ON AUTOMORPHISM GROUPS OF IDEMPOTENT EVOLUTION ALGEBRAS

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ABSTRACT. We study the automorphism group of an idempotent evolution algebra, show that any finite group can be the automorphism group of an evolution algebra, and describe certain evolution algebras with given automorphism groups. In particular, we classify *n*-dimensional idempotent evolution algebras whose automorphism group is isomorphic to the symmetric group S_n , and classify idempotent evolution algebras with maximal diagonal automorphism subgroups.

1. INTRODUCTION

Evolution algebras are non-associative and commutative algebras motivated by the evolution laws of genetics [10]. These algebras are related to different fields and present many interesting properties [1, 2, 3, 4, 5, 7, 10]. Here we consider finite dimensional evolution algebras over a field \mathbb{F} . According to [10], an *n*-dimensional evolution algebra \mathcal{E} over \mathbb{F} can be defined by using a natural basis e_1, \ldots, e_n and a structure matrix $A = (a_{ij}), a_{ij} \in \mathbb{F}, 1 \leq i, j \leq n$, such that

(1.1) $e_i e_j = 0$, if $i \neq j$ and $(e_1^2, ..., e_n^2) = (e_1, ..., e_n)A$.

Note that in general, natural bases are not unique and different natural bases lead to different structure matrices [7, 10]

We call an evolution algebra \mathcal{E} idempotent if $\mathcal{E}^2 = \mathcal{E}$. From (1.1), it is clear that

(1.2) $\mathcal{E}^2 = \mathcal{E} \Leftrightarrow e_1^2, ..., e_n^2$ form a basis of $\mathcal{E} \Leftrightarrow A$ is nonsingular.

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We denote by $\mathcal{E}(A)$ the evolution algebra with the structure matrix A if we need to specify A, and denote by Γ_A (or $\Gamma_{\mathcal{E}}$) the graph whose adjacency matrix is obtained from A by replacing all nonzero entries of A by 1. The vertices of Γ_A will be just $e_1, ..., e_n$. We call Γ_A the graph associated with \mathcal{E} . Note that Γ_A is a digraph.

The automorphism group of an idempotent finite dimensional evolution algebra has been studied in [7, 8] via the associated graph Γ_A . In particular, it was shown in [7] that the automorphism group of a finite dimensional idempotent evolution algebra is finite.

In section 2, we revisit the main results of [7, 8] on the automorphism groups of finite dimensional idempotent evolution algebras using a different approach to gain more insight on these automorphism groups. The classifications of evolution algebras in dimensions ≤ 4 have been attempted, however, the classification lists are long even for these low dimensions [2, 3] (even with incomplete classifications). In this paper, we consider the problem from a different viewpoint: classify evolution algebras with a given automorphism group. Since the automorphism group of an idempotent evolution algebra is finite (usually not finite otherwise [8]), it is natural to consider idempotent evolution algebras associated with a given finite group. In section 3, we show that any finite group can be the automorphism group of an evolution algebra, and identify certain evolution algebras with given groups. In particular, we give a classification of *n*-dimensional idempotent evolution algebras that have S_n as the automorphism group. In section 4, we show that the maximal diagonal automorphism subgroup that an n-dimensional idempotent evolution algebra can have is the cyclic group C_{2^n-1} of order $2^n - 1$. Under the assumption that the field \mathbb{F} is algebraically closed of characteristic 0, we show that, up to isomorphism, there exists only one *n*-dimensional idempotent evolution algebra whose diagonal automorphism subgroup is C_{2^n-1} , and its automorphism group is $C_{2^n-1} \rtimes C_n$.

2. Automorphisms of an idempotent evolution algebra

Let \mathcal{E} (resp. \mathcal{E}') be an idempotent evolution algebra with a natural basis $e_1, ..., e_n$ (resp. $e'_1, ..., e'_n$) and the structure matrix $A = (a_{ij})$ (resp. $B = (b_{ij})$). If $\phi : \mathcal{E} \to \mathcal{E}'$ is an isomorphism, then $(\phi(e_1), ..., \phi(e_n))$ is a natural basis of \mathcal{E}' with the structure matrix A. Let $P = (p_{ij})$ be the matrix of bases change in \mathcal{E}' defined by $(\phi(e_1), ..., \phi(e_n)) = (e'_1, ..., e'_n)P$. Then we have the following from [10]:

(2.1)
$$BP^{(2)} = PA$$
 and $B(P * P) = 0$,

where $P^{(2)} = (p_{ij}^2)$, and $P * P = (c_{ij}^k)$ is an $n \times \frac{n(n-1)}{2}$ matrix whose rows are indexed by k and columns are indexed by the pairs (i, j) such that $1 \leq i < j \leq n$. The entries are defined by $c_{ij}^k = p_{ki}p_{kj}, i < j$. Since we assumed that \mathcal{E}' is idempotent, B is nonsingular, so P * P = 0, which implies that for each row k $(1 \leq k \leq n)$ of P, there exists exactly one nonzero element, say $p_{kk'}$. By the fact that $\det(P) \neq 0$, we see that there exists a permutation $\sigma \in S_n$ such that $\sigma(k') = k, 1 \leq k \leq n$. Thus we have the following:

Theorem 2.1. Two idempotent evolution algebras \mathcal{E} and \mathcal{E}' are isomorphic if and only if there exists a permutation $\sigma \in S_n$ and an $n \times n$ matrix $P = (p_{ij})$, such that $p_{ij} \neq 0 \Leftrightarrow i = \sigma(j)$ and $BP^{(2)} = PA$.

Now consider $\mathcal{G} = \operatorname{Aut}(\mathcal{E})$ for an idempotent evolution algebra \mathcal{E} . For $g \in \mathcal{G}$, let $G = (g_{ij})$ be the matrix of g with respect to a natural basis e_1, \ldots, e_n , that is, $g(e_i) = \sum_{i=1}^n g_{ki}e_k, 1 \leq i \leq n$. Applying Theorem 2.1 to the setting A = B, G = P, we have the following ([7], Corollary 4.7):

Corollary 2.1. For any $g \in \mathcal{G}$, there exists an element σ of the symmetric group S_n such that $g(e_i) = d_i e_{\sigma(i)}$ for some $0 \neq d_i \in \mathbb{F}$, $1 \leq i \leq n$.

For $g \in \mathcal{G}$, let $D_g = \text{diag}(d_1, ..., d_n)$ be the diagonal matrix determined by g, and let P_{σ} be the permutation matrix corresponds to σ , where the d_i 's and the σ are determined by g as in Corollary 2.1. Then in matrix form, we have

(2.2)
$$g : (e_1, ..., e_n) \longrightarrow (e_1, ..., e_n) P_{\sigma} D_g.$$

Define $\phi : \mathcal{G} \longrightarrow S_n$ by $\phi(g) = \sigma$, where σ corresponds to the permutation matrix P_{σ} determined by g as in (2.2). Note that

(2.3)
$$P_{\sigma}^{-1} \operatorname{diag}(d_1, ..., d_n) P_{\sigma} = \operatorname{diag}(d_{\sigma(1)}, ..., d_{\sigma(n)})$$

This can be seen by reducing it to the case of a transposition. Note also that

 $P_{\sigma_2}^{-1}P_{\sigma_1}^{-1}\text{diag}(d_1,...,d_n)P_{\sigma_1}P_{\sigma_2} = \text{diag}(d_{\sigma_1\sigma_2(1)},...,d_{\sigma_1\sigma_2(n)}).$ This is because if $P_{\sigma_1}^{-1}\text{diag}(d_1,...,d_n)P_{\sigma_1} = \text{diag}(d'_1,...,d'_n)$, where $d'_i = d_{\sigma_1(i)}$, then

$$P_{\sigma_2}^{-1} \operatorname{diag}(d'_1, ..., d'_n) P_{\sigma_2} = \operatorname{diag}(d'_{\sigma_2(1)}, ..., d'_{\sigma_2(n)}) \\ = \operatorname{diag}(d_{\sigma_1 \sigma_2(1)}, ..., d_{\sigma_1 \sigma_2(n)}).$$

Thus for $g_1, g_2 \in \mathcal{G}$, we have

(2.4)
$$(P_{\sigma_1}D_{g_1})(P_{\sigma_2}D_{g_2}) = P_{\sigma_1}P_{\sigma_2}D' = P_{\sigma_1\sigma_2}D'$$

for some diagonal matrix D', which implies that ϕ is a group homomorphism and ker (ϕ) consists of the diagonal automorphisms of \mathcal{E} .

Furthermore, if $g(e_i) = d_i e_{\sigma(i)}, 1 \leq i \leq n$, then from $g(e_i^2) = d_i^2 e_{\sigma(i)}^2$ and $e_i^2 = \sum_{j=1}^n a_{ji} e_j$ (see (1.1)), we have

(2.5)
$$\sum_{j=1}^{n} a_{ji} d_j e_{\sigma(j)} = d_i^2 \sum_{j=1}^{n} a_{j\sigma(i)} e_j = d_i^2 \sum_{j=1}^{n} a_{\sigma(j)\sigma(i)} e_{\sigma(j)}.$$

Since $(e_1, ..., e_n)$ is a basis, (2.5) holds if and only if

(2.6)
$$d_j a_{ji} = d_i^2 a_{\sigma(j)\sigma(i)}, \ \forall \ i, j$$

Then since $d_i \neq 0$, $1 \leq i \leq n$, $a_{ji} \neq 0$ if and only if $a_{\sigma(j)\sigma(i)} \neq 0$. Thus g induces a graph automorphism of Γ_A via σ .

Let $\operatorname{Aut}(\Gamma_A)$ be the graph automorphism group of Γ_A . Then $\operatorname{Aut}(\Gamma_A)$ is a subgroup of S_n .

For our convenience, we recall the following from [10]. Let \mathcal{E} be an arbitrary *n*-dimensional evolution algebra with a natural basis e_1, \ldots, e_n and the structure matrix A. For a linear endomorphism g of \mathcal{E} , let G be the matrix of g with respect to e_1, \ldots, e_n . Then

(2.7) Aut(
$$\mathcal{E}$$
) = { $G \mid \det(G) \neq 0, AG^{(2)} = GA, \text{and } A(G * G) = 0$ }.

Note that for a diagonal matrix D, A(D*D) = 0 is always true since D*D = 0, and $D^{(2)} = D^2$. If $D = \text{diag}(d_1, ..., d_n) \in \text{ker}(\phi) \subset \text{Aut}(\mathcal{E})$, then $AD^{(2)} = DA$ is equivalent to

(2.8)
$$d_j^2 a_{ij} = d_i a_{ij}, \quad 1 \le i, j \le n$$

If \mathcal{E} is idempotent, then $\det(A) \neq 0$, which implies that there exists a permutation $\tau \in S_n$ such that for each $1 \leq j \leq n$, $a_{\tau(j)j} \neq 0$. This in turn implies that $d_j^2 = d_{\tau(j)}$ by (2.8). Let the order of τ be t. If t = 1, i.e. τ is the identity, then $d_j^2 = d_j$, so all $d_j = 1$ since $d_j \neq 0$. If t > 1, then

$$d_j^{2^t} = (d_j^2)^{2^{t-1}} = d_{\tau(j)}^{2^{t-1}} = \dots = d_{\tau^t(j)} = d_j, \ \forall \ 1 \le j \le n.$$

This implies that all d_j are roots of $x^{2^t-1} - 1$ since $d_j \neq 0$.

Let t_A be the smallest such t. That is, t_A is the minimum order of the permutations $\tau \in S_n$ such that $a_{\tau(1)1} \cdots a_{\tau(n)n} \neq 0$. Then our discussions have proved the following theorem, which summarizes the main results of [7, 8] on the automorphism group of a finite dimensional idempotent evolution algebra (cf. Theorem 4.8 of [7], and Theorem 3.2 of [8]).

Theorem 2.2. Let \mathcal{E} be an n-dimensional idempotent evolution algebra with natural basis e_1, \ldots, e_n and structure matrix A, let \mathcal{G} be the

automorphism group of \mathcal{E} , and let $\mathcal{D} \subset \mathcal{G}$ be the subgroup of diagonal automorphisms.

(1) The subgroup \mathcal{D} is a normal subgroup of \mathcal{G} . The diagonal entries of an element of \mathcal{D} are roots of $x^{2^{t_A}-1}-1$. In particular, \mathcal{D} is a finite group of odd order.

(2) The quotient group \mathcal{G}/\mathcal{D} is isomorphic to a subgroup of $Aut(\Gamma_A)$. In particular, \mathcal{G} is finite.

It is clear that, in general, not every graph automorphism of the associated graph Γ_A induces an automorphism of the evolution algebra \mathcal{E} . However, we have the following:

Theorem 2.3. Let $A = (a_{ij})$ be the adjacency matrix of a graph Γ with n vertices. If an evolution algebra \mathcal{E} has A as the structure matrix, then every element of $Aut(\Gamma)$ induces an element of $\mathcal{G} = Aut(\mathcal{E})$. If in addition A is nonsingular, that is, \mathcal{E} is idempotent, then $\mathcal{G}/\mathcal{D} \cong Aut(\Gamma)$.

Proof. Let \mathcal{E} be defined by A with the natural basis $e_1, ..., e_n$. If $P_{\sigma} = (p_{ij})$ is the matrix of a permutation $\sigma \in S_n$ with respect to the basis $e_1, ..., e_n$, that is

$$\sigma : (e_1, \dots, e_n) \longrightarrow (e_{\sigma(1)}, \dots, e_{\sigma(n)}) = (e_1, \dots, e_n) P_{\sigma},$$

then the entries $p_{ij} = 1$ if $\sigma(j) = i$ (equivalently, $j = \sigma^{-1}(i)$) and 0 otherwise. Thus,

(2.9) P_{σ} defines an element of Aut(Γ) $\Leftrightarrow P_{\sigma}A = AP_{\sigma}$.

This can be seen as follows. The permutation σ induces an automorphism of Γ if and only if $a_{ij} = a_{\sigma(i)\sigma(j)}$, or equivalently, $a_{\sigma^{-1}(i)j} = a_{i\sigma(j)}$, $\forall i, j$. On the other hand,

$$(P_{\sigma}A)_{ij} = \sum_{k} p_{ik}a_{kj} = p_{i\sigma^{-1}(i)}a_{\sigma^{-1}(i)j} = a_{\sigma^{-1}(i)j},$$

$$(AP_{\sigma})_{ij} = \sum_{k} a_{ik}p_{kj} = a_{i\sigma(j)}p_{\sigma(j)j} = a_{i\sigma(j)}.$$

So $a_{ij} = a_{\sigma(i)\sigma(j)}, \forall i, j \Leftrightarrow P_{\sigma}A = AP_{\sigma}.$

Since for any $\sigma \in S_n$, $e_{\sigma(i)}e_{\sigma(j)} = 0$ $(i \neq j)$ is always true, so by (2.7), we have

(2.10) σ induces an automorphism of $\mathcal{E} \Leftrightarrow AP_{\sigma}^{(2)} = P_{\sigma}A$.

But $P_{\sigma}^{(2)} = (p_{ij}^2) = P_{\sigma}$ since $p_{ij} = 0$ or 1. Thus (2.9) and (2.10) are equivalent, and hence every graph automorphism of Γ induces an automorphism of \mathcal{E} . The second part of the theorem follows from part (2) of Theorem 2.2.

3. Evolution Algebras with given automorphism groups

We now turn to the question of whether every finite group can be the automorphism group of an evolution algebra.

Theorem 3.1. Let \mathbb{F} be a field of characteristic 0. Given any finite group G, there exists a finite dimensional idempotent evolution algebra \mathcal{E} over \mathbb{F} such that $Aut(\mathcal{E}) \cong G$.

Proof. A well-known result due to Frucht [9] says that for any finite group G, there exists a graph Γ such that $\operatorname{Aut}(\Gamma) \cong G$. Suppose Γ has n vertices. Let the adjacency matrix of Γ be B. For any nonnegative integer x, set $A(x) = B + xI_n$, where I_n is the identity matrix of size n. Then for $\sigma \in S_n$,

$$(3.1) P_{\sigma}A(x) = A(x)P_{\sigma} \Leftrightarrow P_{\sigma}B = BP_{\sigma}$$

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So by (2.9), the graph automorphism group of the graph corresponding to A(x) is the same as that of Γ for any nonnegative integer x. Now $\det(A(x))$ is a polynomial of degree n in x, so it has at most n roots in \mathbb{F} . Since $\operatorname{char}(\mathbb{F}) = 0$, \mathbb{F} contains a copy of \mathbb{Z} , thus there is a positive integer $m \in \mathbb{F}$ such that A(m) is nonsingular.

Define an *n*-dimensional evolution algebra \mathcal{E} by using A(m) as the structure matrix together with a natural basis e_1, \ldots, e_n . Then \mathcal{E} is idempotent. Since all the diagonal entries of A(m) are nonzero, the identity permutation e satisfies $a_{e(1)1} \cdots a_{e(n)n} \neq 0$, so $t_{A(m)} = 1$ (see the paragraph just before Theorem 2.2), and thus Theorem 2.2 (1) implies that the subgroup \mathcal{D} of diagonal automorphisms of \mathcal{E} is trivial. Now (2.10), (3.1), and Theorem 2.3 together imply $\operatorname{Aut}(\mathcal{E}) \cong \operatorname{Aut}(\Gamma) \cong G$.

We denote by $\mathcal{E}(\Gamma)$ the evolution algebra defined by using the adjacency matrix of a graph Γ as its structure matrix with respect to a natural basis.

Example 3.1. Let K_n be the complete graph with n-vertices without self-loop, then $Aut(K_n) \cong S_n$. Abusing notation, we also denote the adjacency matrix of K_n by K_n . Suppose that $char(\mathbb{F})$ does not divide n-1. For n > 1, K_n has 0 on the diagonal and 1 at all other places, thus $det(K_n) = (-1)^{n-1}(n-1) \neq 0$. So for n > 1, $\mathcal{E}(K_n)$ is idempotent. If $D = diag(d_1, ..., d_n) \in \mathcal{D}$, then for each pair $i \neq j$, since both the (i, j) and the (j, i) entries of K_n are equal to 1, we must have $d_i^2 = d_j$ and $d_i = d_j^2$ by (2.8). Thus all d_i 's are roots of $x^3 - 1$.

For n = 2, $Aut(K_2) \cong S_2 = \mathbb{Z}_2$. If $char(\mathbb{F}) = 3$, then since $x^3 - 1 = (x-1)^3$, \mathcal{D} is trivial, and $Aut(\mathcal{E}(K_2)) \cong \mathbb{Z}_2$. If $char(\mathbb{F}) \neq 3$ and $x^3 - 1$ splits in \mathbb{F} , then \mathcal{D} is generated by $diag(\xi, \xi^2)$, where ξ is a primitive

root of $x^3 - 1$, so $\mathcal{D} \cong \mathbb{Z}_3$. Let σ be the generator of $Aut(K_2)$, then $\sigma^{-1}diag(\xi,\xi^2)\sigma = diag(\xi^2,\xi)$, and so $Aut(\mathcal{E}(K_2)) \cong S_3$ in this case.

For $n \geq 3$, by using distinct triples i, j, k and (2.8), we see that $diag(d_1, ..., d_n) \in \mathcal{D}$ if and only if all $d_i = 1$. So \mathcal{D} is trivial and $Aut(\mathcal{E}(K_n)) \cong S_n$ by Theorem 2.3.

Example 3.1 shows that for an *n*-dimensional idempotent evolution algebra \mathcal{E} , Aut(\mathcal{E}) can be bigger than S_n . We next give a classification of *n*-dimensional idempotent evolution algebras whose automorphism groups are exactly S_n .

Lemma 3.1. Let Γ be a graph with n-vertices (self-loops are allowed) and let $A = (a_{ij})$ be its adjacency matrix. If $Aut(\Gamma) \cong S_n$, then $A = aK_n + bI_n$, where $a, b \in \{0, 1\}$.

Proof. This is clear. Under the assumption, either all vertices have no self-loop, or they all have; and if $a_{ij} \neq 0$ for some pair $i \neq j$, then for for any other pair $r \neq s$, there is a $\sigma \in S_n$ such that $\sigma(i) = r$ and $\sigma(j) = s$.

Lemma 3.2. For $n \geq 4$, if \mathcal{E} is an n-dimensional idempotent evolution algebra such that $Aut(\mathcal{E}) \cong S_n$, then the diagonal automorphism subgroup \mathcal{D} is trivial and $Aut(\Gamma_{\mathcal{E}}) \cong S_n$.

Proof. For n = 4, S_4 has two nontrivial normal subgroups: the alternating subgroup A_4 and $V_4 = \{(1), (12)(34), (13)(24), (14)(23)\}$, they both have even orders. Since the order of \mathcal{D} is an odd number by Theorem 2.2 (1), \mathcal{D} must be trivial. For $n \geq 5$, the only nontrivial normal subgroup of S_n is the alternating subgroup A_n , so \mathcal{D} is also trivial. Now Theorem 2.2 (2) completes the proof. \Box

Let \mathcal{E} be an *n*-dimensional idempotent evolution algebra with a natural basis $e_1, ..., e_n$, and the structure matrix $A = (a_{ij})$. Assume that $\mathcal{G} = \operatorname{Aut}(\mathcal{E}) = S_n$. Consider the cases n < 4.

The case n = 1 is trivial. Let n = 2. Since $\mathcal{D} \subset S_2 = \mathbb{Z}_2$ and $|\mathcal{D}|$ is an odd number, \mathcal{D} must be trivial, and so $\operatorname{Aut}(\Gamma_A) \cong \mathbb{Z}_2$. Thus $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, a \neq 0, a^2 - b^2 \neq 0$; or $a = 0, b \neq 0$ and $x^3 - 1$ has only one root in \mathbb{F} . These evolution algebras are isomorphic to one of the evolution algebras given by the structure matrices $C = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}, c^2 \neq 1$, or isomorphic to the one given by the structure matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In the former case, the isomorphism is given by $e_i \to ae'_i, i = 1, 2$, where (e'_1, e'_2) is the natural basis corresponding to the structure matrix C and c = b/a. In the latter case, the isomorphism is given by $e_i \to be'_i$, i = 1, 2. We will show in Theorem 3.2 that the isomorphism classes of these evolution algebras are represented by $\begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}$, $c^2 \neq 1$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (the structure matrices depend on the choice of the natural bases, so it needs to show that the algebras represented by these matrices are non-isomorphic).

Now consider the case n = 3. Let $\operatorname{diag}(d_1, d_2, d_3) \in \mathcal{G}$. If at least two of the diagonal elements $a_{ii}, 1 \leq i \leq 3$, of A are nonzero, then by (2.8) and the fact that A is nonsingular, we see that all $d_i = 1$, which implies that \mathcal{D} is trivial and $\operatorname{Aut}(\Gamma_{\mathcal{E}}) \cong S_3$. That will imply (use (2.9))

(3.2)
$$A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}, a \neq 0, b \neq a, -a/2 \text{ (char}(\mathbb{F}) \neq 2).$$

The isomorphism classes of these 3-dimensional idempotent evolution algebras are represented by the following family of structure matrices (see Theorem 3.2):

(3.3)
$$A = \begin{pmatrix} 1 & c & c \\ c & 1 & c \\ c & c & 1 \end{pmatrix}, \ c \neq 1, -1/2 \ (char(\mathbb{F}) \neq 2).$$

The isomorphism is given by $e_i \to ae'_i$, $1 \le i \le 3$, where (e'_1, e'_2, e'_3) is a natural basis of the evolution algebra with the structure matrix given by (3.3) and c = b/a.

Assume that there is only one $a_{ii} \neq 0$. Then without lost of generality, we can assume $a_{11} = a_{22} = 0$ and $a_{33} \neq 0$. If \mathcal{D} is trivial, we again have $\operatorname{Aut}(\Gamma_{\mathcal{E}}) \cong S_3$, that will imply all $a_{ii} \neq 0$, a contradiction. So \mathcal{D} is nontrivial, then as a normal subgroup of S_3 , $\mathcal{D} \cong \mathbb{Z}_3$. Let $D = \operatorname{diag}(d_1, d_2, d_3)$ be a generator of \mathcal{D} . Then $d_3 = 1$ since $a_{33} \neq 0$, and both d_1 and d_2 are primitive roots of $x^3 - 1$. Since $\mathcal{G}/\mathcal{D} \cong \mathbb{Z}_2$, there exists an element of the form $P_{(12)}D$ in \mathcal{G} (see (2.2)). By using $A(P_{(12)}D)^{(2)} = P_{(12)}DA$ (see (2.10)), we see that $a_{12} \neq 0 \Leftrightarrow a_{21} \neq 0$; $a_{13} \neq 0 \Leftrightarrow a_{23} \neq 0$; and $a_{31} \neq 0 \Leftrightarrow a_{32} \neq 0$. Since we assumed that $a_{11} = a_{22} = 0$, we must have $a_{12} \neq 0$, otherwise Awould be singular. Also, since $d_3 = 1$, any of $a_{13}, a_{23}, a_{31}, a_{32}$ is nonzero would imply $d_1 = d_2 = 1$. So all these entries must be 0. Now use $A(P_{(12)}D)^{(2)} = P_{(12)}DA$ again, we see under the assumption that \mathcal{D} is nontrivial, we have

(3.4)
$$A = \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & b \end{pmatrix}, \quad a \neq 0, b \neq 0.$$

All these evolution algebras are isomorphic to the one with a = b = 1by the map $e_i \to ae'_i$, $i = 1, 2, e_3 \to be'_3$.

Assume that all $a_{ii} = 0$, let $D = \text{diag}(d_1, d_2, d_3)$ be a diagonal automorphism. Argue as before by using the fact that $a_{ij} \neq 0$ implies $d_i = d_j^2$ and that A is nonsingular, we see that $d_i = 1, 1 \leq i \leq 3$. Thus \mathcal{D} is trivial, and that leads to

(3.5)
$$A = \begin{pmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{pmatrix}, \quad a \neq 0.$$

So up to isomorphism, there is only one such evolution algebra represented by the one with a = 1.

We now ready to classify all *n*-dimensional idempotent evolution algebras \mathcal{E} such that $\operatorname{Aut}(\mathcal{E}) = S_n$.

Theorem 3.2. The following is a complete list of non-isomorphic ndimensional idempotent evolution algebras whose automorphism group is S_n .

- (1) For n = 1, there is only one isomorphic class given by the structure matrix (1).
- (2) For n = 2, the non-isomorphic classes are represented by the structure matrices $\begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}$, $c^2 \neq 1$; and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (only if $x^3 1$ has one root in \mathbb{F}).
- (3) For n = 3, the non-isomorphic classes are represented by the following structure matrices (note that the class given by the matrix in the middle only exists if x³ − 1 has 3 distinct roots in F):

$$\begin{pmatrix} 1 & c & c \\ c & 1 & c \\ c & c & 1 \end{pmatrix}, c \neq 1, -1/2 \ (char(\mathbb{F}) \neq 2), \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} (char(\mathbb{F}) \neq 2).$$

(4) For $n \ge 4$, the non-isomorphic classes are represented by the following structure matrices $(char(\mathbb{F}) \nmid n-1 \text{ for the second and }$

the third cases):

$$\begin{pmatrix} 1 & c & \cdots & c \\ c & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c \\ c & \cdots & c & 1 \end{pmatrix}, c \neq 1, 1/(1-n), \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}$$

Proof. Let \mathcal{E} be an *n*-dimensional idempotent evolution algebra with the structure matrix A with respect to a natural basis $e_1, ..., e_n$ such that $\operatorname{Aut}(\mathcal{E}) = S_n$. To prove the theorem, we first find the structure matrices for $n \geq 4$. By Lemma 3.1 and Lemma 3.2, we have

(3.6)
$$A = \begin{pmatrix} a & b & \cdots & b \\ b & a & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \cdots & b & a \end{pmatrix}$$
such that $\det(A) \neq 0$.

Since det $(A) = (a + (n - 1)b)(a - b)^{n-1}$, we have det $(A) \neq 0 \Leftrightarrow a \neq b, (1-n)b$. Denote the matrix of (3.6) by A(a, b) and the corresponding evolution algebra by $\mathcal{E}(a, b)$. If $a \neq 0$, then $\mathcal{E}(a, b)$ is isomorphic to $\mathcal{E}(1, b/a)$ by $e_i \to ae'_i, 1 \leq i \leq n$. If $a = 0, \mathcal{E}(0, b)$ is isomorphic to $\mathcal{E}(0, 1)$ by $e_i \to be'_i, 1 \leq i \leq n$.

It remains to prove that $\mathcal{E}(1,c), c \neq 1, 1/(1-n)$, and $\mathcal{E}(0,1)$ are pairwise non-isomorphic evolution algebras.

Let $K_n = (\gamma_{ij})$ (the adjacency matrix of the complete graph with n vertices without self-loop). Then $\gamma_{ii} = 0, 1 \leq i \leq n$ and $\gamma_{ij} = 1, i \neq j$. Note that $A(1, c) = I_n + cK_n$ and $A(0, 1) = K_n$.

Assume that $\mathcal{E}(0,1)$ is isomorphic to some $\mathcal{E}(1,c)$. Then by Theorem 2.1 and equation (2.1), there exists a matrix $P = (p_{ij})$ and a permutation $\sigma \in S_n$ such that

$$p_{ij} \neq 0 \Leftrightarrow i = \sigma(j) \text{ and } K_n P^{(2)} = P + cPK_n.$$

Thus for all $1 \leq i, j \leq n$, from

$$(K_n P^{(2)})_{ij} = \sum_{k=1}^n \gamma_{ik} p_{kj}^2 = \gamma_{i\sigma(j)} p_{\sigma(j)j}^2 \quad \text{and}$$
$$(P + cPK_n)_{ij} = p_{ij} + c \sum_{k=1}^n p_{ik} \gamma_{kj} = p_{ij} + c \gamma_{\sigma^{-1}(i)j},$$

we have $\gamma_{i\sigma(j)}p_{\sigma(j)j}^2 = p_{ij} + c\gamma_{\sigma^{-1}(i)j}$. Set $i = \sigma(j)$, we have $p_{\sigma(j)j} = 0$, which is a contradiction. Thus $\mathcal{E}(0,1)$ cannot be isomorphic to any $\mathcal{E}(1,c)$.

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Now assume that $\mathcal{E}(1,c)$ is isomorphic to $\mathcal{E}(1,b)$ for some $c \neq b$. Again apply Theorem 2.1 and equation (2.1), let $P = (p_{ij})$ and $\sigma \in S_n$ be such that

$$p_{ij} \neq 0 \Leftrightarrow i = \sigma(j) \text{ and } P^{(2)} + bK_n P^{(2)} = P + cPK_n.$$

Then similar to the discussions above, for all $1 \leq i, j \leq n$, we have $p_{ij}^2 + \gamma_{i\sigma(j)}p_{\sigma(j)j}^2 = p_{ij} + c\gamma_{\sigma^{-1}(i)j}$. Choose $i = \sigma(j)$, we have $p_{\sigma(j)j}^2 = p_{\sigma(j)j}$, which implies $p_{\sigma(j)j} = 1, 1 \leq j \leq n$, that is, P is the permutation matrix of σ . Thus since $P^{(2)} = P$, we have $bK_nP = cPK_n$. But that would lead to a contradiction since $c \neq b$ and $K_nP = PK_n$ by (2.9). Therefore $\mathcal{E}(1,c), c \neq 1, 1/(1-n)$ are pairwise non-isomorphic. Finally, note that the proof covers the case n = 2, 3 too (recall that we only need to show the listed matrices define non-isomorphic algebras for these two cases).

4. DIAGONAL AUTOMORPHISM SUBGROUP OF AN IDEMPOTENT EVOLUTION ALGEBRA

Let \mathcal{E} be an idempotent evolution algebra with a natural basis $e_1, ..., e_n$ and the structure matrix $A = (a_{ij})$, and let \mathcal{D} be the diagonal automorphism subgroup of Aut(\mathcal{E}). We consider the problem of which idempotent evolution algebra \mathcal{E} processes the maximal possible \mathcal{D} . To simplify our discussions, we assume that \mathbb{F} is algebraically closed of characteristic 0 in this section.

Let $\sigma \in S_n$ be such that $a_{\sigma(j)j} \neq 0, 1 \leq j \leq n$. Suppose that

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_p = (i_1 \cdots i_r)(j_1 \cdots j_s) \cdots (p_1 \cdots p_t)$$

is the decomposition of σ into disjoint cycles. For $D = \text{diag}(d_1, ..., d_n)$ in \mathcal{D} , we can divide $\{d_1, ..., d_n\}$ into disjoint subsets

$$\{d_{i_1}, ..., d_{i_r}\}, \{d_{j_1}, ..., d_{j_s}\}, ..., \{d_{p_1}, ..., d_{p_t}\}$$

according to the decomposition of σ . Then by (2.8), we see that the d_{i_h} 's are roots of $x^{2^r-1}-1$, the d_{j_h} 's are roots of $x^{2^s-1}-1$, and so on. Furthermore, we have $d_{i_h}^2 = d_{\sigma_1(i_h)} = d_{i_{h+1}}$, and so on.

If $\tau \in S_n$ is another permutation such that $a_{\tau(j)j} \neq 0, 1 \leq j \leq n$, let $\tau = \tau_1 \tau_2 \cdots \tau_q$ be its disjoint decomposition. If, say, for some $1 \leq h \leq r$, i_h appears in τ_1 , then d_{i_h} is also a root of $x^{2^{t-1}} - 1$, where $t = o(\tau_1)$. This will imply that d_{i_h} is a common root of $x^{2^{r-1}} - 1$ and $x^{2^{t-1}} - 1$, and so it is a root of $x^d - 1$, where $d = \gcd(2^r - 1, 2^t - 1)$. Therefore, for a fixed n, the maximal possible D occur when there exists only one σ such that $a_{\sigma(j)j} \neq 0, 1 \leq j \leq n$. This can also be seen by the fact that more nonzero entries of A would impose more constraints on the possible value of the d_i 's. Thus, given n, the maximal possible \mathcal{D} occur among

those \mathcal{E} such that the structure matrices satisfy $a_{ij} \neq 0 \Leftrightarrow i = \sigma(j)$ for some $\sigma \in S_n$.

Let A be a structure matrix such that $a_{ij} \neq 0 \Leftrightarrow i = \sigma(j)$ for some fixed $\sigma \in S_n$. If $\sigma = \sigma_1 \sigma_2 \cdots \sigma_p$ $(p \ge 1)$ is the decomposition of σ into the product of disjoint cycles cycles of length ≥ 2 , let $o(\sigma_i) = n_i, 1 \le i \le p$. Then $\sum_{i=1}^p n_i \le n$. From the above discussions, we see that in this case, the maximal possible order of \mathcal{D} is

$$(2^{n_1} - 1)(2^{n_2} - 1) \cdots (2^{n_p} - 1) < 2^{n_1} 2^{n_2} \cdots (2^{n_p} - 1) < 2^{\sum_{i=1}^p n_i} - 1 \le 2^n - 1.$$

On the other hand, if σ is a cycle of length n, then from $d_j^2 = d_{\sigma(j)}$, $1 \leq j \leq n$, we see that D is determined by d_1 . Since we assume that \mathbb{F} is algebraically closed of characteristic 0, in this case, \mathcal{D} is isomorphic to the cyclic group C_{2^n-1} formed by the roots of $x^{2^n-1}-1$.

Theorem 4.1. Assume the base field \mathbb{F} is algebraically closed of characteristic 0. Let \mathcal{E} be an idempotent evolution algebra with a natural basis $e_1, ..., e_n$ and the structure matrix $A = (a_{ij})$, and let \mathcal{D} be the diagonal automorphism subgroup of $Aut(\mathcal{E})$.

- (1) The maximal possible order of \mathcal{D} is $2^n 1$. This maximal order is achieved if and only $a_{ij} \neq 0 \Leftrightarrow i = \sigma(j)$ for some fixed cyclic permutation $\sigma \in S_n$ of length n; and in this case, \mathcal{D} is cyclic of order $2^n - 1$ and $Aut(\mathcal{E}) \cong C_{2^n-1} \rtimes C_n$.
- (2) All n-dimensional idempotent \mathcal{E} with maximal \mathcal{D} are isomorphic to the one represented by the structure matrix $A = P_{\sigma}$, where $\sigma = (12 \cdots n)$.

Proof. It remains to prove (2) and $\operatorname{Aut}(\mathcal{E}(P_{\sigma})) \cong C_{2^{n}-1} \rtimes C_{n}$, where $\sigma = (12 \cdots n)$. Since $\Gamma_{P_{\sigma}}$ is a cyclic digraph, $\operatorname{Aut}(\Gamma_{P_{\sigma}}) \cong C_{n}$. So Theorem 2.2 implies $\operatorname{Aut}(\mathcal{E}(P_{\sigma})) \cong C_{2^{n}-1} \rtimes C_{n}$.

To prove (2), let B be the structure matrix of an n-dimensional evolution algebra with a diagonal automorphism subgroup of order $2^n - 1$. Then there exists $b = (b_1, ..., b_n)^T \in \mathbb{F}^n$ such that $b_1 \cdots b_n \neq 0$ and $B = P_{\tau} \operatorname{diag}(b_1, ..., b_n)$ for some cyclic permutation τ of length n. By relabeling if necessary, we can assume that $\tau = \sigma = (12 \cdots n)$. Thus we need to prove that $\mathcal{E}(B) \cong \mathcal{E}(P_{\sigma})$ for $B = P_{\sigma} \operatorname{diag}(b_1, ..., b_n)$. By Theorem 2.1, it suffices to find a nonsingular diagonal matrix $D = \operatorname{diag}(d_1, ..., d_n)$ such that $BD^{(2)} = DP_{\sigma}$. This is equivalent to solving the following equation for the d_i 's:

$$P_{\sigma} \operatorname{diag}(b_1, \dots, b_n) \operatorname{diag}(d_1^2, \dots, d_n^2) = \operatorname{diag}(d_1, \dots, d_n) P_{\sigma}.$$

That is, to solve

$$diag(b_1d_1^2, ..., b_nd_n^2) = P_{\sigma}^{-1}diag(d_1, ..., d_n)P_{\sigma} = diag(d_{\sigma(1)}, ..., d_{\sigma(n)}) = diag(d_2, ..., d_n, d_1).$$

Note that if we know d_1 , then we can find all d_i , i > 1, by

(4.1)
$$d_2 = b_1 d_1^2, \ d_3 = b_2 d_2^2 = b_2 b_1^2 d_1^4, \ \dots, \ d_n = (\prod_{i=1}^{n-1} b_i^{2^{n-i-1}}) d_1^{2^{n-1}}.$$

The constraint for d_1 is

$$d_1 = b_n b_{n-1}^2 \cdots b_1^{2^{n-1}} d_1^{2^n} = (\prod_i^n b_i^{2^{n-i}}) d_1^{2^n}.$$

Thus, if we take a root of the polynomial $(\prod_{i=1}^{n} b_i^{2^{n-i}}) x^{2^n-1} - 1$ as d_1 , then we can find a nonsingular diagonal matrix D that satisfies $BD^{(2)} = DP_{\sigma}$ by (4.1).

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References

- Y. Cabrera Casado, M. Siles Molina, and M.V. Velasco, Evolution algebras of arbitrary dimension and their decompositions, Linear Algebra Appl. 495 (2016), 122-162.
- [2] Y. Cabrera Casado, M. Siles Molina, and M.V. Velasco, Classification of three-dimensional evolution algebras, Linear Algebra Appl. 524 (2017), 68-108.
- [3] Y. Cabrera Casado, M. Kanuni, and M. Siles Molina, Basic ideals in evolution algebras, Linear Algebra Appl. 570 (2019), 148-180.
- [4] Y. Cabrera Casado, M. Kanuni, and M. Siles Molina, Classification of four dimensional perfect non-simple evolution algebras, arXiv:1801.03856
- [5] José M. Casas, Manuel Ladra, Bakhrom A. Omirov, and Utkir A. Rozikov, On evolution algebras, Algebra Colloq. 21 (2014), 331-342.
- [6] C. Costoya, P. Ligouras, A. Tocino, and A. Viruel, Regular evolution algebras are universally finite, Proc. Amer. Math. Soc. 150 (2022), 919-925.

- [7] A. Elduque and A. Labra, Evolution algebras and graphs. J. Algebra Appl. 14, no. 7 (2015), 1550103.
- [8] A. Elduque and A. Labra, Evolution algebras, automorphisms, and graphs, Linear Multilinear Algebra 69 (2021), 331-342.
- [9] R. Frucht, Herstellung von Graphen mit vorgegebener abstrakter Gruppe, Compositio Math. 6 (1938), 239-250.
- [10] J. P. Tian, Evolution Algebras and Their Applications, Lecture Notes in Mathematics, 1921, Springer-Verlag, Berlin, 2008.

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