TECHNICAL REPORT NO. 12

August 1966

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Part of the research underlying this report was undertaken for the Pilot Program in Environmental Systems Analysis, NIH, 1 P10 ES 00098-01, Walter R. Lynn, Director, at Cornell and for the Office of Naval Research Contract Nonr-1228(10), Project NR 047-021 at Northwestern University and for the U. S. Army Research Office-Durham, Contract No. DA-31-124-ARO-D-322 also at Northwestern University.

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COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK

September, 1966

Abstract

L. S. Shapley has found a necessary and sufficient condition for the non-emptiness of the core of a characteristic function n-person game stating that the core is non-empty if and only if a certain system of linear inequalities on minimal balanced collection of finite sets is consistent. Using some well known constructs of linear programming, we associate to any n-person game two dual linear programming problems in which the constraint set of the primal includes the core of the game, and characterize the non-emptiness of the core in terms of properties of dual optimal solutions of these problems. We then prove the Shapley conjecture on sharpness of the set of proper minimal balanced inequalities with respect to core feasibility of proper n-person games. Using the Farkas-Minkowski Theorem, we obtain a characterization of redundant inequalities with respect to core feasibility and express the rate of growth of the game as a sequence of lower bounds for successive game values corresponding to increasing subsets of the collection of N players, which vitiates the possibility of constraint redundancy. If all game values are non-negative, the characteristic growth rate induces a partial ordering on game values corresponding to subsets of N.

1. Introduction

Recently L. S. Shapley [4] found a necessary and sufficient condition for the non-emptiness of the core of a characteristic function n-person game, which states that the core is non-empty if and only if a certain system of linear inequalities on minimal balanced collections of finite sets is consistent. One application of this result is that if all minimal balanced collections of order n are known, then the question of non-emptiness of the core of an n-person game can be answered by examining whether each minimal balanced collection satisfies its respective balanced linear inequality or not. Following in this direction, Peleg [5] has set forth an inductive combinatorial method for constructing minimal balanced collections of order $n+1$ from those of order n. Thus, following these lines, it would appear that one would need to construct minimal balanced collections of increasing order ad infinitum in order to investigate questions such as non-emptiness of a given core or relations between types of incidence matrices and cores, etc.

Our approach here to these and other matters is quite different. Using some well known constructs of linear programming, such as the theorem on the association of extreme points with linearly independent sets [1] and the opposite sign theorem [1], we reprove some of the results of Shapley [4] on the relations between balanced sets, minimal balanced sets, and extreme points in the space of weight vectors for an appropriate incidence matrix. We associate to any n-person game two dual linear programming problems in which the constraint set of the primal problem includes the core of the game, and characterize the non-emptiness of the core in terms of properties of dual optimal solutions of these problems. We also prove the Shapley conjecture (see [4], p. 15) on sharpness of the set of proper minimal balanced inequalities with respect to determining whether the core of a proper n-person game is empty or not. Using the Farkas-Minkowski Theorem, we characterize redundant inequalities with respect to core feasibility, and characterize the growth rate of the game which vitiates constraint redundancy.

2. Games, Balanced Sets, and Solutions Space for an Appropriate Incidence Matrix

Let $N = \{1,2,\ldots,n\}$. Following Shapley [3] a game v is a function from the subsets of N to the reals such that $v(\emptyset)=0$. The core of v is defined to be the set of all additive functions x such that $x(S) \geq v(S)$, all $S \subset N$ and $x(N) = v(N)$. A set $\{S_1,\ldots,S_p\}$ of distinct, non-empty, proper subsets of N is said to be balanced if there exists positive weights $\omega_1,\ldots,\omega_p$ such that $\sum_{j/i \in S_j} \omega_j = 1$ all $i \in N$.[1/] This definition has been succinctly stated in terms of an incidence matrix associated with this set of subsets by Peleg [5]. A minimal balanced set is one that includes no other balanced set. For our purposes, however, we introduce one incidence matrix Y corresponding to all subsets of N except $\emptyset$ and $\{1,2,3,\ldots,n\}$. We assume that

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[1/] See Shapley [4] page 1.

all of these subsets are indexed in a convenient way, say, listing subsets of one element first, then those containing two elements, etc. Thus, we obtain an indexing $S_1, S_2, \ldots, S_{2^n-2}$ and we may define $y_{ij} = \begin{cases} 1 & j \in S_i \\ 0 & j \notin S_i \end{cases}$ for $1 \leq i \leq 2^n-2$ and $1 \leq j \leq n$.[1/] Let Y denote this matrix of 2^n-2 rows and n columns. Let $\Lambda = \{\omega \mid \omega^T Y = e_n^T, \ \omega \geq 0\}$. Observe that each row vector, R_i, of Y corresponds precisely to one and only one subset of N.

<u>Proposition 1</u> Λ is spanned by its extreme points and hence is a convex polyhedron. [Lemma 2, Shapley [3]).

<u>Proof</u> We may write $\Lambda = \{\omega \mid \sum_{i=1}^{2^n-2} R_i \omega_i = e, \ \omega \geq 0\}$. Since each R_i is non-zero and non-negative, any non-trivial expression of the zero vector, say $R_i \alpha_i = 0$ implies some α_r and α_s are of opposite sign. Hence by the opposite sign theorem,[2/] Λ is spanned by its extreme points. Since there are only finitely many of these, Λ is a convex polyhedron.

<u>Proposition 2</u> There is a one-to-one correspondence between all minimal balanced sets and extreme points of Λ.[3/]

<u>Proof</u> Given any extreme point $\omega \epsilon \Lambda$, let $I = \{ i \mid \omega_i > 0\}$.

[1/] See Peleg [5], page 155

[2/] See Charnes-Cooper [1] page 282.

[3/] See Shapley [4] page 11.

Then by the theorem on the association of extreme points with linearly independent sets,[1/] the set $\{R_i | i\varepsilon I\}$ is linearly independent and hence contains no proper subset which is also in Λ, and therefore corresponds to a minimal balanced collection. On the other hand given any minimal balanced collection with weights $\{\omega_i > 0 | i\varepsilon I\}$, the associated rows must be linearly independent, for otherwise some subset of these rows is also feasible. Hence $\{\omega_i > 0 | i\varepsilon I\}$ is an extreme point of Λ.

Proposition 3 Any balanced set is the union of the minimal balanced sets that it contains [Shapley [3] p. 10].

Proof Any balanced set, ω is a member of Λ and hence if ω is not an extreme point, then by repeated application of the opposite sign property ω may be expressed as a convex combination of extreme points of Λ, i.e. initially we may write $\omega = \mu\omega^{(1)} + (1-\mu)\omega^{(2)}$ where $0 < \mu < 1$, $\omega^{(1)}, \omega^{(2)} \varepsilon \Lambda$ and $\omega^{(1)}$ and $\omega^{(2)}$ each have at least one more zero coordinate than ω. Since non-zero coordinate positions of ω appear among those of $\omega^{(1)}$ or $\omega^{(2)}$, ω is the union of the balanced sets associated with $\omega^{(1)}$ and $\omega^{(2)}$. The process is now repeated if necessary on $\omega^{(1)}$ and $\omega^{(2)}$ until extreme points are encountered. Thus, at the conclusion of this process the balanced set associated with ω will be the union of those associated with the extreme points at termination.

[1/] Charnes-Cooper [1] page 245.

We remark that this "purification" algorithm which incorporates other features[1/] has already been coded and could be immediately applied to the problem of decomposing balanced sets into unions of minimal balanced sets.

3. Characterization of the Core of a Game by Linear Programming

We observe that any additive function x which is defined on N is completely determined by its values on the integers $1,2,\ldots,n$. Furthermore, x is in the core of a game v if and only if

$$(1)\quad Yx \geq V \qquad \text{and} \qquad (2)\quad e^T x = v(N), \text{ where}$$

Y is the incidence matrix defined above and

$V = [v(S_1), v(S_2), \ldots, v(S_{2^n-2})]^T$ where we follow exactly the same ordering used to define Y. However, in order to use the power of linear programming, we shall replace condition (2) by the weaker one, (2') $e^T x \geq v(N)$ and construct the following dual linear programs associated with a given game n-person game v:

(I)		(II)	
min	$e^T x$	max	$w^T V + \eta v(N)$
subject to	$Yx \geq V$	subject to	$w^T Y + \eta e^T = e^T$
	$e^T x \geq v(N)$		$w^T \geq 0, \quad \eta \geq 0.$

1/ See Charnes-Kortanek-Raike [3]

<u>Proposition 4</u> For any n-person game whatever, problems (I) and (II) possess dual optimal solutions. The core of the game, v, is empty if and only if for any optimal solution x^* to (I), $e^T x^* > v(N)$.

<u>Proof</u> For any game v, consistency of (I) is most easily seen by observing that Y contains the identity matrix I_n as its first n rows, and $e^T = (1,1,\ldots,1)$. Problem (II) is also consistent as seen by taking $w^T = 0$ and $\eta = 1$. Hence by the dual theorem of linear programming there exist dual (extreme point) optimal solutions to (I) and (II). This completes the proof of the first assertion. For the second assertion of the Proposition, observe that if the core is empty, then there is no x satisfying (1) and (2). Hence at any (I)-optimum, x^*, we must have $e^T x^* > v(N)$. On the other hand, if at an optimum for (I), $e^T x^* > v(N)$, then since $e^T x^*$ is a minimum for $e^T x$ over (1) and (2'), there is no x satisfying (2).

Actually, the values of η at extreme points are quite limited as the following proposition shows.

<u>Proposition 5</u> Let $(\bar{\omega}, \bar{\eta})$ be any extreme point feasible solution to (II). Then $\bar{\eta}$ is 0 or 1.

<u>Proof</u> Since $\bar{\omega} \geq 0$, it follows that $\bar{\omega}^T Y \geq 0$ and therefore $\bar{\eta} \leq 1$. But if $0 < \bar{\eta} < 1$, then $\bar{\omega}^T = 0$, for otherwise there would be at least two distinct ways of expressing e^T as a linear combination with respect to the set of linearly independent vectors associated with the extreme point $(\bar{\omega}, \bar{\eta})$, which is a

contradiction. Hence $\bar{\eta} = 0$ or 1.

Duality Features, Sensitivity Analysis, and Non-empty Cores

Proposition 6 Given any game v, with an empty core, it is always possible to obtain a game v' which has non-empty core by changing at least one value of v.

Proof If v has an empty core, then at a dual optimum $(x^*;\omega^*,\eta_*)$, for (I) - (II), it follows that $e^T x^* > v(N)$ by Proposition 4. Hence by complementary slackness, $\eta_* = 0$ which implies $\omega^{*T} Y = e^T$.

Therefore ω^* is a minimal balanced collection. Hence v may be given a core immediately simply by changing one imputation alone, namely by increasing $v(N)$. Q.E.D.

In considering changes in a game v (having empty core) which may lead to a game with non-empty core, it may happen in some applications that certain of the values of v are required to remain unchanged. For example, perhaps it may not be possible to change the value of $v(N)$. In such situations we still have at our disposal the vector ω^* which forms a set of dual evaluators for changes in V. Since each component of ω^* is greater than zero, it follows that negative marginal changes in components of V, i.e. those components which are permitted to be changed and correspond to components of ω^*, will effect a strict decrease in the objective function $e^T x$ of (I). Thus, the power of sensitivity analysis[1/] of linear programming may be brought to bear on the problem of rendering games without core to ones which have non-empty core. These features can incorporate restrictions of the type already mentioned, i.e., maintaining some of the original game values while permitting others to vary freely, or perhaps subject to other linear inequality constraints.

Observe that Proposition 4 is equivalent to Theorem 2, p. 11 of Shapley [4], which states the necessary and sufficient condition for nonempty core in terms of an upper bound, $v(N)$ for all extreme points of the convex polyhedron Λ. Clearly, if there were an extreme point $\varpi \epsilon \Lambda$ satisfying $\varpi^T V > v(N)$, then at any dual optimum $(x^*; \omega^*, \eta_*)$ it follows that $e^T x^* \geq \varpi^T V > v(N)$ and $\eta_* = 0$. On the other hand if $e^T x^* > v(N)$, then $\eta_* = 0$ implying $\omega^* \epsilon \Lambda$ and $e^T x^* = \omega^{*T} V > v(N)$. Thus Proposition 4 is completely equivalent to Shapley's Theorem 2. The question of emptiness of the core of an n-person game is equivalent to whether the optimal value of problem (I) is strictly greater than $v(N)$ or not.

Proper Games and Sharpness of Proper Minimal Balanced Collections

A game is called <u>proper</u> if the set function v is super ad-

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1/ See Charnes-Cooper [1] and [2] for simultaneous considerations of data variations and their programming consequences.

ditive, i.e.,

$$v(S) + v(T) \leq v(S \cup T) \quad \text{for all } S, T \subseteq N$$

with $S \cap T = \emptyset$. A minimal balanced collection is <u>proper</u> if no two of its elements are disjoint. Accordingly we identify an extreme point $\omega \epsilon \Lambda$ as proper if the sets corresponding to the rows of Y associated with positive components of ω satisfy the pairwise intersecting property above. We shall find it convenient to let e_T denote the row in Y corresponding to a given set $T \subseteq N$.

<u>Proposition 7</u> [Shapley [4], Theorem 3] The proper game v has a non-empty core if and only if $\omega^T V \leq v(N)$ for all proper extreme points $\omega \epsilon \Lambda$.

<u>Proof</u> One of the implications is obvious by Proposition 4. We now show that it suffices to examine only proper extreme points of Λ by eliminating redundant inequalities of problem (I) with respect to a given optimal solution for (I) in a manner which does not destroy dual (II) feasibility. To this end, let $(x^*; \omega^*, \eta_*)$ be a dual optimal solution to problems (I) and (II) respectively, where $\omega^{*T} = \{\omega^*_{S_j} : 1 \leq k \leq p\}$, $\bar{S} = \{S_j : 1 \leq k \leq p\}$, and $\omega^*_{S_k}$ is the positive weight corresponding to the set S_k, $1 \leq k \leq p$. To obtain the required reductions, we proceed as follows. Let Q be any complement with respect to any set properly containing S_k, i.e., $T = S_k \cup Q$, $T \supsetneq S_k$. We examine cases regarding the known inequality $e_Q^T x^* \geq v(Q)$.

<u>Case 1</u> $e_Q^T x^* > v(Q)$. We may eliminate the inequality $e_Q^T x \geq v(Q)$. from (I) without affecting dual optimality.

<u>Case 2</u> $e_Q^T x^* = v(Q)$. If $Q \not\in \bar{S}$, we may eliminate the inequality

$e_Q^T x \geq v(Q)$ without affecting dual optimality since $\omega_Q = 0$. If, however, $Q = S_j \in \bar{S}$, then $v(T) \leq e_T^T x^* = e_{S_k}^T x^* + e_{S_j}^T x^* =$

$v(S_k) + v(S_j) \leq v(T)$, since $S_k \cap S_j = \emptyset$, $e_{S_k}^T x^* = v(S_k)$,

$e_{S_j}^T x^* = v(S_j)$ by complementary slackness, and v is proper.

Hence $v(T) = e_T^T x^*$. Thus, if $T = N$, then we may eliminate all inequalities from (I) except $e^T x \geq v(N)$ without affecting optimality of x^*. If on the other hand $T \neq N$, then we may remove the inequality associated with S_k or S_j (or both) without affecting dual optimality following the Shapley construction ([4], p. 14), where the set T may now be introduced with appropriate positive weight. Thus, whether in case 1 or case 2 above, we may remove inequalities from (I) which do not affect optimality of x^*, and the process stops when an inequality system is attained which corresponds to sets which have pairwise non-empty intersections. Q.E.D.

Let $\bar{Y}$ be all the rows of Y which correspond to sets which have pairwise non-empty intersection, and let $\bar{\Lambda}$ be the polyhedron associated with the matrix $\bar{Y}$. We shall call the set of extreme points of $\bar{\Lambda}$ universal with respect to the property of core feasibility of any proper game of corresponding dimension. Thus, analogous to Shapley [4], we obtain the smaller set, $\bar{\Lambda}$, which is universal for proper games of order n in terms of Proposition 7. The question Shapley raises and which we propose to answer is whether $\bar{\Lambda}$ is sharp, i.e. is there yet a smaller set of extreme points within $\bar{\Lambda}$ which is universal for proper n-person games? Shapley's conjecture in the affirmative appears to be correct as we shall now show.

It will suffice to construct for arbitrary n, a proper game which singles out any a priori specified proper extreme point $\bar{\omega}\varepsilon\bar{\Lambda}$ such that $\bar{\omega}^T\bar{V} > v(N)$ and $\bar{\omega}^T\bar{V} \leq v(N)$ for all other extreme points in $\bar{\Lambda}$.

Proposition 8 The extreme points of $\bar{\Lambda}$ defined above are sharp, in the sense that no proper subset of extreme points of $\bar{\Lambda}$ is a universal set for determining core feasibility of proper games of order n.

Proof Let $\bar{\omega}^T = \{\omega_{S_1},\ldots,\omega_{S_p}\}$ be any extreme point of $\bar{\Lambda}$ so that $\bar{S} = \{S_1,\ldots,S_p\}$ is a proper minimal balanced collection with weights $\bar{\omega}^T$.

Let $Z_{S_k} = \{S - S_k | S \supsetneq S_k\}$, $1 \leq k \leq p$. Thus, Z_{S_k} is the set of complements of S_k with respect to sets properly containing it. Let $\bar{Z} = \bigcup_{k=1}^{p} Z_{S_k}$. The values for v are assigned as follows:

$$v(S) = \begin{cases} 1 & ; \text{ if } S\varepsilon\bar{S} \\ -\mu & ; \text{ if } S\varepsilon\bar{Z} \\ -\mu+1; & \text{ if } S\notin\bar{S}\cup\bar{Z} \end{cases} ; \quad v(\emptyset) = 0,$$

where $\mu > 1$ is to be specified later.

We show that (1) v is well-defined and (2) v is a proper game.

Lemma 1 v as defined above is a well-defined function on subsets of N.

Proof It suffices to show that $\bar{S}\cap\bar{Z} = \emptyset$. Suppose not, i.e.

$S \varepsilon Z_{S_k}$ and $S = S_j$. Then there exists Q such that $Q \supsetneq S_k$ and $Q - S_k = S_j$. But $S_k \cap S_j \neq \emptyset$ since ϖ^T is proper and therefore $Q - S_k = S_j$ is impossible.

<u>Lemma 2</u> v is a proper game.

<u>Proof</u> Let Q and R be any non-empty subsets of N and $Q \cap R = \emptyset$. We must show $v(Q) + v(R) \leq v(Q \cup R)$.

<u>Case 1</u> Q or R in $\bar{S}$. First, observe that not both Q and R in $\bar{S}$; otherwise we contradict properness of ϖ. Thus, we may assume $Q = S_k \varepsilon \bar{S}$ and $R \notin \bar{S}$. But $R = Q \cup R - Q$ since $Q \cap R = \emptyset$, and this implies that $R \varepsilon Z_{S_k}$. Hence $v(R) = -\mu$, and $v(Q) + v(R) = 1 - \mu$. Claim now that $Q \cup R \notin \bar{S} \cup \bar{Z}$. On the contrary, if $Q \cup R \varepsilon \bar{S}$, then we can make R a member of $\bar{S}$ by simply transferring the weight of $Q \cup R$ to Q and to R, eliminating $Q \cup R$ from $\bar{S}$. This is a contradiction since $R \notin \bar{S}$. Thus, $Q \cup R \notin \bar{S}$. We show now that $Q \cup R \notin \bar{Z}$. If to the contrary $Q \cup R \varepsilon \bar{Z}$, say $Q \cup R \varepsilon \bar{Z}_{S_j}$ then there exists $T \supsetneq S_j$ for some j such that $T - S_j = Q \cup R = S_k \cup R$ which again is impossible since $S_j \cap S_k \neq \emptyset$ for any j. Hence $Q \cup R \notin \bar{S} \cup \bar{Z}$, and therefore $v(Q \cup R) = -\mu + 1$. Hence $v(Q) + v(R) = 1 - \mu \leq v(Q \cup R) = -\mu + 1$.

<u>Case 2</u> $Q \notin \bar{S}$ and $R \notin \bar{S}$. If $Q \cup R \notin \bar{Z}$, then $v(Q) + v(R) \leq -2\mu + 2 \leq -\mu + 1 \leq v(Q \cup R)$, since $\mu > 1$. On the other hand, if $Q \cup R \varepsilon \bar{Z}$, say $Q \cup R \varepsilon Z_{S_k}$, then there exists $T \supsetneq S_k$ such that $T - S_k = Q \cup R$. But this implies $T - S_k - Q = R$

since $Q \cap R = \emptyset$, which implies $R \varepsilon Z_{S_k}$. Similarly, we conclude $S \varepsilon Z_{S_k}$. Hence $v(Q) + v(R) = -2\mu \leq -\mu = v(Q \cup R)$. Thus in all cases $v(Q) + v(R) \leq v(Q \cup R)$, for $Q, R \subseteq N$ with $Q \cap R = \emptyset$. Therefore, v is superadditive and therefore a proper game.

Observe that $N = \{1,2,\ldots,n\} \not\in \overline{S} \cup \overline{Z}$ and therefore $v(N) = -\mu + 1$. Clearly increasing the value of $v(N)$ will not destroy the properness of the game v. Therefore we may and do redefine $v(N) = 0$.

<u>Determination of a Value for μ</u>

Let $\omega_1 = \{\omega_{11},\ldots,\omega_{1t(1)}\},\ldots,\ \omega_k = \{\omega_{k1},\ldots,\omega_{kt(k)}\}$ be all the extreme points of $\overline{\Lambda}$ other than $\overline{\omega}$, where for each i, $1 \leq i \leq k$, it follows that $1 \leq k(i) \leq n$ since n is the rank of Y. Let $\alpha = \min_{i,j}\{\omega_{ij}\}$ and $\sigma_i = \omega_{i1} + \ldots + \omega_{it(i)}$ for $1 \leq i \leq k$. Then $\alpha > 0$ and there exists real $\mu > 1$ such that

$$(\mu-1)\alpha \geq \sigma_i \quad \text{for all } i, \quad 1 \leq i \leq k$$

<u>Lemma 3</u> If $\omega_i \varepsilon \overline{\Lambda}$, $\omega_i \neq \overline{\omega}$, then $\omega_i^T\overline{V} \leq 0$.

<u>Proof</u> Given $\omega_i \neq \overline{\omega}$, then $\omega_i^T\overline{V} \leq \sigma_i + (-\mu+1)\omega_{ij}$ for some j, $1 \leq j \leq t(i)$ since at least one positive component of ω_i corresponds to a subset of N not in $\overline{S}$, i.e., a subset in $\overline{Z}$ or

not in $\bar{S} \cup \bar{Z}$ with game value $\leq -\mu + 1$. But

$$\sigma_i + (-\mu+1)\omega_{ij} \leq \sigma_i + (-\mu+1)\alpha \leq 0 \quad \text{since} \quad \omega_{ij} \geq \alpha > 0 \quad \text{and}$$

$\mu > 1$. Hence $\omega_i^T \bar{V} \leq 0$ for each $\omega_i \varepsilon \bar{\Lambda}$, $\omega_i \neq \bar{\omega}$. Q.E.D.

We complete the proof of Proposition 8 by observing that $\bar{\omega}^T \bar{V}$ is simply the sum of its positive components, and therefore $\bar{\omega}^T \bar{V} > 0 = v(N)$. Hence for our a priori specified extreme point $\bar{\omega} \varepsilon \bar{\Lambda}$, we have constructed a proper game which in terms of linear programming problem (II), has a functional value $> v(N)$ at $\bar{\omega}$, while for all other extreme points in $\bar{\Lambda}$, the (II)-functional value is $\leq v(N)$. Thus, the core of the game is empty, and it is precisely the proper extreme point $\bar{\omega}$ and this point alone which satisfies the condition $\omega^T \bar{V} > v(N)$.

Redundancy, Growth Conditions, and the Farkas-Minkowski Property

In discussing Proposition 7 above, we discovered that for any proper game it is necessary to examine only proper extreme points in order to ascertain the existence of a core. The technique of proof was to show that with respect to an optimal solution x^* of problem (I), certain inequalities could be deleted from (I) without affecting the optimality of x^*, until a system of inequalities remained whose corresponding sets satisfied the pairwise non-intersecting property. In general, however, the inequalities which are deleted are not of themselves redundant, i.e., any one of these may not be a consequence of some subsystem of inequalities. It is possible for a proper game to have no redundant inequalities. In this section we characterize redundancy by introducing a natural ordering of values of the game corresponding to increasing subsets of N, which for positive games induces a partial ordering on subsets of N.

Proposition 9 For any $S \subseteq N$, $e_S^T x \geq v(S)$ can only be the implicand of inequalities of the form

$$e_Q^T x \geq v(Q), \quad \text{where} \quad Q \subseteq S.$$

Proof Assume $e_S^T x \geq v(S)$ whenever $e_Q^T x \geq v(Q)$ for $Q \neq S$ $Q \in \overline{Q}$, a collection of subsets in N. Then by the Farkas-Minkowski Theorem, there exists $\lambda_Q \geq 0$ such that $e_S = \sum_Q e_Q \lambda_Q$ and $v(S) \leq \sum_Q v(Q)\lambda_Q$. But since $\lambda_Q \geq 0$ and e_Q consists of zeros and ones, any positive positions in any e_Q outside of coordinate positions corresponding to S could never be annihilated, and therefore would contradict the expression of e_S. Hence $Q \subseteq S$.

Proposition 10 (Characterization of Redundancy) $\text{Max} \sum_{Q \subset S} v_Q \lambda_Q$, subject to $\sum_{Q \subset S} e_Q \lambda_Q = e_S$, $\lambda_Q \geq 0$ exists and is denoted by M(S). $e_S^T x \geq v(S)$ is redundant if and only if $v(S) \leq M(S)$.

Proof If $e_S^T x \geq v(S)$ is redundant, then $e_S^T x \geq v(S)$ whenever $e_Q^T x \geq v(Q)$, $Q \subset S$. Hence, for problem (I_S) with dual (II_S) below, (I_S) is consistent and bounded below.

(I_S)	(II_S)
$\min \; e_S^T x$	$\max \sum_Q v(Q)\lambda_Q$
s.t. $e_Q^T x \geq v(Q), \quad Q \subsetneq S$	s.t. $\sum_Q e_Q \lambda_Q = e_S$
	$\lambda_Q \geq 0$

By the Farkas-Minkowski Theorem, there exists feasible λ_Q such that $v(S) \leq \sum_Q v(Q)\lambda_Q \leq M(S)$, proving the first implication. On the other hand suppose $v(S) \leq M(S)$. Then by dual optimality, for any x satisfying $e_Q^T x \geq v(Q)$, $Q \subsetneq S$, it follows that $e_S^T x \geq M(S) \geq v(S)$, which proves redundancy. Q.E.D.

<u>Proposition 11</u>. If the game v is not strictly proper, then there exists at least one redundant inequality.

<u>Proof</u> Suppose no constraint is redundant. Then for any $S \subsetneq N$, $v(S) > M(S)$ by Proposition 10. In particular for any Q, R, $Q \cap R = \emptyset$, we have

$$v(Q \cup R) > M(Q \cup R) \geq v(Q) + v(R). \quad \text{Q.E.D.}$$

<u>Remark</u> There are strictly proper games, however, that have redundant constraints.

<u>Proposition 12</u> The operator M defined on a proper game is itself proper, i.e. given S_1, S_2, $S_1 \cap S_2 = \emptyset$, then

$$M(S_1) + M(S_2) \leq M(S_1 \cup S_2)$$

<u>Proof</u> Consider the two problems,

(II_T)

$$\text{Max} \sum_{R \subset T} v(R)\lambda_R$$

$$\text{s.t.} \sum_{R \subset T} e_R \lambda_R = e_T$$

$$\lambda_R \geq 0$$

$II_{(S_{1,2})}$

$$\text{Max} \sum_{Q \subset S_1} v(Q)\lambda_Q + \sum_{P \subset S_2} v(P)\lambda_P$$

$$\text{s.t.} \sum_{Q \subset S_1} e_Q \lambda_Q + \sum_{P \subset S_2} e_P \lambda_P = e_T$$

$$\lambda_Q \ \lambda_P \geq 0$$

where $T = S_1 \cup S_2$. Clearly the optimal value of $(II_{S_{1,2}})$ is $M(S_1) + M(S_2)$ since $S_1 \cap S_2 = \emptyset$. However, following the Shapley [4] (p. 14) construction, we can do better by transferring weights to move into problem (II_T). Specifically, in considering terms $v(Q)\lambda_Q + v(P)\lambda_P$ of $(II_{S_{1,2}})$, if $\lambda_Q = \lambda_P$ we set $R = Q \cup P$ and $\lambda_R = \lambda_Q + \lambda_P$ to obtain a possibly bigger solution for (II_T). If $\lambda_Q > \lambda_P$, then assign weight λ_P to $v(T)$ and $\lambda_Q - \lambda_P$ to $v(Q)$ and delete e_P, to obtain an improved solution for (II_T). Therefore $M(T) \geq M(S_1) + M(S_2)$.

Corollary If v takes on positive values, then the operator M induces a partial ordering on subsets of N, given by $R \subset S \quad M(R) \leq M(S)$.

An Example

We return to the game constructed in the proof of Proposition 8 and with slight changes exhibit a strictly proper game with no redundant inequalities. Here, let $\overline{S} = \{S_j : 1 \leq j \leq n\}$ with weights $\lambda_{S_j} = \frac{1}{n-1}$, where $S_j = N - \{j\}$. Then for any subset

$Q \subsetneq N$, $Q \subseteq S_j$ for some j. But this implies $Q \not\in \bar{S}$ (otherwise transfer weights as usual to obtain a contradiction on minimality of $\bar{S}$). Claim that $e_S^T x \geq v(S)$ is not redundant, where $S \subsetneq N$. Clearly this is true if S consists of one element. If S has more than one element, then $S \not\in \bar{Z}$ in addition to $S \not\in \bar{S}$. Hence by definition $v(S) = -\mu + 1$. Now consider M(S) determined by max $\sum_{Q \subseteq S} v(Q)\lambda_Q$, subject to $\sum_{Q \subseteq S} e_Q \lambda_Q = e_S$, $\lambda_Q \geq 0$. Clearly, $\sum_Q \lambda_Q > 1$ for any feasible solution, and $v(Q) \leq -\mu + 1$ since $Q \not\in \bar{S}$. Therefore

$M(S) = \sum_Q v(Q)\lambda_Q^* \leq (-\mu+1)\sum_Q \lambda_Q^* < -\mu + 1 = v(S)$ at an optimum. Hence, $e_S^T x \geq v(S)$ is not redundant by Proposition 10. Finally, upon setting $v(N) = \sum v(S_j)\lambda_{S_j} = \frac{n}{n-1} + \delta$, for some $\delta > 0$, it follows that $e^T x \geq v(N)$ is also not redundant. By choice of $\mu > 1$, $\sum_Q v(Q)\lambda_Q \leq 0$ on all extreme points except $\bar{S}$, and the maximum achieved at $\bar{S}$, $\frac{n}{n-1}$, is strictly less than v(N). It is easy to check that v as defined is strictly proper. In fact, in case 2 of the proof of Proposition 8 we already have strict inequality. In case 1, we see now that if $Q = S_k \in \bar{S}$, and $R \not\in \bar{S}$, $Q \cap R \neq \emptyset$, then $Q \cup R = N$ and

$v(Q) + v(R) = 1 - \mu + 1 < 0 < v(N)$ and strict inequality occurs in this case also. Hence v is strictly proper, has no redundant inequalities, and by the redefinition of v(N) has nonempty core.

Thus, the sequence $\{M(S): S \subseteq N\}$ permits a sequential designation of values of a game such that if $v(S) > M(S)$, no inequalities are redundant. The sequence, any member of which depends on previous game values, expresses the rate of growth as a lower bound for successive game values corresponding to larger and larger subsets of the collection of N players which vitiates any possibility of constraint redundancy.

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