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On BCV-Sasakian Manifold

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Abstract

C. Baikousssis, D.E. Blair[1] made a study of Legendre curves in contact metric manifolds. J. I. Inoguchi, T. Kumamoto, N. Ohsugi, and Y. Suyama[2] studied fundamental properties of Heisenberg 3-spaces. M. Belkhelfa, I.E. Hirica, R. Rosca, L. Verstlraelen[3] obtained a complete characterization of surfaces with paralel second fundamental form in 3-dimensional Bianchi-Cartan-Vranceanu spaces(BCV).

In this paper, making use of method in paper of C. Baikousssis, D.E. Blair and M. Belkhelfa, I.E. Hirica, R. Rosca, L. Verstlraelen we obtained helices and their characterizations in BCV-Sasakian spaces such that the circular helices in BCV$-$Sasakian space correspond to the circles in E^3, the circular helices in Eucliden space correspond to the circular helices in BCV$-$Sasakian space and these helices are non-geodesical BCV$-$ Legendre curves. We have seen calculable that the covariant derivative of vector field Y with respect to vector field X without christoffel symbols in BCV-Sasakian spaces. Also we obtained more general curvature κ and torsion τ of a curve γ in BCV-Sasakian spaces.

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1 Introduction

Let M be a $(2n+1)-$manifold and $\varphi,\ \xi$ and η are $(1,1)$, $(1,0)$ and $(0,1)$ tensors on M, respectively. If these tensors satisfy the following conditions then the structure (η, ξ, φ) is called almost contact structure on M. And (M, η, ξ, φ) is called almost contact manifold

$$\left.\begin{array}{lcl} \varphi^2 & = & -I + \eta\xi \\ \varphi\xi & = & 0 \\ rank\ \varphi & = & 2n \\ \eta o \varphi & = & 0 \\ \eta(\xi) & = & 1 \end{array}\right\}. \tag{1}$$

If g is a Riemannian metric on the contact manifold (M, η, ξ, φ), g satisfies the equation

$$\begin{aligned} g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y), \\ g(\xi, X) &= \eta(X) \end{aligned}$$

then the structure $(M, \eta, \xi, \varphi, g)$ is called almost contact metric manifold. Furthermore, if the metric g satisfies the equation

$$g(\varphi X, Y) = -d\eta(X, Y)$$

then the structure $(M, \eta, \xi, \varphi, g)$ is called a contact metric manifold. The necessary and saficiend condition for an almost contact metrik manifold $(M, \eta, \xi, \varphi, g)$ to be a Sasakian manifold is holding the equation

$$(D_X\varphi)Y = g(X, Y)\xi - \eta(Y)X, X, Y \in \chi(M).$$

Moreover, if the tensor

$$\begin{array}{rlcl} N^{(1)}: & \chi(M) \times \chi(M) & \longmapsto & \chi(M) \\ & (X, Y) & \longmapsto & N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi \end{array}$$

on the Sasakian manifold $(M, \eta, \xi, \varphi, g)$ vanishes then the tensor $N^{(1)}$ is called Sasakian tensor and the contact manifold $(M, \eta, \xi, \varphi, g)$ is called Sasakian manifold [3].

2 Bcv-Sasakian Spaces

$\{\mathbb{R}^3, g_{\lambda,\mu}\}$ is called BCV space which denoted by $\mathfrak{M}^3$ or $\mathfrak{M}^3_{\lambda,\mu}$ where $g_{\lambda,\mu}$ is Bianchi-Cartan-Vranceanu (BCV) metric in $\mathbb{R}^3$ and denoted by

$$g_{\lambda,\mu} = \frac{dx_1^2 + dx_2^2}{\{1 + \mu(x_1^2 + x_2^2)\}^2} + \left(dx_3 + \frac{\lambda}{2}\frac{x_2dx_1 - x_1dx_2}{1 + \mu(x_1^2 + x_2^2)}\right)^2 \tag{2}$$

for $\lambda, \mu \in \mathbb{R}$ such that $1 + \mu(x_1^2 + x_2^2) \neq 0$. The dimension of this space is $\dim \mathfrak{M}^3_{\lambda,\mu} = 3$. If $\mu = 0$, $\lambda = 0$, then the space $\mathfrak{M}^3$ is called Euclidean space and denoted by E^3. In the special case that $\mu = 0$, $\lambda \neq 0$ the space $\mathfrak{M}^3$ is called Heisenberg space. Heisenberg space is denoted by N^3 [10]. In 1894 and later in 1928, L. Bianchi classified Riemannian metrics in the $3-$dimensional Euclidean space E^3[13, 14]. In the same year E. Cartan [5] and in 1947 G. Vranceanu [15], published some papers related with these spaces.

According to the metric 2 , an orthonormal basis $\phi = \{e_1, e_2, e_3\}$ of $\chi(\mathfrak{M}^3)$ is denoted by

$$\left.\begin{aligned} e_1 &= \{1 + \mu(x_1^2 + x_2^2)\}\frac{\partial}{\partial x_1} - \frac{1}{2}\lambda x_2 \frac{\partial}{\partial x_3}, \\ e_2 &= \{1 + \mu(x_1^2 + x_2^2)\}\frac{\partial}{\partial x_2} + \frac{1}{2}\lambda x_1 \frac{\partial}{\partial x_3}, \\ e_3 &= \frac{\partial}{\partial x_3}. \end{aligned}\right\} \tag{3}$$

The dual basis θ of ϕ is given by

$$\left.\begin{aligned} \theta^1 &= \frac{dx_1}{1 + \mu(x_1^2 + x_2^2)} , \\ \theta^2 &= \frac{dx_2}{1 + \mu(x_1^2 + x_2^2)} , \\ \theta^3 &= dx_3 + \frac{\lambda}{2}\ \frac{x_2 dx_1 - x_1 dx_2}{1 + \mu(x_1^2 + x_2^2)}. \end{aligned}\right\} \tag{4}$$

For the orthonormal basis $\phi = \{e_1, e_2, e_3\}$ of $\chi(\mathfrak{M}^3)$, if Levi-Civita connection on $\mathfrak{M}^3$ denoted by D, then we have

$$\begin{bmatrix} D_{e_1}e_1 \\ D_{e_1}e_2 \\ D_{e_1}e_3 \end{bmatrix} = \begin{bmatrix} 0 & 2\mu x_2 & 0 \\ -2\mu x_2 & 0 & \frac{\lambda}{2} \\ 0 & -\frac{\lambda}{2} & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

$$\begin{bmatrix} D_{e_2}e_1 \\ D_{e_2}e_2 \\ D_{e_2}e_3 \end{bmatrix} = \begin{bmatrix} 0 & -2\mu x_1 & -\frac{\lambda}{2} \\ 2\mu x_1 & 0 & 0 \\ \frac{\lambda}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

$$\begin{bmatrix} D_{e_3}e_1 \\ D_{e_3}e_2 \\ D_{e_3}e_3 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\lambda}{2} & 0 \\ \frac{\lambda}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

and

$$\begin{aligned}
[e_1, e_2] &= -2\mu x_2 e_1 + 2\mu x_1 e_2 + \lambda e_3 \ , \\
[e_3, e_2] &= 0 \ , \\
[e_1, e_3] &= 0 \ .
\end{aligned}$$

The transformation φ on $\chi(\mathfrak{M}^3)$ given by

$$\begin{array}{llll}
\varphi : & \chi(\mathfrak{M}^3) & \longmapsto & \chi(\mathfrak{M}^3) \\
& e_1 & \longmapsto & \varphi(e_1) = \ e_2 \\
& e_2 & \longmapsto & \varphi(e_2) = \ -e_1 \\
& e_3 & \longmapsto & \varphi(e_3) = \ 0
\end{array}$$

is a linear endomorfizm and the corresponding matrix is given by

$$\varphi = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with respect toorthonormal basis $\phi = \{e_1, e_2, e_3\}$ of $\chi(\mathfrak{M}^3)$. On the space $\mathfrak{M}^3$, if $\lambda \neq 0$

$$\eta = \theta^3 = dx_3 + \frac{\lambda}{2} \frac{x_2 dx_1 - x_1 dx_2}{1 + \mu(x_1^2 + x_2^2)}$$

and $\xi = e_3$, then we have the following relations;

$$\left.\begin{array}{lcl}
\varphi(\xi) & = & 0 \, , \\
\eta(\xi) & = & 1 \, , \\
d\eta(X,Y) & = & \frac{\lambda}{2} g_{\lambda,\mu}(X, \varphi(Y)) \quad ; \quad X, Y \in \chi(\mathfrak{M}^3) \ , \\
(D_X \varphi) Y & = & \frac{\lambda}{2} \left\{ g_{\lambda,\mu}(X,Y)\xi - \eta(Y)X \right\} \ .
\end{array}\right\} \tag{5}$$

The structure $(\mathfrak{M}^3, \varphi, \xi, \eta, g_{\lambda,\mu})$ together with the equations 5 is a Sasakian manifold [2, 16]. From now on for $\lambda \neq 0$ we will call the space as BCV-Sasakian space.

Since D is Levi-Civita connection and for $\forall X, Y \in \chi(\mathfrak{M}^3)$ we have

$$\begin{aligned}
D_X Y &= D_X \{v_1 e_1 + v_2 e_2 + v_3 e_3\} \\
&= \underbrace{v_1 D_X e_1 + v_2 D_X e_2 + v_3 D_X e_3}_{I} + \underbrace{X(v_1) e_1 + X(v_2) e_2 + X(v_3) e_3}_{II},
\end{aligned}$$

where $Y = v_1e_1 + v_2e_2 + v_3e_3$,

$$
\begin{aligned}
I &= v_1 D_X e_1 + v_2 D_X e_2 + v_3 D_X e_3 \\
&= v_1(\omega_{12}(X)e_2 + \omega_{13}(X)e_3) + v_2(-\omega_{12}(X)e_1 + \omega_{23}(X)e_3) + +v_3(-\omega_{13}(X)e_1 - \omega_{23}(X)e_2) \\
&= \det \begin{bmatrix} e_1 & e_2 & e_3 \\ \omega_{23}(X) & -\omega_{13}(X) & \omega_{12}(X) \\ v_1 & v_2 & v_3 \end{bmatrix} \\
&= \{\omega_{23}(X)e_1 - \omega_{13}(X)e_2 + \omega_{12}(X)e_3\} \wedge Y
\end{aligned}
$$

and

$$II = X(v_1)e_1 + X(v_2)e_2 + X(v_3)e_3.$$

In the above expression ω_{ij} $(1 \leq i, j \leq 3)$ are connection forms. The dual basis of orthonormal basis $\phi = \{e_1, e_2, e_3\}$ is $\{\theta_1, \theta_2, \theta_3\}$. Hence

$$
\begin{aligned}
\omega_{12}(X) &= (2\mu x_2\theta^1 - 2\mu x_1\theta^2 - \frac{\lambda}{2}\theta^3)(X), \\
\omega_{13}(X) &= -\frac{\lambda}{2}\theta^2(X), \\
\omega_{23}(X) &= \frac{\lambda}{2}\theta^1(X).
\end{aligned}
$$

We know that $\omega_{ij} = -\omega_{ji}$. In this way,

$$
\begin{aligned}
I &= \left\{ \frac{\lambda}{2}\theta^1(X)\, e_1 + \frac{\lambda}{2}\theta^2(X)\, e_2 + \left(2\mu x_2\theta^1 - 2\mu x_1\theta^2 - \frac{\lambda}{2}\theta^3 \right)(X)\, e_3 \right\} \wedge Y \\
&= \left\{ \frac{\lambda}{2}\left\{\theta^1(X)\, e_1 + \theta^2(X)\, e_2 + \theta^3(X)\, e_3\right\} + \left(2\mu x_2\theta^1(X) - 2\mu x_1\theta^2(X) - \lambda\theta^3(X)\right) e_3 \right\} \wedge Y \\
&= \left\{ \frac{\lambda}{2}X - g_{\lambda,\mu}([e_1, e_2], X)\, e_3 \right\} \wedge Y \\
&= \frac{\lambda}{2}X \wedge Y - g_{\lambda,\mu}([e_1, e_2], X)\, \varphi Y.
\end{aligned}
$$

II looks like the Euclidean connection $\widetilde{D}$, so we may show $X(v_1)e_1 + X(v_2)e_2 + X(v_3)e_3$ as $\widetilde{D}_X Y$, that is, $X(v_1)e_1 + X(v_2)e_2 + X(v_3)e_3 = \widetilde{D}_X Y$. It is obvious that D can be given as

$$D_X Y = \frac{\lambda}{2}X \wedge Y - g_{\lambda,\mu}([e_1, e_2], X)\, \varphi Y + \widetilde{D}_X Y\ . \tag{6}$$

On the other hand we can show that D satisfies the connection rules(See,

[16]). We may define some differential operators on $\chi(\mathfrak{M}^3)$ as follows:

$$\begin{aligned}
\text{grad } f &= \nabla f = \sum_{i=1}^{3} e_i\,[f]\, e_i, \\
\text{div } X &= g_{\lambda,\mu}(\nabla, X) - g_{\lambda,\mu}(\varphi X, [e_1, e_2]), \\
\triangle f &= \text{div}(\text{grad} f) = g_{\lambda,\mu}(\nabla, \nabla f) - g_{\lambda,\mu}(\varphi(\nabla f), [e_1, e_2]),
\end{aligned}$$

for $\forall f \in C^\infty(\mathfrak{M}^3, \mathbb{R}), \forall X \in \chi(\mathfrak{M}^3)$[16].

3 Frenet Vector Fields in BCV−Sasakian Space

Let γ be an arbitrary curve denoted by

$$\begin{array}{rccll}
\gamma : & I & \longmapsto & \mathfrak{M}^3 & \\
 & s & \longmapsto & \gamma(s) & = (\gamma_1(s), \gamma_2(s), \gamma_3(s))
\end{array}$$

in BCV−Sasakian space. Furthermore for the Frenet vector fields $\{V_1, V_2, V_3\}$ and the curveture κ and torsion τ of γ the Frenet equations are denoted by

$$\begin{bmatrix} D_{V_1}V_1 \\ D_{V_1}V_2 \\ D_{V_1}V_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

If $\kappa \neq 0$ then we have

$$V_2 = \frac{1}{\kappa} D_{V_1} V_1 \tag{7}$$

and hence,

$$D_{V_1} V_2 = -\kappa V_1 + \tau V_3 \ .$$

Taking the direction derivative of the both sides of equation 8 with respect to V_1 and rearranging it we obtain

$$\dot{\left(\frac{1}{\kappa}\right)} D_{V_1} V_1 + \frac{1}{\kappa} D_{V_1} D_{V_1} V_1 = -\kappa V_1 + \tau V_3 \ . \tag{8}$$

Again, differentiating both sides of 8 we get

$$\ddot{\left(\frac{1}{\kappa}\right)} D_{V_1} V_1 + \dot{\left(\frac{1}{\kappa}\right)} D^2_{V_1} V_1 + \dot{\left(\frac{1}{\kappa}\right)} D^2_{V_1} V_1 + \frac{1}{\kappa} D^3_{V_1} V_1 = -\dot{\kappa} V_1 - \kappa D_{V_1} V_1 + \dot{\tau} V_3 + \tau D_{V_1} V_3. \tag{9}$$

From 7 we have

$$\begin{aligned} D_{V_1}V_3 &= -\tau V_2 \\ &= -\frac{\tau}{\kappa} D_{V_1}V_1 \ . \end{aligned}$$

By the use of

$$\begin{aligned} \dot{\left(\frac{1}{\kappa}\right)} &= -\frac{\dot{\kappa}}{\kappa^2} \ , \\ \ddot{\left(\frac{1}{\kappa}\right)} &= 2\frac{(\dot{\kappa})^2}{\kappa^3} - \frac{\ddot{\kappa}}{\kappa^2} \end{aligned}$$

and rearranging 9 we obtain the general equation for curves in BCV$-$Sasakian space

$$D^3_{V_1}V_1 - 2\frac{\dot{\kappa}}{\kappa} D^2_{V_1}V_1 + \left(2\left(\frac{\dot{\kappa}}{\kappa}\right)^2 - \frac{\ddot{\kappa}}{\kappa} + \kappa^2 + \tau^2\right) D_{V_1}V_1 + \dot{\kappa}\kappa V_1 - \dot{\tau}\kappa V_3 = 0 \tag{10}$$

In [4] Ç. Camcı obtained the following result for curves in Sasakian space

$$D^3_{V_1}V_1 - 2\frac{\dot{\kappa}}{\kappa} D^2_{V_1}V_1 + \left(2\left(\frac{\dot{\kappa}}{\kappa}\right)^2 - \frac{\ddot{\kappa}}{\kappa} + \kappa^2 + 1\right) D_{V_1}V_1 + \dot{\kappa}\kappa V_1 = 0 \ .$$

(See also[6, 12]). We can give the following result.

Corollary 1 *The equation 10 is a general equation for the curves in BCV$-$Sasakian space.*

Now let us calculate the torsion of a curve in BCV$-$Sasakian space. Let γ be an unite speed curve

$$\begin{array}{cccll} \gamma : & I & \longmapsto & \mathfrak{M}^3 & \\ & s & \longmapsto & \gamma(s) & = (\gamma_1(s), \gamma_2(s), \gamma_3(s)) \end{array}$$

on BCV$-$Sasakian space. Let us calculate Frenet vector fields of γ (in the case $|\eta(\dot{\gamma})| \neq 1$) . Assume that $\eta(\dot{\gamma}) = \sigma$ and $\dot{\gamma}(s) = V_1$. By the use of 2, 7 we get $g_{\lambda,\mu}(D_{V_1}V_1, V_1) = 0$. We may take an orthonormal basis of BCV$-$Sasakian space as

$$\left\{ V_1, \frac{\varphi V_1}{\sqrt{1-\sigma^2}}, \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}} \right\}. \tag{11}$$

Hence we have

$$D_{V_1}V_1 = \alpha\frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}} \tag{12}$$

$\alpha, \beta \in \mathbb{R}$

$$\kappa = \sqrt{\alpha^2 + \beta^2}$$

and

$$D_{V_1}V_1 = \kappa V_2 \Rightarrow V_2 = \frac{1}{\kappa}D_{V_1}V_1.$$

On the other hand, the directional derivative of φV_1 with respect to V_1 is

$$\begin{aligned} D_{V_1}\varphi V_1 &= \varphi D_{V_1}V_1 + \left(D_{V_1}\varphi\right)V_1 \\ &= \varphi\left(\alpha\frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}\right) + \frac{\lambda}{2}\left(\xi - \sigma V_1\right) \\ &= -\frac{\alpha}{\sqrt{1-\sigma^2}}V_1 + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}}\xi - \frac{\beta\sigma}{\sqrt{1-\sigma^2}}\varphi V_1 + \frac{\lambda}{2}\left(\xi - \sigma V_1\right) \end{aligned} \tag{13}$$

and similarly the derivative of $\xi - \sigma V_1$ is

$$\begin{aligned} D_{V_1}\left(\xi - \sigma V_1\right) &= D_{V_1}\xi - \dot{\sigma}V_1 - \sigma D_{V_1}V_1 \\ &= -\frac{\lambda}{2}\varphi V_1 - \dot{\sigma}V_1 - \sigma\left(\alpha\frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}\right) \\ &= -\frac{\lambda}{2}\varphi V_1 - \dot{\sigma}V_1 - \sigma\alpha\frac{\varphi V_1}{\sqrt{1-\sigma^2}} - \sigma\beta\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}. \end{aligned} \tag{14}$$

and also the derivative of $\sigma = \eta(\dot{\gamma})$ is

$$\begin{aligned} \dot{\sigma} &= D_{V_1}\sigma \\ &= D_{V_1}g_{\lambda,\mu}(V_1, \xi) \\ &= g_{\lambda,\mu}(D_{V_1}V_1, \xi) + g_{\lambda,\mu}(V_1, D_{V_1}\xi) \\ &= g_{\lambda,\mu}\left(\alpha\frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}, \xi\right) + g_{\lambda,\mu}(V_1, -\frac{\lambda}{2}\varphi V_1). \\ &= \beta\sqrt{1-\sigma^2} \quad . \end{aligned}$$

Hence we have

$$\beta = \dot{\sigma}\frac{1}{\sqrt{1-\sigma^2}}. \tag{15}$$

The derivatives of components $\dfrac{\alpha}{\sqrt{1-\sigma^2}}$ and $\dfrac{\beta}{\sqrt{1-\sigma^2}}$ are,

$$D_{V_1}\left(\frac{\alpha}{\sqrt{1-\sigma^2}}\right) = \dot{\alpha}\frac{1}{\sqrt{1-\sigma^2}} + \alpha\beta\sigma\frac{1}{1-\sigma^2} \tag{16}$$

$$D_{V_1}\left(\frac{\beta}{\sqrt{1-\sigma^2}}\right) = \dot{\beta}\frac{1}{\sqrt{1-\sigma^2}} + \beta^2\sigma\frac{1}{1-\sigma^2}\ . \tag{17}$$

respectively. Thus we have that

$$\begin{aligned}
D_{V_1}V_2 &= D_{V_1}\left(\frac{1}{\kappa}D_{V_1}V_1\right) \\
&= -\frac{\dot{\kappa}}{\kappa^2}D_{V_1}V_1 + \frac{1}{\kappa}D_{V_1}D_{V_1}V_1 \\
&= -\frac{\dot{\kappa}}{\kappa^2}D_{V_1}V_1 + \frac{1}{\kappa}D_{V_1}\left(\alpha\frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta\frac{\xi-\sigma V_1}{\sqrt{1-\sigma^2}}\right) \\
&= -\frac{\dot{\kappa}}{\kappa^2}D_{V_1}V_1 + \frac{1}{\kappa}D_{V_1}\left(\frac{\alpha}{\sqrt{1-\sigma^2}}\right)\varphi V_1 + \frac{1}{\kappa}\left(\frac{\alpha}{\sqrt{1-\sigma^2}}\right)D_{V_1}\varphi V_1 \qquad (18) \\
&+ \frac{1}{\kappa}D_{V_1}\left(\frac{\beta}{\sqrt{1-\sigma^2}}\right)(\xi-\sigma V_1) + \frac{1}{\kappa}\left(\frac{\beta}{\sqrt{1-\sigma^2}}\right)D_{V_1}(\xi-\sigma V_1)\,.
\end{aligned}$$

Using the equations 12, 13, 14, 15, 16 and 17 we get

$$D_{V_1}V_2 = -\kappa V_1 + \tau V_3\,,$$

where

$$\begin{aligned}
\tau V_3 = &\left(-\frac{\alpha\dot{\kappa}}{\kappa^2} + \frac{\dot{\alpha}}{\kappa} - \frac{\lambda\beta}{2\kappa} - \frac{\alpha\beta\sigma}{\kappa\sqrt{1-\sigma^2}}\right)\frac{\varphi V_1}{\sqrt{1-\sigma^2}} \\
&+ \left(-\frac{\beta\dot{\kappa}}{\kappa^2} + \frac{\dot{\beta}}{\kappa} + \frac{\lambda\alpha}{2\kappa} + \frac{\alpha^2\sigma}{\kappa\sqrt{1-\sigma^2}}\right)\frac{\xi-\sigma V_1}{\sqrt{1-\sigma^2}}.
\end{aligned}$$

Taking the norm of the vectors of the both sides of the last equation we obtain

$$\tau = \frac{\lambda}{2} + \frac{\alpha\dot{\beta}-\dot{\alpha}\beta}{\alpha^2+\beta^2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}}\ . \tag{19}$$

Hence we can give the following result.

Proposition 2 *Let $\gamma \subset \mathfrak{M}^3$ be a Frenet curve in BCV−Sasakian space given by arc-length parameter. Then the torsion τ of γ is given by*

$$\tau = \frac{\lambda}{2} + \frac{\alpha\dot{\beta}-\dot{\alpha}\beta}{\alpha^2+\beta^2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}}.$$

Remark 1 *In [1] D. Blair found that in a 3−Sasakian space the value of τ is*

$$\tau = 1 + \frac{\alpha\dot{\beta} - \dot{\alpha}\beta}{\alpha^2 + \beta^2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}}. \tag{20}$$

As we see that these two values of τ are different, in the special case of $\lambda = 2$ they are the same.

Definition 1 *The subspace D_m of $T_{\mathfrak{M}^3}(m)$*

$$D_m = \{X \in T_{\mathfrak{M}^3}(m) : \eta(X) = 0\}$$

is called contact distribution. 1−dimensional integral submanifold of D_m is called a Legendre curve [3]. According to this definition a Legendre curve on $\mathfrak{M}^3$ may be denoted as

$$\begin{array}{rlcl} \gamma : & I & \longmapsto & D_m \subset \mathfrak{M}^3 \\ & s & \longmapsto & \gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s)) \end{array}$$

$\eta(\dot{\gamma}) = 0$ ve $m = \gamma(s)$ .Since $\mathfrak{M}^3$ is a BCV−Sasakian space, a Legendre curve in $\mathfrak{M}^3$ is called a BCV−Legendre curve [9].

Now let us calculate the torsion of a BCV−Legendre curve.

Theorem 3 *In a BCV−Sasakian manifold, the torsion of a BCV−Legendre curve γ which is not a geodesic is equal to $\frac{\lambda}{2}$.*

Proof. Let the curve γ

$$\begin{array}{rlcl} \gamma : & I & \longmapsto & \mathfrak{M}^3 \\ & s & \longmapsto & \gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s)) \end{array}$$

be a BCV−Legendre curve with arclengthed parameter. From 11 we can choose an orthonormal basis of a BCV−Sasakian space as

$$\{V_1, \varphi V_1, \xi\} \ .$$

Since γ is a BCV−Legendre curve we have $\sigma = \eta(\dot{\gamma}) = 0.$ Using 12 we obtain

$$D_{V_1} V_1 = \kappa \varphi V_1, \quad V_2 = \varphi V_1 \ .$$

and similarly,

$$\begin{aligned} D_{V_1} V_2 &= D_{V_1} \varphi V_1 \\ &= \varphi D_{V_1} V_1 + (D_{V_1}\varphi) V_1 \\ &= -\kappa V_1 + \frac{\lambda}{2}\xi \ . \end{aligned} \tag{21}$$

Furthermore, by the use of $V_2 = \varphi V_1$ and $V_3 = \xi$ we may write

$$\begin{bmatrix} D_{V_1}V_1 \\ D_{V_1}V_2 \\ D_{V_1}V_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \dfrac{\lambda}{2} \\ 0 & -\dfrac{\lambda}{2} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} .$$

which shows that $\tau = \dfrac{\lambda}{2}$ ■

Remark 2 *For the case M is a $3-$Sasakian space, D. Blair [1] found that,*

$$\begin{bmatrix} D_{V_1}V_1 \\ D_{V_1}V_2 \\ D_{V_1}V_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} .$$

A curve which is not a geodesic is a BCV$-$Legendre curve if and only if it starts as a Legendre curve and its torsion is equal to $\dfrac{\lambda}{2}$. This shows us that the special case of $\lambda = 2$ is the result of Blair. More generally and more precisely we have the following theorem.

Theorem 4 *Let γ be a differentiable curve and BCV$-$Sasakian space $\mathfrak{M}^3$ given with condition $\sigma = \eta(\dot{\gamma}) \neq 1$ at one point of $\mathfrak{M}^3$. If $\tau = \dfrac{\lambda}{2}$ and at one point $\sigma = \dot{\sigma} = 0$ then γ is a BCV$-$Legendre curve.*

Proof. For $\sigma = \eta(\dot{\gamma}) \neq 1$, we decomposed $D_{V_1}V_1$ as

$$D_{V_1}V_1 = \alpha \frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}$$

$\alpha, \beta \in \mathbb{R}$ and

$$\tau = \frac{\lambda}{2} + \frac{\alpha\dot{\beta} - \dot{\alpha}\beta}{\alpha^2 + \beta^2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}}. \tag{22}$$

Here α must be different from zero: In above expression if $\alpha = 0$, $D_{V_1}V_1$ is collinear with ξ and in turn that V_1 is collinear with ξ, this is a contradiction. So we suppose $\alpha \neq 0$. We have the equation 15

$$\beta = \dot{\sigma} \frac{1}{\sqrt{1-\sigma^2}}.$$

Making this substitution for β and using the hypothesis $\tau = \frac{\lambda}{2}$ 22 gives us that

$$\alpha \left(\ddot{\sigma} + \frac{2\sigma\dot{\sigma}}{1-\sigma^2} \right) + \alpha^3\sigma - \dot{\alpha}\dot{\sigma} = 0. \tag{23}$$

Clearly $\sigma = 0$ is a solution and $\dot{\sigma} = 0$ implies $\sigma = 0$. Thus we now assume that σ is non-constant. Setting $\upsilon = \frac{\dot{\sigma}}{\alpha}$ this equation becomes

$$\upsilon\frac{d\upsilon}{d\sigma} + \frac{2\sigma\upsilon^2}{1-\sigma^2} + \sigma = 0.$$

Integration gives

$$\dot{\sigma}^2 = \alpha^2\left(1-\sigma^2\right)\left(C\left(1-\sigma^2\right)-1\right)$$

where C is a constant. Now suppose that at one point, $\sigma = \dot{\sigma} = 0$; then since $\alpha \neq 0$ we have $C = 1$. Finally since $\sigma^2 \leq 1$, we have that $\sigma = 0$, a contradiction[1].

■

4 Helices and Their Characterizations in BCV−Sasakian Space

The characterization and classification for a curve on a Riemannian manifold is observed in[9, 11]. For a differentiable curve γ on a Riemannian manifold M^3, we can get the following results;

- If the curvatures κ and τ of γ are all equal to zero, then the curve is a geodesic,
- If the first curvature κ of γ is a non-zero constant and τ is zero, then the curve is a circle,
- If the first curvature κ of γ is a non-zero constant and τ is zero, then the curve is a circle,
- If the curvatures κ and τ of γ are both constant, then γ is an helix,
- If the curvatures κ and τ of γ are not constant but $\frac{\kappa}{\tau}$ is constant then γ is a general helix (inclined curve)[8],
- If $\kappa = 0$ then the curve γ is a straight line. If $\kappa \neq 0$ but is not constand and $\tau = 0$ then the curve γ is a planar curve.

We know that every BCV−Sasakian manifold is a Riemannian manifold. Hence we can get the following results;

1. Let

$$\begin{array}{rccl} \gamma: & I & \longmapsto & \mathfrak{M}^3 \\ & s & \longmapsto & \gamma(s) = \ (\gamma_1(s), \gamma_2(s), \gamma_3(s)). \end{array}$$

be a unit speed curve in BCV−Sasakian space $\mathfrak{M}^3$ given with $0 < \eta(\dot{\gamma}) = \sigma < 1$ condition. By the use of equations 19 and 22 we get

$$\kappa = \sqrt{\alpha^2 + \beta^2}$$

and

$$\tau = \frac{\lambda}{2} + \frac{\alpha\dot{\beta} - \dot{\alpha}\beta}{\alpha^2 + \beta^2} + \frac{\alpha\sigma}{\sqrt{1 - \sigma^2}}. \tag{24}$$

Here we have three cases;

(a) If $\kappa = \tau = 0$, then the curve γ is a geodesic. Because of $\alpha \neq 0$, $\kappa = \sqrt{\alpha^2 + \beta^2} \neq 0$ and than the curve γ can not be a geodesic curve.

(b) If $\kappa = c^{te} \neq 0$ and $\tau = 0$ then the curve γ becomes a circle. Here we have two subcases;

i. Supcase $\kappa \neq 0$ but $\tau = 0$ then α and β are both constants so $\tau = \frac{\lambda}{2} + \frac{\alpha\sigma}{\sqrt{1 - \sigma^2}} = 0$ and we get

$$\lambda = -\frac{2\alpha\sigma}{\sqrt{1 - \sigma^2}}$$

ii. If α and β are neither constants than we have $\alpha^2 + \beta^2 = r^2 = c^{te}$ so by 24 we get

$$\alpha\dot{\beta} - \dot{\alpha}\beta = \mp r^2.$$

Since $\tau = 0$ from 24 we get $\lambda = -\frac{2\alpha\sigma}{\sqrt{1 - \sigma^2}} \pm 2.$

(c) Since we have that $\kappa = \sqrt{\alpha^2 + \beta^2}$ and $\alpha \neq 0$ than $\kappa \neq 0$. On the other hand we know that $\tau \neq 0$ in BCV−Sasakian space. So the curve in $\mathfrak{M}^3$ is not a straight line.

2. Let us consider that the curve

$$\begin{array}{rccl} \gamma: & I & \longmapsto & \mathfrak{M}^3 \\ & s & \longmapsto & \gamma(s) = \ (\gamma_1(s), \gamma_2(s), \gamma_3(s)). \end{array}$$

be a unit speed *BCV − Legendre curve*. In Teorem 3.3 we had shown that the torsion of a BCV−Legendre curve γ which is not a geodesic is equal to $\frac{\lambda}{2}$. Here we have three cases;

(a) In the case $\tau = \dfrac{\lambda}{2}$ $(\lambda \neq 0)$ the curve γ can not be a circle.

(b) If $\dfrac{\kappa}{\tau}$ is a non-zero constant then the curve γ is a helix. Now, let us analyse this. We know that the torsion of a BCV$-$Legendre curve γ which is not a geodesic is equal to $\dfrac{\lambda}{2}$ and also constant. Hence, κ also must be constant and than $\ddot{\kappa} = \dot{\kappa} = 0$. By 10, we get

$$D^3_{V_1} V_1 = -(\frac{\lambda}{2} + \kappa^2) D_{V_1} V_1. \tag{25}$$

By using $V_1 = \dot{\gamma}$, $\triangle = -D_{V_1} D_{V_1}$ and $H = D_{V_1} V_1$ we have

$$\triangle H = (\frac{\lambda}{2} + \kappa^2) H, \text{ (} \Delta \text{ is the Laplacian operator)} \tag{26}$$

where $\dfrac{\lambda}{2} + \kappa^2 =$ constand. So 26 tells us that the curve γ is a circular helix. Therefore equation 26 charactherises that γ is a circular helix.

(c) In order to be γ is a straight line on $\mathfrak{M}^3$ then κ must be zero. So we get

$$g_{\lambda,\mu} (D_{V_1} V_1, \varphi V_1) = 0.$$

On the other hand, since $\tau = \dfrac{\lambda}{2} \neq 0$ then γ can not be a planar curve. So this case does not hold.

Corollary 4.1 *The circular helices in BCV$-$Sasakian space correspond to the circles in E^3.*

Corollary 4.2 *The circular helices in Eucliden space correspond to the circular helices in BCV$-$Sasakian space and these helices are non-geodesical BCV$-$Legendre curves.*

Example 1 *Consider the curve*

$$\begin{array}{rcccl} \gamma : & I & \longmapsto & \mathfrak{M}^3 & \\ & t & \longmapsto & \gamma(t) = & (r\cos t, r\sin t, c)\,. \end{array} , c = c^{te} \in \mathbb{R}$$

An easy calculation gives us that

$$\kappa = \left| \frac{1}{r^3 + r} + \frac{\lambda r}{2r^2 + 2} - 2\mu r^2 \right|$$

and

$$\tau = \frac{\lambda}{2}\left(\frac{1}{r^2+1}\right) - \frac{1}{r^2+1} + 2\mu r^3.$$

So κ *and* τ *are non-zero constants, then the ratio* $\frac{\kappa}{\tau}$ *is also a constant. Then the curve is a general helix.*

Example 2 *Consider the curve*

$$\begin{array}{rcll} \gamma: & I & \longmapsto & \mathfrak{M}^3 \\ & s & \longmapsto & \gamma(s) = \left(r\cos s, r\sin s, \dfrac{\lambda r^2}{2(1+\mu r^2)}s\right). \end{array}$$

An easy calculation gives us that

$$\kappa = \left|\frac{1}{r} - 2\mu r\right|,$$
$$\tau = \frac{\lambda}{2}.$$

We can say that κ, τ and $\frac{\kappa}{\tau}$ are constants. Therefore γ is a circular helix. According to D. Ferus γ is also called a $\omega-$ curve [7].

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