ON BENDING OF ELASTIC PLATES*

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1. Introduction. In two earlier publications[1,2] the author has considered the theory of bending of thin elastic plates with reference to the question of the boundary conditions which may be prescribed along the edges of a plate. The principal result of this work was a new system of differential equations for the deformations and stresses in thin plates. With this system of equations it is possible and necessary to satisfy three boundary conditions along the edges of a plate instead of the two conditions which Kirchhoff has first established for the classical theory.

The physical basis of these results was recognition of the fact that omission of the strain energy of the transverse shears is responsible for the contraction of the three physical boundary conditions into two conditions,** and that the problem can be treated without this omission.

While the subject is of interest from the point of view of the general theory of elasticity,[3,4] it is also of some practical importance, in particular with regard to the problem of stress concentration at the edge of holes in transversely bent plates. For such problems the classical theory leads to results which are not in accordance with experiment as soon as the diameter of the hole becomes so small as to be of the order of magnitude of the plate thickness,[5,6] while the new equations which take transverse shear deformation into account lead to results which are substantially in agreement with experiment.[7]

The main purpose of the present paper is to give an account of the author's earlier derivations[2] in simpler and more general form. While previously an isotropic homogeneous material was assumed, plates of homogeneous or non-homogeneous construction are now considered, with elastic properties which in the direction perpendicular to the plane of the plate are different from the elastic properties in directions parallel to the plane of the plate.

As a further example of application of the present system of equations, we treat the bending of a cantilever plate due to a terminal transverse load. For the homogeneous plate our result represents a minimum energy approximation to St. Venant's

* Received Aug. 7, 1946.

[1] E. Reissner, J. Math. Phys. **23,** 184–191 (1944).

[2] E. Reissner, J. Appl. Mech. **12,** A68–A77 (1945).

** At a free edge the three physical conditions are those of vanishing transverse force, vanishing bending couple and vanishing twisting couple. The two Kirchhoff conditions which take their places are vanishing bending couple and vanishing of the sum of transverse force and edgewise rate of change of twisting couple.

[3] A. E. H. Love, *A treatise on the mathematical theory of elasticity,* 4th ed., Cambridge University Press, Cambridge, 1927, pp. 27–29.

[4] J. J. Stoker, Bull. Am. Math. Soc. **48,** 247–261 (1942).

[5] J. N. Goodier and G. H. Lee, J. App. Mech. **8,** A27–A29 and A189 (1941).

[6] D. C. Drucker, J. Appl. Mech. **9,** A161–A164 (1942).

[7] D. C. Drucker, J. Appl. Mech. **13,** A250–A251 (1946).

solution, while for the non-homogeneous (sandwich) plate the problem appears not to have been discussed previously.

As before, the results are obtained by an application of the basic minimum principle for the stresses and the Lagrangian multiplier method is used to obtain approximate stress strain relations. The discussion of the significance of the Lagrange multipliers is made more precise compared with that given in the earlier work, in accordance with comments which have been made.[8]

2. Statics and strain energy of plates. Let M_x and M_y be the bending couples H the twisting couple and V_x and V_y the transverse shear-stress resultants. Let p be the surface load per unit of area (Fig. 1). The equilibrium conditions for an element $dxdy$ of the plate are then

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+p=0,\qquad \frac{\partial M_x}{\partial x}+\frac{\partial H}{\partial y}-V_x=0,\qquad \frac{\partial H}{\partial x}+\frac{\partial M_y}{\partial y}-V_y=0. \tag{1}$$

Equations (1) hold regardless of the way in which the stresses are distributed over the thickness of the plate. In terms of the stresses,

$$M_x=\int_{-h/2}^{h/2} z\sigma_x dz,\qquad M_y=\int_{-h/2}^{h/2} z\sigma_y dz,\qquad H=\int_{-h/2}^{h/2} z\tau_{xy} dz,$$
$$V_x=\int_{-h/2}^{h/2} \tau_{xz} dz,\qquad V_y=\int_{-h/2}^{h/2} \tau_{yz} dz. \tag{2}$$

Equations (1) are three equations for five unknowns. To obtain further equa-

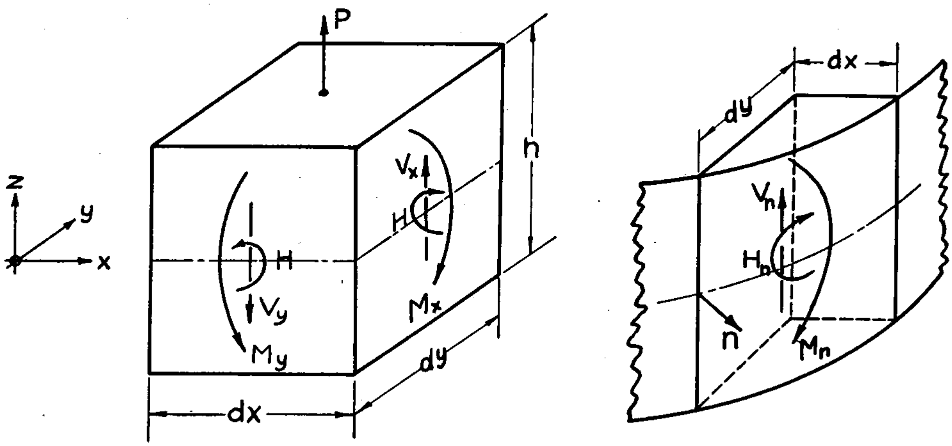

FIG. 1. Infinitesimal elements of a plate in interior and at boundary, showing orientation of stress resultants and couples.

tions, use has to be made of the stress strain relations. This is done here through the means of the basic minimum principle for the stresses (Castigliano's theorem of least work) according to which *the true state of stress is distinguished from all statically correct states of stress by the condition that the complementary energy be a minimum.*[9]

[8] J. N. Goodier, J. Appl. Mech. **13**, A251–A252 (1946).

[9] E. Trefftz in *Handbuch der Physik*, J. Springer, Berlin, 1927, vol. 6, p. 73 and Z. angew. Math. Mech. **15**, 101–108 (1935); I. S. Sokolnikoff and R. D. Specht, *Mathematical theory of elasticity*, McGraw-Hill Book Co., Inc., New York, 1946, pp. 284–287.

For a material obeying Hooke's law, and for given surface stresses or displacements, the complementary energy is the difference of the strain energy Π_s and of the work Π_b, which the surface stresses do over that portion of the surface where the displacements are prescribed.

Appropriate expressions for Π_s and Π_b are

$$\Pi_s = \frac{1}{2}\iint \left\{\frac{1}{(1-\nu^2)D}\left[M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)H\right] - \frac{2}{C_n}p(M_x + M_y) + \frac{1}{C_s}(V_x^2 + V_y^2)\right\} dxdy, \tag{3}$$

$$\Pi_b = \oint (M_n\bar{\beta}_n + H_s\bar{\beta}_s + V_n\bar{w})ds. \tag{4}$$

The values of the constants D, C_n and C_s depend on the properties of the material and on the nature of the stress distribution across the thickness of the plate. Examples of their calculation for homogeneous and non-homogeneous plates will be given later on.

The functions $\bar{\beta}_n$, $\bar{\beta}_s$ and $\bar{w}$ are the generalized boundary displacements of the problem. As Π_b measures the work of the boundary stresses it follows that $\bar{\beta}_n$ must be considered as the angle through which the moment M_n turns. A corresponding definition holds for $\bar{\beta}_s$. For the same reason the quantity $\bar{w}$ is to be considered as the appropriate measure of the transverse deflection of the plate. The precise meaning of $\bar{\beta}_n$, $\bar{\beta}_s$ and $\bar{w}$, in terms of weighted averages of the three components of boundary displacement $\overline{U}_n$, $\overline{U}_s$ and $\overline{W}$, will be obtained in the following by equating the work of the boundary stresses as given by Eq. (4) to the work of the boundary stresses according to the three-dimensional theory and by reducing the expression of the three-dimensional theory to Eq. (4) by introducing the assumed variation of the stresses over the thickness of the plate.

3. Variational derivation of the stress strain relations. To make the complementary energy $\Pi_s - \Pi_b$ a minimum subject to the equations of equilibrium (1), these equations are multiplied by Lagrangian multipliers λ_a, λ_b and λ_c, respectively, and integrated over the plate area. The result is added to $\Pi_s - \Pi_b$ and the variation of the resulting expression is made to vanish:

$$\iint \left\{\frac{M_x - \nu M_y}{(1-\nu^2)D}\delta M_x + \frac{M_y - \nu M_x}{(1-\nu^2)D}\delta M_y + \frac{2(1+\nu)H}{(1-\nu^2)D}\delta H - \frac{p}{C_n}(\delta M_x + \delta M_y) + \frac{1}{C_s}(V_x\delta V_x + V_y\delta V_y) + \lambda_a\delta\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + p\right) + \lambda_b\delta\left(\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} - V_x\right) + \lambda_c\delta\left(\frac{\partial H}{\partial x} + \frac{\partial M_y}{\partial y} - V_y\right)\right\} dxdy + \oint \left\{\bar{\beta}_n\delta M_n + \bar{\beta}_s\delta H_s + \bar{w}\delta V_n\right\}ds = 0. \tag{5}$$

Applying Eq. (5) to a rectangular plate and eliminating variations of derivatives by integration by parts, we find that the boundary values of the Lagrangian multipliers must be

$$\bar{\lambda}_a = \bar{w}, \qquad \bar{\lambda}_b = \bar{\beta}_x, \qquad \bar{\lambda}_c = \bar{\beta}_y.$$

As Eq. (5) also holds for *any part* of the plate if the boundary displacements referring to this part are identified with the displacements occurring in the actual solution of the problem, it follows that the Lagrangian multipliers throughout the plate are related to the generalized displacements in the interior of the plate through the equations

$$\lambda_a = w, \qquad \lambda_b = \beta_x, \qquad \lambda_c = \beta_y. \tag{6}$$

Introducing Eqs. (6) into (5) and integrating by parts, we obtain the variational equation

$$\iint \left\{\left[\frac{M_x - \nu M_y}{(1-\nu^2)D} - \frac{p}{C_n} - \frac{\partial \beta_x}{\partial x}\right]\delta M_x + \left[\frac{M_y - \nu M_x}{(1-\nu^2)D} - \frac{p}{C_n} - \frac{\partial \beta_y}{\partial y}\right]\delta M_y \right.$$
$$+ \left[\frac{2(1+\nu)H}{(1-\nu^2)D} - \frac{\partial \beta_x}{\partial y} - \frac{\partial \beta_y}{\partial x}\right]\delta H + \left[\frac{V_x}{C_s} - \frac{\partial w}{\partial x} - \beta_x\right]\delta V_x$$
$$\left. + \left[\frac{V_y}{C_s} - \frac{\partial w}{\partial y} - \beta_y\right]\delta V_y\right\} dxdy = 0. \tag{7}$$

From (7) follow the generalized stress strain relations of the problem:

$$M_x = D\left(\frac{\partial \beta_x}{\partial x} + \nu\frac{\partial \beta_y}{\partial y} + \frac{1+\nu}{C_n}p\right), \qquad M_y = D\left(\frac{\partial \beta_y}{\partial y} + \nu\frac{\partial \beta_x}{\partial x} + \frac{1+\nu}{C_n}p\right),$$
$$H = \frac{1-\nu}{2}D\left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right), \qquad \beta_x = -\frac{\partial w}{\partial x} + \frac{V_x}{C_s}, \qquad \beta_y = -\frac{\partial w}{\partial y} + \frac{V_y}{C_s}. \tag{8}$$

The conditions along a boundary $f_b(x, y) = 0$ are

$$\beta_n = \bar{\beta}_n \text{ or } M_n = \overline{M}_n, \quad \beta_s = \bar{\beta}_s \text{ or } H_s = \overline{H}_s, \quad w = \bar{w} \text{ or } V_n = \overline{V}_n. \tag{9}$$

Equations (9) are the *three* boundary conditions appropriate to the present theory when displacements or stresses are prescribed. They include the case of a free edge ($\overline{M}_n = \overline{H}_s = \overline{V}_n = 0$) and the case of a built-in edge ($\bar{\beta}_n = \bar{\beta}_s = \bar{w} = 0$). Appropriate conditions for more general edge conditions (such as elastic support) may be derived in a similar way.

The five Eqs. (8) together with the three Eqs. (1) represent a complete system of equations for the eight functions V_x, V_y, M_x, M_y, H, β_x, β_y, w. When $C_s = C_n = \infty$ they reduce to the customary equations of plate theory. To obtain the appropriate (Kirchhoff) form of the boundary conditions in this limiting case one must, however, go back to Eq. (3) and therein make $C_s = \infty$ before carrying out the remaining analysis.

4. Integration of the system of plate equations. It is possible to transform the system of Eqs. (1) and (8) such that integration in terms of harmonic and "wave" functions is possible.

The first of the equations in final form is

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = -p. \tag{10a}$$

Two further equations for V_x, V_y and w are obtained by introducing the first three Eqs. (8) into the last two Eqs. (1) and by observing the remaining Eqs. (8) and (1). The result is

$$V_x - \frac{(1-\nu)D}{2C_s}\nabla^2 V_x = -D\frac{\partial \nabla^2 w}{\partial x} - (1+\nu)D\left(\frac{1}{2C_s} - \frac{1}{C_n}\right)\frac{\partial p}{\partial x}, \tag{10b}$$

$$V_y - \frac{(1-\nu)D}{2C_s}\nabla^2 V_y = -D\frac{\partial \nabla^2 w}{\partial y} - (1+\nu)D\left(\frac{1}{2C_s} - \frac{1}{C_n}\right)\frac{\partial p}{\partial y}, \tag{10c}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Once Eqs. (10a) to (10c) are solved, the remaining five quantities M_x, M_y, H, β_x, β_y are found from Eqs. (8) by differentiations only. The first three Eqs. (8) may be written in the alternate form

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) + (1-\nu)\frac{D}{C_s}\frac{\partial V_x}{\partial x} + D\left(\frac{1+\nu}{C_n} - \frac{\nu}{C_s}\right)p, \tag{10d}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) + (1-\nu)\frac{D}{C_s}\frac{\partial V_y}{\partial y} + D\left(\frac{1+\nu}{C_n} - \frac{\nu}{C_s}\right)p, \tag{10e}$$

$$H = -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y} + \frac{1-\nu}{2}\frac{D}{C_s}\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right), \tag{10f}$$

where β_x and β_y have been taken from the last two Eqs. (8) and use has been made of (10a).

The system (10a) to (10f) is completed by the last two Eqs. (8).

The solution of equations (10a) to (10c) requires finding a particular integral for the load function p and finding sufficiently general solutions of the homogeneous equations. The latter is accomplished, as in the paper quoted in Footnote 2, by satisfying the homogeneous equation (10a) by means of a stress function χ in terms of which

$$V_x = \frac{\partial \chi}{\partial y}, \qquad V_y = -\frac{\partial \chi}{\partial x}. \tag{11}$$

With

$$\frac{1-\nu}{2}\frac{D}{C_s} = k^2 \tag{12}$$

the homogeneous equations (10b) and (10c) become

$$\frac{\partial}{\partial y}(\chi - k^2\nabla^2\chi) = -\frac{\partial}{\partial x}(D\nabla^2 w), \qquad \frac{\partial}{\partial x}(\chi - k^2\nabla^2\chi) = \frac{\partial}{\partial y}(D\nabla^2 w). \tag{13}$$

Since Eqs. (13) are Cauchy-Riemann equations, we have

$$D\nabla^2 w - i(\chi - k^2\nabla^2\chi) = \phi + i\psi = f(x+iy). \tag{14}$$

From $\chi - k^2\nabla^2\chi = -\psi$ follows

$$\chi = \psi_1 - \psi \tag{15}$$

where ψ_1 is the general solution of the "wave" equation (with imaginary velocity of propagation)

$$\psi_1 - k^2\nabla^2\psi_1 = 0. \tag{16}$$

Thus, the stress function χ is a combination of a harmonic function ψ and a wave function ψ_1. And if the harmonic contribution to χ is taken as the imaginary part of a complex function $f(x+iy)$ then $D\nabla^2 w$ is the corresponding real part. From

$$D\nabla^2 w = \phi \tag{17}$$

it follows that, when $p=0$, w itself is a biharmonic function, just as in the theory without transverse shear deformation.

Some applications of these results to the solution of specific problems for isotropic homogeneous plates are to be found in an earlier paper.[2]

5. Homogeneous plates. The values of the constants D, C_n and C_s in the strain energy expression depend on the nature of the plate material. Their determination will now be carried out under the assumption that the material of the plate is subject to the following system of stress strain relations

$$\epsilon_x = \frac{\partial U}{\partial x} = \frac{1}{E}(\sigma_x - \nu\sigma_y) - \frac{\nu_z}{E_z}\sigma_z. \tag{18a}$$

$$\epsilon_y = \frac{\partial V}{\partial y} = \frac{1}{E}(\sigma_y - \nu\sigma_x) - \frac{\nu_z}{E_z}\sigma_z, \tag{18b}$$

$$\gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = \frac{2(1+\nu)}{E}\tau_{xy}, \tag{18c}$$

$$\epsilon_z = \frac{\partial W}{\partial z} = \frac{1}{E_z}[\sigma_z - \nu_z(\sigma_x + \sigma_y)], \tag{18d}$$

$$\gamma_{xz} = \frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} = \frac{1}{G_z}\tau_{xz}, \tag{18e}$$

$$\gamma_{yz} = \frac{\partial W}{\partial y} + \frac{\partial V}{\partial z} = \frac{1}{G_z}\tau_{yz}. \tag{18f}$$

Equations (18) stipulate that the plate is isotropic with respect to directions parallel to the plane of the plate but has elastic properties in the direction normal to the plane of the plate which are different.

The strain energy for a plate of thickness h with the stress strain relations (18) is given by

$$\Pi_s = \frac{1}{2}\iint\int_{-h/2}^{h/2}\left\{\frac{1}{E}[\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + 2(1+\nu)\tau_{xy}^2] + \frac{1}{E_z}[\sigma_z^2 - 2\nu_z\sigma_z(\sigma_x+\sigma_y)] + \frac{1}{G_z}[\tau_{xz}^2 + \tau_{yz}^2]\right\} dz\,dx\,dy. \tag{19}$$

Equation (19) is reduced to (3) by appropriate assumptions regarding the variation of the stresses across the thickness of the plate. It is rational to assume that the bending stresses vary linearly over the thickness of the plate, while the transverse shear stresses vary parabolically:

$$\sigma_x = \frac{M_x}{h^2/6}\frac{z}{h/2}, \qquad \sigma_y = \frac{M_y}{h^2/6}\frac{z}{h/2}, \qquad \tau_{xy} = \frac{H}{h^2/6}\frac{z}{h/2} \tag{20}$$

$$\tau_{xz} = \frac{V_x}{2h/3}\left[1 - \left(\frac{z}{h/2}\right)^2\right], \qquad \tau_{yz} = \frac{V_y}{2h/3}\left[1 - \left(\frac{z}{h/2}\right)^2\right]. \tag{21}$$

Equations (20) and (21) satisfy two of the three three-dimensional differential equations of equilibrium provided the stress couples and stress resultants satisfy (1). From the third of the three-dimensional equilibrium equations, and from the condition that the load p is acting on the face $z = +h/2$, the transverse normal stress σ_z is obtained:

$$\sigma_z = \frac{3p}{4}\left[\frac{2}{3} + \frac{z}{h/2} - \frac{1}{3}\left(\frac{z}{h/2}\right)^3\right]. \tag{22}$$

Substituting Eqs. (20) to (22) into Eq. (19) for Π_s, we find that the integration with respect to z may be carried out and that (19) reduces to (3).* The values of the constants D, C_n and C_s are found to be

$$D = \frac{Eh^3}{12(1-\nu^2)}, \qquad C_n = \frac{5}{6}\frac{E_z h}{\nu_z}, \qquad C_s = \frac{5}{6}G_z h. \tag{23}$$

Equations (23) are introduced into Eqs. (10). There occurs in particular

$$k^2 = \frac{1-\nu}{2}\frac{D}{C_s} = \frac{1}{10}\frac{Eh^2}{2(1+\nu)G_z},$$
$$D\left(\frac{1+\nu}{C_n} - \frac{\nu}{C_s}\right) = \frac{1}{10}\frac{Eh^2}{1-\nu^2}\left(\frac{\nu_z(1+\nu)}{E_z} - \frac{\nu}{G_z}\right). \tag{24}$$

For an isotropic material ($E_z = E$, $\nu_z = \nu$, $G_z = E/2(1+\nu)$) the terms in (24) reduce to the values for these quantities which were first obtained in an earlier paper of the author[2] and Eqs. (10a) to (10f) reduce to Eqs. (I) to (VI) of the earlier paper.

In order to determine the significance of the generalized displacements β_x, β_y and w, we write the work of the surface stresses in the form

$$\Pi_b = \oint \int_{-h/2}^{h/2} [\sigma_n \overline{U}_n + \tau_{ns}\overline{U}_s + \tau_{nz}\overline{W}]dzds, \tag{25}$$

where $\overline{U}_n$, $\overline{U}_s$ and $\overline{W}$ are the actual displacement components of a point of the boundary. Substituting (20) and (21) into (25), we have

$$\Pi_b = \oint \int_{-h/2}^{h/2} \left\{\frac{M_n}{h^2/6}\frac{z}{h/2}\overline{U}_n + \frac{H_s}{h^2/6}\frac{z}{h/2}\overline{U}_s + \frac{V_n}{2h/3}\left[1 - \left(\frac{z}{h/2}\right)^2\right]\overline{W}\right\} dzds. \tag{26}$$

Comparison of Eqs. (26) and (4) gives

* With the exception of a term containing p^2 which disappears when the variation is carried out and which is therefore not evaluated explicitly.

$$\bar{\beta}_n = \frac{6}{h^2}\int_{-h/2}^{h/2} \bar{U}_n \frac{z}{h/2}\, dz, \qquad \bar{\beta}_s = \frac{6}{h^2}\int_{-h/2}^{h/2} \bar{U}_s \frac{z}{h/2}\, dz,$$
$$\bar{w} = \frac{3}{2h}\int_{-h/2}^{h/2} \bar{W}\left[1 - \left(\frac{z}{h/2}\right)^2\right] dz. \tag{27}$$

As Eqs. (26) and (4) hold for any portion of the plate it follows from Eqs. (27) that throughout the interior of the plate.

$$\beta_x = \frac{6}{h^2}\int_{h/2}^{h/2} U \frac{z}{h/2}\, dz, \qquad \beta_y = \frac{6}{h^2}\int_{h/2}^{h/2} V \frac{z}{h/2}\, dz,$$
$$w = \frac{3}{2h}\int_{-h/2}^{h/2} W\left[1 - \left(\frac{z}{h/2}\right)^2\right] dz. \tag{28}$$

From Eqs. (28) it is concluded that β_x and β_y represent quantities which are equivalent to but not identical with components of change of slope of the normal to the undeformed middle surface, while w is a weighted average, taken over the thickness, of the transverse displacements of the points of the plate. Thus, according to the third Eq. (28), *the present theory leads to approximate values not for the deflection of the middle surface of the plate but for a weighted average across the thickness of the deflections of all points of the plate which lie on a normal to the middle surface.*

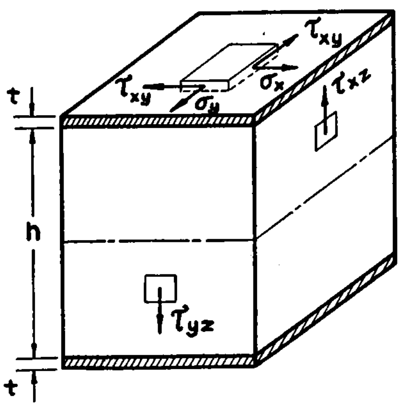

FIG. 2. Infinitesimal element of sandwich plate, showing dimensions and relevant components of stress.

6. Sandwich plates. We consider a composite plate consisting of a core layer of thickness h and of two face layers of thickness t. It is assumed that t is small compared with h and that the core material is much more flexible than the face material. Under these assumptions the transverse shears are predominantly taken by the core plate while the bending stresses are primarily taken by the face plates (Fig. 2).

We take for the strain energy of the composite plate the following expression*

$$\Pi_s = \frac{t}{E_f}\iint [\sigma_{x,f}^2 + \sigma_{y,f}^2 - 2\nu\sigma_{x,f}\sigma_{y,f} + 2(1+\nu)\tau_{xy,f}]\, dx\, dy$$
$$+ \frac{1}{2G_c}\iint\int_{-h/2}^{h/2} [\tau_{xz,c}^2 + \tau_{yz,c}^2]\, dz\, dx\, dy, \tag{29}$$

where the subscript f refers to the face layers. The stresses in the face plates are taken to be uniform across the thickness and the relations between stresses and couples are then,

* While the assumptions made in what follows should give an accurate picture (within the linear theory of bending) for combinations such as a foamy core substance and aluminum face plates they will not be sufficiently accurate for plates composed for instance of two different kinds of wood.

$$\sigma_{x,f} = \pm \frac{M_x}{t(h+t)}, \qquad \sigma_{y,f} = \pm \frac{M_y}{t(h+t)}, \qquad \tau_{xy,f} = \pm \frac{H}{t(h+t)} \cdot \tag{30}$$

As no stresses σ_x, σ_y, τ_{xy} are assumed to be acting through the core material, it follows from the differential equations of equilibrium that the transverse shear stresses do not vary across the thickness of the core,

$$\tau_{xz,c} = \frac{V_x}{h}, \qquad \tau_{yz,c} = \frac{V_y}{h} \cdot \tag{31}$$

Substituting Eqs. (30) and (31) into (29), we obtain

$$\Pi_s = \int\int\left\{\frac{1}{t(h+t)^2E_f}[M_x^2 + M_y^2 - 2\nu M_xM_y + 2(1+\nu)H^2] + \frac{1}{2G_ch}[V_x^2 + V_y^2]\right\} dxdy. \tag{32}$$

Comparison of equation (32) with equation (4) shows that for the sandwich plate the constants occurring in the system of differential equations (10) are given in terms of the dimensions and elastic properties of the plate as follows

$$D = \frac{1}{2}\frac{(h+t)^2E_f}{(1-\nu^2)}, \qquad C_n = \infty, \qquad C_s = hG_c; \tag{33}$$

$$k^2 = \frac{1-\nu}{2}\frac{D}{C_s} = \frac{t(h+t)^2E_f}{4h(1+\nu)G_c}, \quad D\left(\frac{\nu}{C_s} - \frac{1+\nu}{C_n}\right) = \frac{(\nu t(h+t)^2E_f}{2h(1-\nu)(1+\nu)G_c} \tag{34}$$

The magnitude of the effect of transverse shear deformation is primarily determined by the magnitude of the quantity k. Comparing the first Eq. (24) for the isotropic homogeneous plate with the first Eq. (34) for the sandwich plate it is seen that the effect is of greater importance for the sandwich plate than for the isotropic plate whenever

$$\frac{th}{2}\frac{E_f}{G_c} > \frac{2(1+\nu)}{5}h^2,$$

or whenever the ratio E_f/G_c is greater than the ratio h/t.

The significance of the Lagrangian multipliers β_x, β_y and w in the present case is determined in the same manner as for the homogeneous plate. One finds here, instead of Eqs. (28), that in terms of the components of displacement U, V, W,

$$\beta_x = \frac{1}{h}\left[U\left(\frac{h}{2}\right) - U\left(-\frac{h}{2}\right)\right], \qquad \beta_y = \frac{1}{h}\left[V\left(\frac{h}{2}\right) - V\left(-\frac{h}{2}\right)\right],$$
$$w = \frac{1}{h}\int_{-h/2}^{h/2} W dz. \tag{35}$$

7. Plate equations in polar coordinates. For the applications to stress concentration problems it is convenient to have Eqs. (10) in terms of plane polar coordinates r, θ. Appropriate transformation leads to

$$\frac{\partial r V_r}{\partial r} + \frac{\partial V_\theta}{\partial \theta} = - r p, \tag{36a}$$

$$V_r - k^2\left[\nabla^2 V_r - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} - \frac{1}{r^2} V_r\right] = - D\frac{\partial \nabla^2 w}{\partial r} - (1+\nu)D\left(\frac{1}{2C_s} - \frac{1}{C_n}\right)\frac{\partial p}{\partial r}, \tag{36b}$$

$$V_\theta - k^2\left[\nabla^2 V_\theta + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} - \frac{1}{r^2} V_\theta\right]$$
$$= - D\frac{1}{r}\frac{\partial \nabla^2 w}{\partial \theta} - (1+\nu)D\left(\frac{1}{2C_s} - \frac{1}{C_n}\right)\frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{36c}$$

$$M_r = - D\left[\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] + 2k^2\frac{\partial V_r}{\partial r} + D\left(\frac{1+\nu}{C_n} - \frac{\nu}{C_s}\right)p, \tag{36d}$$

$$M_\theta = - D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right]$$
$$+ 2k^2\left[\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r}\right] + D\left(\frac{1+\nu}{C_n} - \frac{\nu}{C_s}\right)p, \tag{36e}$$

$$H_{r\theta} = - (1-\nu)D\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) + k^2\left[\frac{1}{r}\frac{\partial V_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)\right] \tag{36f}$$

$$\beta_r = - \frac{\partial w}{\partial r} + \frac{V_r}{C_s}, \tag{36g}$$ $$\beta_\theta = - \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{V_\theta}{C_s}\cdot \tag{36h}$$

Equations (36d) to (36f) have been given in the paper quoted in Footnote 2 for the case of the isotropic homogeneous plate. Equations (36b) and (36c) have not previously been given. They are included in order to facilitate the obtaining of particular integrals of the system of equations for load functions of the form $p = \cos n\theta f(r)$.

Equations (11) which define the stress function χ for the solution of the homogeneous equations take on the form

$$V_r = \frac{1}{r}\frac{\partial \chi}{\partial \theta}, \qquad V_\theta = - \frac{\partial \chi}{\partial r}\cdot \tag{37}$$

Equations (15) and (16) remain unchanged:

$$\chi = \psi_1 - \psi \tag{15}$$

where $\psi(r, \theta)$ is a harmonic function and ψ_1 now satisfies the equation

$$\psi_1 - k^2\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\psi_1 = 0. \tag{16}$$

Also, as before

$$D\nabla^2 w = \phi, \tag{17}$$

where now

$$\phi(r, \theta) + i\psi(r, \theta) = f(re^{i\theta}). \tag{14}$$

Suitable expressions for ϕ, ψ and ψ_1 are given in Eqs. (42) to (46) of the paper quoted in Footnote 2. In the present formulation which includes nonisotropic, nonhomogeneous plates the quantity $h/\sqrt{10}$ in these equations is replaced by the quantity k defined in Eq. (12) above.

8. Bending of cantilever plate by terminal transverse load. As an example of the application of the formulas of this paper we may treat Saint Venant's problem of flexure of a beam with rectangular cross section.[10,11,12] Taking a plate of width $2a$ and length l, held at $x=0$, acted upon by a force P at $x=l$ and free of stress along the edges $y=\pm a$, Saint Venant's semi-inverse procedure amounts to setting

$$M_y = V_y = p = 0. \tag{38}$$

From (10a),

$$\frac{\partial V_x}{\partial x} = 0, \qquad V_x = V_x(y), \tag{39}$$

and from (10c),

$$\frac{\partial D\nabla^2 w}{\partial y} = 0, \qquad D\nabla^2 w = f(x). \tag{40}$$

Introducing (39) and (40) into (10b), we obtain

$$V_x - k^2 \frac{d^2 V_x}{dy^2} = -\frac{df}{dx}. \tag{41}$$

From Eq. (41) it follows, in view of (39) and (40), that

$$V_x = C + A \cosh \frac{y}{k}, \tag{42}$$

$$D\nabla^2 w = -Cx + B. \tag{43}$$

From (43) follows

$$Dw = -C\frac{x^3}{6} + B\frac{x^2}{2} + \phi(x, y), \tag{44}$$

where $\phi(x, y)$ is a harmonic function which, according to (38) and (10e), is determined from the relation

$$M_y = -\left[\frac{\partial^2 \phi}{\partial y^2} + \nu\left(\frac{\partial^2 \phi}{\partial x^2} - Cx + B\right)\right] = 0. \tag{45}$$

Evaluation of Eq. (45) leads to the relation

$$Dw = \frac{-1}{1-\nu}\left(C\frac{x^3}{6} - B\frac{x^2}{2}\right) + \frac{\nu}{1-\nu}\frac{y^2}{2}(Cx - B) + Fx + I. \tag{46}$$

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[10] S. Timoshenko, Proc. London Math. Soc. (2) **20**, 398–407 (1922).
[11] S. Timoshenko, *Theory of elasticity*, McGraw-Hill Book Co., Inc., New York, 1934, pp. 292–298.
[12] A. E. H. Love, *loc. cit.*, pp. 327–346.

From Eqs. (42) and (46) follows for the relevant stress couples as defined by equations (10e) and (10f),

$$M_x = (1 + \nu)(Cx - B), \quad (47) \qquad H = -\nu Cy + Ak \sinh (y/k). \quad (48)$$

The five constants of integration A, B, C, F, I are determined by the following five conditions

$$w = \beta_x = 0 \text{ for } x = 0, \ y = 0 \quad (49)$$

$$M_x = 0, \qquad \int_{-a}^{a} V_x dy = P \quad \text{for} \quad x = l, \quad (50)$$

$$H = 0 \quad \text{for} \quad y = \pm a. \quad (51)$$

It is apparent that as in Saint Venant's theory it is not possible to satisfy the condition of complete restraint at the fixed end and also the actual distribution of the terminal load cannot be prescribed but only its resultant. As a consequence of this the solution has general validity only at distances from the ends $x=0$ and $x=l$ which are at least of the order of magnitude of the width $2a$ of the plate.

With β_x from the fourth Eq. (8), Eqs. (49) become

$$I = 0, \qquad \frac{F}{D} = \frac{C + A}{C_s}. \quad (52)$$

Equations (50) become, with (47) and (42),

$$Cl - B = 0, \qquad 2aC + 2kA \sinh (a/k) = P. \quad (53)$$

Equation (51) becomes, with (48),

$$-\nu aC + Ak \sinh (a/k) = 0. \quad (54)$$

Solving Eqs. (52) to (54) and substituting into Eqs. (42), (46), (47) and (48), we obtain the following relations for the stresses and deflections

$$V_x = \frac{P}{2a}\left[1 + \frac{\nu}{1+\nu}\left(\frac{(a/k)\cosh (y/k)}{\sinh (a/k)} - 1\right)\right], \quad (55)$$

$$w = \frac{Pl^3}{2aD(1-\nu^2)}\left\{\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{1}{6}\left(\frac{x}{l}\right)^3 - \frac{\nu}{2}\left(\frac{y}{l}\right)^2\left(1 - \frac{x}{l}\right) + 2\frac{k^2}{l^2}\left(1 + \nu\frac{a/k}{\sinh (a/k)}\right)\frac{x}{l}\right\}, \quad (56)$$

$$M_x = \frac{P}{2a}\left(\frac{x}{l} - 1\right), \quad (57) \qquad H = \frac{P}{2}\,\frac{-\nu}{1+\nu}\left[\frac{y}{a} - \frac{\sinh (y/k)}{\sinh (a/k)}\right]. \quad (58)$$

Of particular interest is the distribution of shear stress as given by equations (55) and (58). For an isotropic plate the results are similar to a known approximate solution[10,11] for the beam with rectangular cross section. They reduce in fact to this known solution for large values of a/h.

The maximum transverse shear occurs at the ends $y = \pm a$ of the plate,

$$V_{x,\max} = \frac{P}{2a}\left[1 + \frac{\nu}{1+\nu}\left(\frac{a/k}{\tanh a/k} - 1\right)\right]. \tag{59}$$

The factor in brackets is the correction to the result of elementary beam theory. The following table gives numerical values of this factor for the isotropic plate, the orthotropic plate and the sandwich plate, in comparison with the exact values for the isotropic plate and the known approximate values for the isotropic plate.[10,11]

TABLE I. Values of Stress Concentration Factor for Transverse Shear in Cantilever Plate.

a/k	.790	1.581	3.162	6.324	9.486	12.648
$(a/h)_{\text{isotropic}}$ $(a/h)\sqrt{G_z/G}$ $(a/h)\sqrt{\frac{hG_c}{5tG_f}}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4
Eq. (59), $\nu_f=\frac{1}{3}$	1.050	1.180	1.545	2.331	3.121	3.91
Eq. (59), $\nu_f=\frac{1}{4}$	1.040	1.144	1.436	2.065	2.682	3.33
Appr.,[10,11] $\nu=\frac{1}{4}$	1.040	1.143	1.426	1.934		
Exact, $\nu=\frac{1}{4}$	1.033	1.126	1.396	1.988	2.582	3.176

The magnitude of the shear τ_{xy} parallel to the faces of the plate follows from equation (58). τ_{xy} is greatest at the points $(\pm h/2, \pm\eta)$ with η determined from

$$\cosh\frac{\eta}{k} = \frac{\sinh (a/k)}{a/k}. \tag{60a}$$

For sufficiently large values of a/k (practically when $a/k>3$) Eq. (60a) becomes

$$\frac{\eta}{a} = 1 - \frac{\ln (a/k)}{a/k}, \tag{60b}$$

and the corresponding value of $H_{\max}$ is

$$H_{\max} = -\frac{\nu}{1+\nu}\frac{P}{2}\left[1 - \frac{\ln (a/k) + 1}{a/k}\right]. \tag{61a}$$

For homogeneous plates Eq. (61a) gives for the shear stress $\tau_{xy,\max} = 6H_{\max}/h^2$,

$$\frac{2A}{3P}\tau_{xy,\max} = \frac{4}{\sqrt{10}}\sqrt{\frac{G}{G_z}}\frac{\nu}{1+\nu}\left[\frac{a}{k} - \ln\frac{a}{k} - 1\right]. \tag{61b}$$

The following table contains some values of η and of the factor in brackets in the expression for H.

TABLE II. Location and Magnitude of Maximum Shear Stress Couple and Face-Parallel Shear Stress ($\nu=.25$).

a/k	.790	1.581	3.162	6.324	12.65	∞
η/a	.578	.594	.634	.71	.80	1
$\frac{\eta}{a}\frac{\sinh (\eta/k)}{\sin (a/k)}$	.038	.129	.32	.45	.72	1
$\frac{2A}{3P}\sqrt{\frac{G_z}{G}}\tau_{xy,\max}$	.008	.0[illegible]2	.256	.72	2.30	∞
exact[13] $G_z=G$				.968	2.452	∞

By means of (59) and (58) we may calculate the ratio of maximum shear parallel to the plane of the plate and maximum transverse shear. For a homogeneous plate we have, in view of (20) and (21),

$$\frac{\tau_{xy}(\eta,\, h/2)}{\tau_{xz}(a,\,0)} = \frac{4\nu}{1+\nu}\,\frac{a}{h}\left[\frac{\eta}{a} - \frac{\sinh \eta/k}{\sinh a/k}\right]\left[1 + \frac{\nu}{1+\nu}\left(\frac{a/k}{\tanh a/k} - 1\right)\right]^{-1} \tag{62a}$$

and with h/a from equation (24a)

$$\frac{\tau_{xy}(\eta,\, h/2)}{\tau_{xz}(a,\,0)} = \frac{4}{\sqrt{10}}\sqrt{\frac{G}{G_z}}\,\frac{\nu}{1+\nu}\,\frac{a}{k}\left[\frac{\eta}{a} - \frac{\sinh \eta/k}{\sinh a/k}\right]\left[1 + \frac{\nu}{1+\nu}\left(\frac{a/k}{\tanh a/k} - 1\right)\right]^{-1} \tag{62b}$$

From equation (62b) follows in particular the limit relation

$$\lim_{a/k\to\infty} \frac{\tau_{xy}(\eta,\, h/2)}{\tau_{xz}(a,\,0)} = \frac{4}{\sqrt{10}}\sqrt{\frac{G}{G_z}} = 1.266\sqrt{\frac{G}{G_z}} \tag{62c}$$

which is independent of Poisson's ratio. Equation (62c) shows the interesting fact that, for very thin plates, the horizontal shear may be larger than the transverse shear even for isotropic plates. We have confirmed this result for an isotropic plate by an exact calculation[13] in which the factor 1.266 is replaced by a factor 1.342.

The analogue of Eq. (62a) for sandwich plates is obtained, by means of (30), (31) and the first Eq. (34). One finds

$$\frac{\tau_{xy,f}(\eta)}{\tau_{xz,c}(a)} = \frac{\nu}{1+\nu}\sqrt{\frac{hG_f}{2tG_c}}\,\frac{a}{k}\left[\frac{\eta}{a} - \frac{\sinh \eta/k}{\sinh a/k}\right]\left[1 + \frac{\nu}{1+\nu}\left(\frac{a/k}{\tanh a/k} - 1\right)\right]^{-1}. \tag{63}$$

Table III contains values of the stress ratio as given by equations (62a) and (63) for a range of values of a/k and when $\nu = \frac{1}{4}$.

TABLE III. Values of Ratio of Maximum Horizontal Shear Stress to Maximum Transverse Shear Stress for Homogeneous Plates ($\nu = 1/4$) and for Sandwich Plates.

a/k	1.581	3.162	6.324	12.65	30	100	∞
$\sqrt{\frac{16tG_c}{5hG_f}}\frac{\max \tau_f}{\max \tau_c} = \sqrt{\frac{G_z}{G}}\frac{\max \tau_{xy}}{\max \tau_{xz}}$	.046	.179	.470	.695	.950	1.15	1.266

Finally, it may be indicated which form the solution assumes in plate theory without the transverse shear terms. Equations (55) and (58) become

$$V_x = \frac{P}{2a}\,\frac{1}{1+\nu}, \qquad H = -\frac{P}{2}\,\frac{y}{a}\,\frac{\nu}{1+\nu} \tag{64a, b}$$

Eq. (57) remains unchanged and Eq. (56) for the deflection loses the terms involving k.

The load P is thus carried *in part* by transverse shears distributed uniformly across the width of the plate and in part by means of concentrated forces at the edges $y = \pm a$ of the plate. As one would expect, no estimate is possible within the frame of the simpler theory without the transverse shear terms of the actual magnitude of the shear stresses which balance the applied load.

[13] E. Reissner and G. B. Thomas, J. Math. Phys. **25**, 241–243 (1946).