, we find*

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial}+\frac{\partial v}{\partial y}\right)+g\frac{\partial v}{\partial x},\\ \frac{\partial^2 v}{\partial t^2} &= c^2\,\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+(\gamma-1)g\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+g\frac{\partial v}{\partial y}.\end{aligned}\right\} \tag{21}$$

* It may be noticed, parenthetically, that in the case of an isothermal atmosphere where c is constant, these equations are satisfied by

$$u = e^{-(\gamma-1)gy/c^2}f(ct-x), \qquad v = 0.$$

If we now write

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\chi, \qquad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\zeta, \tag{22}$$

we deduce from (21), by differentiation,

$$\frac{\partial^2\chi}{\partial t^2}=c^2\nabla^2\chi+\left(\frac{d.c^2}{dy}+\gamma g\right)\frac{\partial\chi}{\partial y}+g\frac{\partial\zeta}{\partial x}, \tag{23}$$

$$\frac{\partial^2\zeta}{\partial t^2}=-\left\{\frac{d.c^2}{dy}-(\gamma-1)g\right\}\frac{\partial\chi}{\partial x}, \tag{24}$$

where

$$\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2. \tag{25}$$

The latter equation shows that an irrotational motion is not possible unless

$$\frac{d.c^2}{dy}=(\gamma-1)g, \quad\text{or}\quad \frac{d\theta_0}{dy}=\frac{(\gamma-1)g}{\gamma \mathrm{R}}, \tag{26}$$

which we have seen to be the case of convective equilibrium. We note also that

$$\frac{\mathrm{D}p}{\mathrm{D}t}=-\gamma p_0\chi, \tag{27}$$

by (20).

Eliminating ζ between (23) and (24), we obtain

$$\frac{\partial^4\chi}{\partial t^4}=c^2\nabla^2\frac{\partial^2\chi}{\partial t^2}+\left(\frac{d.c^2}{dy}+\gamma g\right)\frac{\partial^3\chi}{\partial y\,\partial t^2}-g\left\{\frac{d.c^2}{dy}-(\gamma-1)g\right\}\frac{\partial^2\chi}{\partial x^2}. \tag{28}$$

If we assume that x and t occur only through a factor $e^{i(kx+\sigma t)}$, the equations (21) take the forms

$$\left.\begin{aligned}\sigma^2u+igkv&=-ikc^2\chi,\\ -igku+\sigma^2v&=-c^2\frac{\partial\chi}{\partial y}-\gamma g\chi,\end{aligned}\right\} \tag{29}$$

whence

$$\left.\begin{aligned}(\sigma^4-g^2k^2)\,u&=ik\left\{gc^2\frac{\partial\chi}{\partial y}+(\gamma g^2-\sigma^2c^2)\,\chi\right\},\\ (\sigma^4-g^2k^2)\,v&=-\sigma^2c^2\frac{\partial\chi}{\partial y}-g\,(\gamma\sigma^2-k^2c^2)\,\chi\right\}.\end{aligned}\right\} \tag{30}$$

From these, or from (28), we have

$$c^2\frac{\partial^2\chi}{\partial y^2}+\left(\frac{d.c^2}{dy}+\gamma g\right)\frac{\partial\chi}{\partial y}+\left[\sigma^2-k^2c^2-\left\{\frac{d.c^2}{dy}-(\gamma-1)g\right\}\frac{gk^2}{\sigma^2}\right]\chi=0. \tag{31}$$

7. So far, the vertical distribution of temperature is arbitrary. In the case of temperature diminishing upwards with a *uniform* gradient, to which we now proceed, there is an upper limit to the atmosphere. If we take the origin of y at this level, we have

$$\theta_0=\beta y, \tag{32}$$

where β is the gradient in question. It easily follows from (10) and (11) that

$$\rho_0 \propto y^n, \qquad p_0 \propto y^{n+1}, \tag{33}$$

where

$$n = g/\mathrm{R}\beta - 1. \tag{34}$$

Also

$$c^2 = \gamma \mathrm{R}\beta y = \gamma g y/(n+1). \tag{35}$$

Hence

$$\left.\begin{aligned} (\sigma^4 - g^2k^2)\,u &= \frac{i\gamma g^2 k}{n+1}\left[y\frac{\partial \chi}{\partial y} + (n+1)\,\chi - \frac{\sigma^2}{gk}ky\chi\right], \\ (\sigma^4 - g^2k^2)\,v &= \frac{-\gamma g^2 k}{n+1}\left[\frac{\sigma^2}{gk}\left\{y\frac{\partial \chi}{\partial y} + (n+1)\,\chi\right\} - ky\chi\right], \end{aligned}\right\} \tag{36}$$

and

$$y\frac{\partial^2 \chi}{\partial y^2} + (n+2)\frac{\partial \chi}{\partial y} + \left\{\frac{n+1}{\gamma}\frac{\sigma^2}{gk} + \frac{n\gamma - n - 1}{\gamma}\frac{gk}{\sigma^2} - ky\right\} k\chi = 0. \tag{37}$$

The meaning of the factor $(n\gamma - n - 1)/\gamma$, which appears in one of these terms, is to be noticed; viz. we have

$$\frac{n\gamma - n - 1}{\gamma} = \frac{\beta_1}{\beta} - 1, \tag{38}$$

where β_1 is the temperature-gradient in a state of convective equilibrium, as given by (8).

To solve (37) we put

$$\chi = e^{ky}\phi, \tag{39}$$

and obtain

$$y\frac{\partial^2 \phi}{\partial y^2} + (n + 2 + 2ky)\frac{\partial \phi}{\partial y} + 2\alpha k\phi = 0, \tag{40}$$

where

$$2\alpha = \frac{n+1}{\gamma}\frac{\sigma^2}{gk} + \left(\frac{\beta_1}{\beta} - 1\right)\frac{gk}{\ } + n + 2. \tag{41}$$

This is integrable by series, the solution which is finite for $y = 0$ being

$$\phi = 1 - \frac{\alpha}{1\,.\,n+2}(2ky) + \frac{\alpha\,.\,\alpha+1}{1\,.\,2\,.\,n+2\,.\,n+3}(2ky)^2 - \ldots; \tag{42}$$

or, in the notation of Dr. E. W. Barnes,*

$$\phi = {}_1\mathrm{F}_1(\alpha;\, n+2;\, -2ky). \tag{43}$$

The remaining solution of (40) is of the form

$$\phi\int\frac{e^{-2ky}dy}{y^{n+2}\phi^2}, \tag{44}$$

where ϕ stands for the series in (42). This is not admissible in the present

* See, for example, 'Camb. Trans.,' vol. 20, p. 253, where references to other papers are given.

If we had assumed $\chi = e^{-ky}\phi$, in place of (39), we should have found

$$\phi = {}_1\mathrm{F}_1(n+2-\alpha;\, n+2;\, 2ky).$$

The comparison verifies a well-known identity; see Barnes, *loc. cit.*

question, since it becomes infinite as y^{-n-1} for infinitesimal values of y, whereas the condition to be satisfied at the upper boundary is $\mathrm{D}p/\mathrm{D}t = 0$, or $y^{n+1}\chi = 0$; see equations (27), (33).

The formulæ (36) now become

$$\left.\begin{aligned}(\sigma^4-g^2k^2)\,u &= \frac{i\gamma g^2 k}{n+1}\left[y\frac{\partial\phi}{\partial y}+(n+1)\,\phi+\left(1-\frac{\sigma^2}{gk}\right)ky\,\phi\right]e^{ky},\\ (\sigma^4-g^2k^2)\,v &= -\frac{\gamma g^2 k}{n+1}\left[\frac{\sigma^2}{gk}\left\{y\frac{\partial\phi}{\partial y}+(n+1)\,\phi\right\}-\left(1-\frac{\sigma^2}{gk}\right)ky\phi\right]e^{ky},\end{aligned}\right\} \qquad (45)$$

the factor $e^{i(\sigma t+kx)}$ being omitted here, as elsewhere, for brevity.

The condition that $v = 0$ at the lower boundary, where $y = h$, say, taken in conjunction with (41), determines the values of α and σ, the wave-length $(2\pi/k)$ being supposed given.*

8. In the case of oscillations about convective equilibrium we have

$$\beta = \beta_1, \qquad n = 1/(\gamma-1). \qquad (46)$$

It follows from (24) that $\partial^2\zeta/dt^2 = 0$; hence either $\zeta = 0$, *i.e.* the motion is irrotational, or the period is infinitely long.

The conditions to which the *steady* rotational motions thus indicated are subject follow most directly from (21). These equations are now equivalent to

$$\left.\begin{aligned}\frac{\partial}{\partial x}\left\{c^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+gv\right\} &= 0,\\ \frac{\partial}{\partial y}\left\{c^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+gv\right\} &= 0,\end{aligned}\right\} \qquad (47)$$

by (26). Hence

$$(\gamma-1)\,y\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+v = \text{const.} \qquad (48)$$

The choice of two functions, u, v, to satisfy this equation, together with the two boundary conditions, can be made in an infinite variety of ways.

The remaining types of disturbance are periodic in character. The formulæ (42) and (45) apply, with

$$2\alpha = \frac{n\sigma^2}{gk}+n+2, \qquad (49)$$

in place of (41). Since (42) makes

$$y\frac{\partial\phi}{\partial y}+(n+1)\phi = (n+1)\left\{1-\frac{\alpha}{1\,.\,n+1}(2ky)+\frac{\alpha\,.\,\alpha+1}{1\,.\,2\,.\,n+1\,.\,n+2}(2ky)^2-\ldots\right\}, \qquad (50)$$

* The case of apparent failure, where $\sigma^2 = gk$, does not arise. This would require, by (36),

$$y\frac{\partial\chi}{\partial y}+(n+1-ky)\,\chi = 0, \quad \text{or} \quad \chi = Cy^{-n-1}e^{ky},$$

which violates the condition at the upper boundary.

the condition that $v = 0$ for $y = h$ may be written

$$(n+1)\frac{\sigma^2}{gk}\,{}_1F_1(\alpha\,;\ n+1\,;\ -2kh)-\left(1-\frac{\sigma^2}{gk}\right)kh\,{}_1F_1(\alpha\,;\,n+2\,;\,-2kh) = 0. \tag{51}$$

A complete discussion of the equations (49) and (51) is out of the question, but the limiting form to which the results tend as the wave-length increases is easily ascertained. In the first place, it appears that when kh is small we have

$$\frac{\sigma^2}{gk} = \frac{kh}{n+1}, \tag{52}$$

approximately, since this ensures, by (49), that αkh is also small. If H denote the virtual height of the atmosphere, as defined above (§ 1), we have

$$\mathrm{H} = h^{-n}\int_0^h y^n dy = h/(n+1). \tag{53}$$

The limiting value of the wave-velocity V is accordingly given by

$$\mathrm{V}^2 = \sigma^2/k^2 = g\mathrm{H}. \tag{54}$$

The bearing of this result has been discussed in the introduction.

The formulæ (45), (50), and (39) now lead to

$$\left.\begin{aligned} u &\propto i\,(n+1-k^2hy+\tfrac{1}{2}k^2y^2), \\ v &\propto k\ (y-h), \end{aligned}\right\} \tag{55}$$

the factor $e^{i(\sigma t+kx)}$ being understood. These values fulfil, as they ought, the irrotational condition

$$\frac{\partial u}{\partial y} = \frac{\partial v}{dx} = ikv. \tag{56}$$

Since the ratio of the amplitude of v to that of u is of the order kh, the motion is mainly horizontal, and the present type of waves may accordingly be characterised as "longitudinal."

The remaining solutions of (49) and (51), when kh is small, involve finite as distinguished from infinitely small values of αkh. As will be seen presently (§ 11), they approximate to the character of waves propagated vertically in the atmosphere.

9. In the general case, where n is not restricted to the precise value $1/(\gamma-1)$, the relation between α and σ is as in (41). When kh is small we have still a longitudinal wave for which σ^2/gk is of the order kh, subject to a certain condition. The equation (51) leads again to the result expressed by (52) or (54), and substituting in (41) we find that the implied assumption that αkh is also small will be justified provided $\beta_1/\beta-1$ be small, *i.e.* provided the temperature-gradient falls only a little short of the convective value β_1.

The limiting form of (42), when no assumption is made as to the order of magnitude of αkh, is

$$\phi = 1 - \frac{2\alpha ky}{1 \cdot n+2} + \frac{(2\alpha ky)^2}{1 \cdot 2 \cdot n+2 \cdot n+3} - \ldots, \tag{57}$$

or in Dr. Barnes' notation,

$$\phi = {}_0\mathrm{F}_1(n+2\,;\, -2\alpha ky), \tag{58}$$

whilst (50) becomes

$$y\frac{\partial \phi}{\partial y} + (n+1)\phi = (n+1)\,{}_0\mathrm{F}_1(n+1\,;\, -2\alpha ky). \tag{59}$$

It appears from (41), without making as yet any special assumption as to the smallness of $\beta_1/\beta - 1$, that when αkh is finite, whilst kh is small, the ratio σ^2/gk will be very small or very great.

In the former case we have

$$\frac{\sigma^2}{gk} = \left(\frac{\beta_1}{\beta} - 1\right) \cdot \frac{1}{2\alpha} \tag{60}$$

ultimately, and the condition (51) becomes

$$(n+1)\left(\frac{\beta_1}{\beta} - 1\right){}_0\mathrm{F}_1(n+1\,;\, -2\alpha kh) - 2\alpha kh\,{}_0\mathrm{F}_1(n+2\,;\, -2\alpha kh) = 0. \tag{61}$$

Since

$${}_0\mathrm{F}_1(n+1\,;\, -z) = \Pi(n)\, z^{-\frac{1}{2}n} \mathrm{J}_n(2z^{\frac{1}{2}}), \tag{62}$$

in the notation of Bessel's functions, this may be written

$$\tfrac{1}{2}\,\omega \mathrm{J}_{n+1}(\omega) = \left(\frac{\beta_1}{\beta} - 1\right)\mathrm{J}_n(\omega), \tag{63}$$

provided

$$\omega^2 = 8\alpha kh. \tag{64}$$

If ω be a root of (63), the corresponding frequency of oscillation is given by

$$\frac{\sigma^2}{gk} = \left(\frac{\beta_1}{\beta} - 1\right) \cdot \frac{4kh}{\omega^2}, \tag{65}$$

and the wave-velocity by

$$\mathrm{V}^2 = \left(\frac{\beta_1}{\beta} - 1\right) \cdot \frac{4(n+1)g\mathrm{H}}{\omega^2}, \tag{66}$$

H being defined as before by (53). This result again is accurate as a limiting form for increasing wave-length.

10. The equation (63) might be discussed, when n or $2n$ is integral, with the help of the tables of Bessel's functions, but it may be sufficient to consider the case where the ratio $(\beta_1 - \beta)/\beta$ is small. It may be noticed that the formula embraces *all* the modes of the present class, the longitudinal waves already discussed corresponding to the case of ω infinitesimal. The roots of (63) which relate to the remaining modes are now given by

$$\mathrm{J}_{n+1}(\omega) = 0, \tag{67}$$

approximately; and in particular the first of these slightly exceeds the first finite root of (67).

In convective equilibrium we have $n = 2{\cdot}5$, if $\gamma = 1{\cdot}40$. The first finite root of $J_{7/2}(\omega) = 0$ is $\omega_1 = 7$, very nearly. Hence for oscillations about a state of very nearly neutral equilibrium we have

$$V = \left(\frac{\beta_1}{\beta_1} - 1\right)^{\frac{1}{2}} \times 0{\cdot}53\sqrt{(gH)}.$$

In the case of $n = 3$, which makes $(\beta_1 - \beta)/\beta = \frac{1}{7}$, a first approximation, given by (67), is $\omega_1 = 7{\cdot}586$, and a second is easily found to be $\omega_1 = 7{\cdot}624$. This leads to

$$V = 0{\cdot}198\sqrt{(gH)},$$

which is about one-fifth the velocity of the longitudinal type of waves.

As to the character of these slow rotational* modes, we find from (24)

$$\zeta = -i\left(\frac{\beta_1}{\beta} - 1\right) \cdot \frac{\gamma}{n+1} \cdot \frac{gk}{\sigma^2} \cdot \chi, \tag{68}$$

or, by (65),

$$\zeta = -\frac{i\gamma}{n+1} \cdot \frac{\omega^2}{4kh} \cdot \chi. \tag{69}$$

Having regard to the kinematical meaning of the functions ζ, χ, as defined by (22), we see that the rotational quality in the relative motion of a fluid element predominates over the dilatational. We learn also from (45) that when $y = 0$ the amplitude of v is to that of u in the ratio σ^2/gk, which is small. Since v vanishes at the lower boundary, we infer that the vertical component of the velocity is in general relatively small. The distribution of horizontal velocity depends ultimately on the function

$$y\frac{\partial\phi}{\partial y} + (n+1)\phi,$$

which varies as

$$(\omega y^{\frac{1}{2}}/h^{\frac{1}{2}})^{-n} J_n(\omega y^{\frac{1}{2}}/h^{\frac{1}{2}}),$$

if ω be the relevant root of (63), or less accurately of (67). In the case of the first root, after the small one, this expression changes sign once, and once only, as y increases from 0 to h. For $n = 3$, the change of sign occurs for $\omega y^{\frac{1}{2}}/h^{\frac{1}{2}} = 6{\cdot}379$, or $y/h = 0{\cdot}70$.

The general character of the types of disturbance at present under consideration is most easily apprehended in the case of a "standing" oscillation. If on the preceding expressions we superpose others which

* The rotational character is, of course, present also in the longitudinal waves, unless $\beta = \beta_1$ exactly, though to relatively slight extent.

differ only in the sign of σ, and reject the imaginary parts, we find, on discarding all but the most important terms, that

$$\left.\begin{aligned} u &\propto \left\{ y\frac{\partial\phi}{\partial y} + (n+1)\phi \right\} \sin kx \cos \sigma t, \\ v &\propto -ky\,\phi \cos kx \cos \sigma t. \end{aligned}\right\} \qquad (70)$$

The differential equation of the lines of (oscillatory) motion, viz.

$$v\,dx - u\,dy = 0, \qquad (71)$$

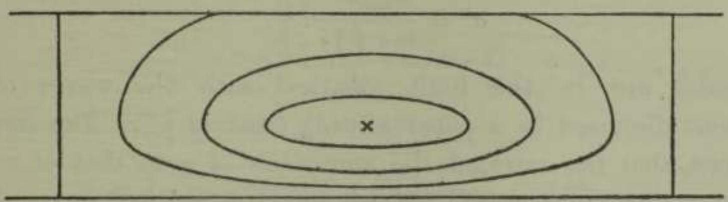
is accordingly satisfied by

$$y^{n+1}\phi \sin kx = \text{const.}, \qquad (72)$$

or

$$y^{\frac{1}{2}(n+1)} J_{n+1}(\omega y^{\frac{1}{2}}/h^{\frac{1}{2}}) \sin kx = C. \qquad (73)$$

If we put $C = 0$ we get the lines $y = 0$, $y = h$, $kx = s\pi$, but the former of these is only an approximation. The annexed figure indicates, without any attempt at minute accuracy, the general arrangement of the lines in the case of the lowest finite root of (67).



In the modes corresponding to the higher roots there are horizontal nodal planes ($v = 0$), in addition to the lower boundary.

Returning for a moment to the more important "longitudinal" type of motion first considered, we note that the formulæ (52), (54), cease to be accurate, even as limiting forms, when the ratio β_1/β differs appreciably from unity. The formulæ (65) and (66) will, however, still apply, ω being the lowest root of (63). As a numerical example, take the case where the temperature-gradient has only one-half the convective value, so that

$$(\beta_1 - \beta)/\beta = 1, \qquad n = 6.$$

I find that the lowest root of

$$\tfrac{1}{2}\omega J_7(\omega) = J_6(\omega)$$

is $\omega = 4{\cdot}96$, approximately, whence

$$V = 1{\cdot}07\sqrt{(gH)}.$$

The result must, of course, in any case be less than $\sqrt{(\gamma g H)}$, or $1{\cdot}18\sqrt{(gH)}$. The change of wave-velocity is accompanied by a change in the character of the oscillation, the variation of horizontal velocity with altitude now becoming sensible.

The preceding formulæ might also be used to estimate the rapidity of falling away from the state of unstable equilibrium which prevails when $\beta > \beta_1$, the value of σ^2 given by (65) being then negative.

11. The modes for which σ^2/gk is large are easily accounted for. We have from (41)

$$\frac{\sigma^2}{gk} = \frac{2\gamma\alpha}{n+1}, \tag{74}$$

and from (45)

$$y\frac{\partial\phi}{\partial y} + (n+1)\phi = 0, \tag{75}$$

these being approximations which gain indefinitely in accuracy with increase of wave-length. On the present supposition that αkh is finite, notwithstanding the smallness of kh, (75) reduces to

$$J_n(\omega) = 0, \tag{76}$$

provided $\omega^2 = 8\alpha kh$, as in (64). If ω be a root of this equation, the corresponding frequency is given by

$$\sigma^2 = \frac{\gamma\omega^2}{4(n+1)} \cdot \frac{g}{h}. \tag{77}$$

These modes are in the limit identical with the waves of vertical displacement discussed in a paper already cited in § 2. The formulæ (45) show, in fact, that the ratio of the amplitude of u to that of v is for the most part of the order gk/σ^2. If we put $h = \frac{1}{4}\gamma R\beta\tau_1^2 = \frac{1}{4}\gamma g\tau_1^2/(n+1)$, the equation (76) takes the form $J_n(\sigma\tau_1) = 0$, which is identical with equation (88) of the paper referred to.

12. It may be worth while, for the sake of the contrast, to give the theory of the oscillations of a heterogeneous but *incompressible* fluid, whose equilibrium density has a similar distribution.

We have now

$$\rho_0\frac{\partial u}{\partial t} = \frac{\partial p}{\partial x}, \qquad \rho_0\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + g\rho, \tag{78}$$

$$\frac{\partial\rho}{\partial t} + v\frac{d\rho_0}{dy} = 0, \tag{79}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{80}$$

If we put

$$p = p_0 + \varpi, \qquad \rho = \rho_0 + \delta, \tag{81}$$

as before, we have

$$\rho_0\frac{\partial u}{\partial t} = -\frac{\partial\varpi}{\partial x}, \qquad \rho_0\frac{\partial v}{\partial t} = -\frac{\partial\varpi}{\partial y} + g\delta. \tag{82}$$

$$\frac{\partial\delta}{\partial t} = -v\frac{d\rho_0}{dy}. \tag{83}$$

From (80) we have

$$u = -\frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\psi}{\partial x}, \tag{84}$$

ψ being the stream-function. Substituting in (82), (83), and eliminating ϖ and δ, we find*

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi + \frac{1}{\rho_0}\frac{d\rho_0}{dy}\left(\frac{\partial^3\psi}{\partial y\,\partial t^2}+g\frac{\partial^2\psi}{\partial x^2}\right) = 0. \tag{85}$$

If x and t occur only in the form $e^{i(\sigma t+kx)}$, we have

$$\frac{\partial^2\psi}{\partial y^2} - k^2\psi + \frac{1}{\rho_0}\frac{d\rho_0}{dy}\left(\frac{\partial\psi}{\partial y}+\frac{gk^2}{\sigma^2}\psi\right) = 0. \tag{86}$$

Also
$$v = ik\psi, \tag{87}$$

and
$$\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\partial\varpi}{\partial t} + v\frac{dp_0}{dy} = i\sigma\varpi + g\rho_0 v = \frac{i\sigma^2\rho_0}{k}\left(\frac{\partial\psi}{\partial y}+\frac{gk^2}{\sigma^2}\psi\right). \tag{88}$$

If we now assume that
$$\rho_0 \propto y^n, \tag{89}$$

we have
$$y\frac{\partial^2\psi}{\partial y^2} + n\frac{\partial\psi}{\partial y} + \left(\frac{ngk}{\sigma^2} - ky\right)k\psi = 0, \tag{90}$$

or writing
$$\psi = e^{ky}\phi, \tag{91}$$

$$y\frac{\partial^2\phi}{\partial y^2} + (n+2ky)\frac{\partial\phi}{\partial y} + n\left(1+\frac{gk}{\sigma^2}\right)k\phi = 0. \tag{92}$$

The solution which is finite for $y = 0$ is

$$\phi = 1 - \frac{\alpha}{1\,.\,n}(2ky) + \frac{\alpha\,.\,\alpha+1}{1\,.\,2\,.\,n\,.\,n+1}(2ky)^2 - \ldots, \tag{93}$$

or
$$\phi = {}_1\mathrm{F}_1(\alpha\,;\,n\,;\,-2ky), \tag{94}$$

if
$$2\alpha = n\left(1+\frac{gk}{\sigma^2}\right). \tag{95}$$

The second solution becomes infinite as y^{-n+1} for $y = 0$, and is therefore excluded, in virtue of (88), by the condition that $\mathrm{D}p/\mathrm{D}t$ must vanish at the upper boundary. Since, by (87), $\psi = 0$ for $y = h$, we have

$${}_1\mathrm{F}_1(\alpha\,;\,n\,;\,-2kh) = 0. \tag{96}$$

This determines α, and the value of σ^2 follows from (95).

It is obvious that, when kh is small, (96) is not satisfied by finite values of α. If α be large, but so that αkh remains finite, the equation (96) tends to the form

$${}_0\mathrm{F}_1(n\,;\,-2\alpha kh) = 0, \tag{97}$$

or
$$\mathrm{J}_{n-1}(\omega) = 0, \tag{98}$$

provided
$$\omega^2 = 8\alpha kh. \tag{99}$$

If ω be a root of (98), we have

$$\frac{\sigma^2}{gk} = \frac{n}{2\alpha} = \frac{4n}{\omega^2}\,.\,kh, \tag{100}$$

* *Cf.* Love, 'Lond. Math. Soc. Proc.' (1), 1891, vol. 22, p. 307.

and therefore, for the wave-velocity,

$$V^2 = \frac{\sigma^2}{k^2} = \frac{4n(n+1)}{\omega^2} \cdot gH. \tag{101}$$

Thus, if, for the sake of comparison with § 7, we put $n = 2{\cdot}5$, we have $\omega_1 = 4{\cdot}491$, whence

$$V = 1{\cdot}32 \sqrt{(gH)}. \tag{102}$$

That the frequency should be increased by the incompressibility was to be expected; that the effect is so considerable is due to the great modification which is caused in the character of the fundamental modes.

The modes corresponding to the higher roots of (98) have horizontal nodal planes, and the frequencies form, by (101), a *descending* series,* as in the case of (65).

Waves at a Surface of Discontinuity.

13. When we proceed to examine the case of waves propagated along a horizontal plane where the equilibrium temperature is discontinuous, it may be sufficient to suppose the temperature uniform throughout each of the regions, above and below this plane, to which the influence of the waves extends. The plane in question is taken as the plane $y = 0$, and the dependent variables relating to the upper region will be distinguished by accents.

The formulæ of §6 will therefore apply to the lower region, with the simplification that c is a constant; so that (31) becomes

$$c^2 \frac{\partial^2 \chi}{\partial y^2} + \gamma g \frac{\partial \chi}{\partial y} + \left\{ \sigma^2 - k^2 c^2 + \frac{(\gamma - 1) g^2 k^2}{\sigma^2} \right\} \chi = 0. \tag{103}$$

This is satisfied by

$$\chi = C e^{\lambda y}, \tag{104}$$

provided

$$\lambda^2 + \frac{\gamma g}{c^2} \lambda + \left\{ \frac{\sigma^2}{c^2} - k^2 + \frac{(\gamma - 1) g^2 k^2}{\sigma^2 c^2} \right\} = 0. \tag{105}$$

We are seeking for a type of motion analogous to that of waves on the interface of two *liquids* of different densities, in which case the values of λ are $\pm k$. We assume, provisionally, that in our case also the roots of (105) are real and of opposite signs; moreover, since the disturbance is to vanish for $y = \infty$, the negative sign is to be taken.

For the upper region we shall have

$$\chi = C' e^{\lambda' y}, \tag{106}$$

with a similar determination of λ'; but the *positive* root is now the appropriate one.

* *Cf.* Rayleigh, 'Lond. Math. Soc. Proc.' (1), 1883, vol. 14, p. 170; 'Scientific Papers,' vol. 2, p. 200.

If η denote the ordinate of the surface of separation, as affected by the waves, we have

$$\partial\eta/\partial t = v \tag{107}$$

for $y = 0$; and the pressure in either fluid at the point (x, η) is to be found by putting $y = 0$ in the corresponding value of the expression

$$p_0 + \varpi + \frac{\partial p_0}{\partial y}\eta. \tag{108}$$

Differentiating with respect to t, we see that $\mathrm{D}p/\mathrm{D}t$ must be continuous at the interface.* This involves, by (27), the continuity of χ, so that the constants C, C′ in (104) and (106) must be equal. Again, it follows from (107) that v must be continuous, whence, by (30),

$$\sigma^2c^2\lambda - gk^2c^2 + \gamma g\sigma^2 = \sigma^2c'^2\lambda' - gk^2c'^2 + \gamma g\sigma^2. \tag{109}$$

This, together with the two equations of the type (105), determines the values of λ, λ', and σ.

To obtain a solution, we denote the two equal members in (109) by μ; thus

$$\lambda = \frac{\mu}{\sigma^2c^2} + \frac{gk^2}{\sigma^2} - \frac{\gamma g}{c^2}, \qquad \lambda' = \frac{\mu}{\sigma^2c'^2} + \frac{gk^2}{\sigma^2} - \frac{\gamma g}{c'^2}. \tag{110}$$

Substituting in (105), we have

$$\mu^2 + g(2k^2c^2 - \gamma\sigma^2) + (\sigma^4 - g^2k^2)(\sigma^2 - k^2c^2)c^2 = 0, \tag{111}$$

with a similar equation in which c is replaced by c'. Writing these two equations in the form

$$\mu^2 + \mathrm{P}\mu + \mathrm{Q} = 0, \qquad \mu^2 + \mathrm{P}'\mu + \mathrm{Q}' = 0, \tag{112}$$

and eliminating μ, we have

$$(\mathrm{P}-\mathrm{P}')(\mathrm{PQ}'-\mathrm{P}'\mathrm{Q}) + (\mathrm{Q}-\mathrm{Q}')^2 = 0. \tag{113}$$

Now

$$\left.\begin{aligned} \mathrm{P}-\mathrm{P}' &= 2gk^2(c^2-c'^2), \\ \mathrm{Q}-\mathrm{Q}' &= (c^2-c'^2)(\sigma^4-g^2k^2)\{\sigma^2-k^2(c^2+c'^2)\}, \\ \mathrm{PQ}'-\mathrm{P}'\mathrm{Q} &= g(c^2-c'^2)(\sigma^4-g^2k^2)\{\gamma\sigma^4-\gamma\sigma^2k^2(c^2+c'^2)+2k^4c^2c'^2\}. \end{aligned}\right\} \tag{114}$$

Hence

$$(\sigma^4-g^2k^2)\{\sigma^2-k^2(c^2+c'^2)\}^2 + 2g^2k^2\{\gamma\sigma^4-\gamma\sigma^2k^2(c^2+c'^2)+2k^4c^2c'^2\} = 0. \tag{115}$$

This is of the fourth degree in σ^2, but one root only is relevant to the present question. The common root of (112) is

$$\mu = \frac{\mathrm{PQ}'-\mathrm{P}'\mathrm{Q}}{\mathrm{Q}-\mathrm{Q}'} = \gamma g\sigma^2 + \frac{2gk^4c^2c'^2}{\sigma^2-k^2(c^2+c'^2)}, \tag{116}$$

* This might almost have been assumed at once; but it is to be observed that it would not give the correct condition to be satisfied at the common boundary of two *currents*.

whence

$$\left.\begin{aligned}\frac{\lambda}{k} &= \frac{gk}{\sigma^2}\cdot\frac{\sigma^2-k^2(c^2-c'^2)}{\sigma^2-k^2(c^2+c'^2)},\\ \frac{\lambda'}{k} &= \frac{gk}{\sigma^2}\cdot\frac{\sigma^2-k^2(c'^2-c^2)}{\sigma^2-k^2(c^2+c'^2)}.\end{aligned}\right\} \tag{117}$$

If we now write

$$a^2 = \tfrac{1}{2}(c^2+c'^2), \qquad b^2 = \tfrac{1}{2}(c'^2-c^2), \tag{118}$$

$$x = \sigma^2/k^2a^2, \qquad \omega = g/ka^2, \tag{119}$$

the equation (115) becomes

$$x^2(x-2)^2+\omega^2\{(2\gamma-1)x^2-4(\gamma-1)x-4b^4/a^4\} = 0; \tag{120}$$

whilst

$$\frac{\lambda}{k} = \frac{x+2b^2/a^2}{x(x-2)}\omega, \qquad \frac{\lambda'}{k} = \frac{x-2b^2/a^2}{x(x-2)}\omega. \tag{121}$$

It is to be noticed that

$$\frac{b^2}{a^2} = \frac{c'^2-c^2}{c'^2+c^2} = \frac{\rho-\rho'}{\rho+\rho'}, \tag{122}$$

where ρ, ρ', are the equilibrium densities at the plane $y = 0$, on the two sides.

For sufficiently small wave-lengths, ω and x are very small, and the root of (120) with which we are concerned is $x = \omega b^2/a^2$, approximately, whence

$$\sigma^2 = \frac{\rho-\rho'}{\rho+\rho'}gk, \qquad \lambda = -k, \qquad \lambda' = k, \tag{123}$$

as in the case of superposed incompressible fluids.*

To examine the matter further, the simplest procedure is to tabulate the function

$$\omega^2 = \frac{x^2(x-2)^2}{4b^4/a^4+4(\gamma-1)x-(2\gamma-1)x^2} \tag{124}$$

for a series of suitable values of x. The only case of real interest is where the discontinuity of temperature is very slight, so that b^2/a^2 is a small fraction. The following table gives a few results calculated on the supposition that $b^2/a^2 = \frac{1}{100}$, with $\gamma = 1{\cdot}40$. The abrupt step in temperature then amounts to $\frac{1}{50}$ of the mean of the temperatures (absolute) above and below.

x.	ω^2.	ω.	$10\sqrt{(x/\omega)}$.	$-\lambda/k$.	λ'/k.	Wave-length (metres).	Period (seconds).
10^{-6}	$0{\cdot}9960 \times 10^{-8}$	$0{\cdot}9980 \times 10^{-4}$	1·0005	0·998	0·998	7·0	21·2
10^{-5}	$0{\cdot}9615 \times 10^{-6}$	$0{\cdot}9806 \times 10^{-3}$	1·010	0·981	0·980	69·2	65·9
10^{-4}	$0{\cdot}7143 \times 10^{-4}$	$0{\cdot}8452 \times 10^{-2}$	1·088	0·850	0·841	597·0	180·0
10^{-3}	$2{\cdot}004 \times 10^{-3}$	$0{\cdot}4476 \times 10^{-1}$	1·495	0·470	0·425	3160·0	301·0

* Stokes, 'Camb. Trans.,' 1847, vol. 8, p. 441; 'Math. and Phys. Papers,' vol. 1, p. 197.

The fourth column gives the ratio of the frequency to that of waves of the same length on the surface of separation of two homogeneous liquids with the same discontinuity of density, as given by (123), viz., the ratio is

$$\frac{\sigma}{\sqrt{(gk)\,.\,b/a}} = \frac{10\sqrt{x}}{\sqrt{\omega}},$$

on the present suppositions. The seventh column is calculated from $2\pi/k = 2\pi a^2\omega/g$, taking $a = 332$ metres per second; and the last from $2\pi/\sigma = 2\pi a\omega/g\sqrt{x}$. It is seen that, with increasing wave-length, the wave-velocity tends more and more to exceed the value estimated on the assumption of the homogeneity and incompressibility of the two fluids. At the same time, the disturbance tends to become, relatively as well as absolutely, less and less concentrated in the neighbourhood of the plane of discontinuity.

14. In this question, again, it is of some interest to compare the case of waves on the common boundary of two liquids, each of which, though incompressible, has a similar gradation of density. We therefore write, in (86),

$$\frac{1}{\rho_0}\frac{d\rho_0}{dy} = q, \text{ a constant.} \tag{125}$$

If we assume that

$$\psi = Ce^{\lambda y}, \tag{126}$$

we derive

$$\lambda^2 + q\lambda + \left(\frac{gq}{\sigma^2} - 1\right)k^2 = 0. \tag{127}$$

These formulæ may be supposed to relate to the lower region; for the upper region we write q', λ', C' for q, λ, C, respectively.

The continuity of v involves, by (87), that of ψ, so that $C' = C$. Also, in virtue of the continuity of Dp/Dt, we have from (88)

$$\rho\,(\sigma^2\lambda + gk^2) = \rho'\,(\sigma^2\lambda' + gk^2), \tag{128}$$

where ρ, ρ', are the densities just below and just above the plane $y = 0$.

If the two fluids had been portions of the same gas at different temperatures we should have had

$$q = \gamma g/c^2, \qquad q' = \gamma g/c'^2, \tag{129}$$

and therefore

$$\rho/\rho' = q/q'. \tag{130}$$

Now from (127) we have

$$q\,(\sigma^2\lambda + gk^2) = \sigma^2\,(k^2 - \lambda^2). \tag{131}$$

Hence, if we adopt the relation (130) for the sake of the comparison, we must have $\lambda^2 = \lambda'^2$, or, taking account of the signs, $\lambda = -\lambda'$. This leads to

$$\frac{\lambda'}{k} = -\frac{\lambda}{k} = \frac{q-q'}{q+q'}\cdot\frac{gk}{\sigma^2}, \tag{132}$$

and

$$\left(\frac{\sigma^2}{gk}\right)^2 - \frac{2\,qq'}{k\,(q+q')}\,\frac{\sigma^2}{gk} - \left(\frac{q-q'}{q+q'}\right)^2 = 0. \tag{133}$$

The positive root of this quadratic in σ^2/gk is to be taken, since it is the only one which gives the proper signs to λ, λ', it being assumed that $\rho > \rho'$ and therefore $q > q'$. For infinitesimal values of q, q', we reproduce the relations (123).

In order to make the variations of density follow exactly the same law as in the atmospheric problem of § 13 we must give to q, q', the values (129). In the notation of (118), (119), we have then

$$x^2 - \gamma\omega^2 x - \omega^2 b^4/a^4 = 0. \tag{134}$$

The following table, like the former one, refers to the case of $b^2/a^2 = 1/100$. In order that the comparison may be for the same series of wave-lengths, those values of ω are chosen which were obtained in the previous numerical work. The significance of the column headed $10\sqrt{(x/\omega)}$ is the same as on p. 570. The comparison shows the usual effect of a constraint in increasing the frequency.

ω.	$10\sqrt{(x/\omega)}$.	λ'/k.	Wave-length (metres).	Period (seconds).
0.9980×10^{-4}	1·003	0·993	7·0	21·2
0.9806×10^{-3}	1·035	0·934	69·2	64·3
0.8452×10^{-2}	1·324	0·570	597·0	148·0
0.4476×10^{-1}	2·534	0·156	3160·0	178·0