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*BEST ERROR BOUNDS FOR APPROXIMATION BY PIECEWISE POLYNOMIAL FUNCTIONS**

by

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1. INTRODUCTION.

It is the purpose of this paper to study approximation methods which use piecewise polynomial functions. Important examples of such methods are interpolation, best approximation and finite element approximation by polynomials splines, Hermite and finite element functions. They are characterized by the use of families of subspaces of functions S_q^h such that the restriction of any function in S_q^h to an element, typically an interval, a triangle, a simplex or a quadrilateral, is a polynomial of a degree no greater than some constant q in each variable separately. The superscript h denotes a mesh size, i.e., the maximum distance between neighboring spline knots or the largest diameter of any element. We denote the corresponding mesh by M^h and the approximating function by f^h.

There exists a very large literature on error estimates for approximations by piecewise polynomial functions. Many of these results give bounds for the error, and its derivatives, for sufficiently smooth functions. A typical result shows that a L_p-or maximum-norm of the error is $O(h^{q+1})$ if the function f which we approximate is sufficiently smooth. We call such methods optimally accurate. Such results can of course be anticipated from a simple heuristic argument. A good method which uses piecewise polynomials of degree q should, on each element, have an error comparable to

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that of a Taylor series of the same degree centered at some point of the element. We also expect that any polynomial of degree q should be reproduced exactly while some polynomials of degree q+1 should introduce an $O(h^{q+1})$ error.

Many approximation methods are also known to be stable. By stability we mean that the norm of the difference $(f_1+f_2)^h-f_2^h$ is uniformly bounded in terms of the norm of f_1.

In this paper we study the extent by which the rate of convergence suffers when we approximate functions which are not smooth enough to allow the use of the standard error estimates. We first give a precise, direct, Jackson-type, theorem assuming only that the method is stable, optimally accurate and quasi-linear, see Section 3.

Our main technical tools come from the theory of Fourier multipliers and Besov spaces as developed by Hörmander [10], Löfström [11] and Peetre [14,18]. The Besov spaces can be regarded as generalizations of the spaces needed in the precise formulation of the classical theory of best approximation by trigonometric polynomials, see Sections 2 and 4. For the classical theory, developed by Jackson, Bernstein and Zygmund, see Shapiro [20] or Timan [26]. One of our main technical ideas is the use of Peetre's [15] elegant alternative definition of the Besov spaces, see also Peetre [14,18] and Grevholm [8]. For other applications of similar ideas in theoretical numerical analysis, see Brenner, Thomée and Wahlbin [2] and references listed therein.

The Besov spaces can also be characterized by interpolation between Sobolev spaces. Part of our direct theorem can be derived by such an argument, see Hedstrom and Varga [9]. Our proof requires no previous knowledge of interpolation in Banach spaces.

We also give an inverse, Bernstein-Zygmund-type, theorem. This result shows that our direct theorem is the best possible when the mesh is chosen without regard to the particular function which is approximated. For the detailed assumptions and a discussion see Section 3.

Proofs of our direct and inverse theorems are not given in this paper. since they have recently appeared in Widlund [27]. In the last few years a number of papers have appeared which contain these results in special cases. For a discussion of this literature see Widlund [27].

We believe our contribution primarily lies in that we have shown that much of the special structure of the subspaces S_q^h can be ignored.

As an illustration we now compare the best approximation of continuous 2π-periodic functions by trigonometric polynomials and periodic splines of degree q using the same number of parameters. We assume that the spline knots are uniformly distributed and that we take no advantage of possible knowledge of the location of singularities of f or its derivatives. In other words we allow the poorest possible shift of the mesh. The subspaces of 2π-periodic functions which give an $O(n^{-r})$ rate of convergence are then identical for the two methods if $r < q+1$ and it is larger for the first method when $r = q+1$. Furthermore the spline method is saturated in that it has a rate of convergence with an exponent $r > q+1$ only for constant functions while, as is well-known, the subspaces of functions which give an order r convergence using trigonometric polynomials are dense in the space of continuous functions for all positive r.

In Section 4, we consider certain additional questions in the important special case of p=2. Our direct theorem and a simple inclusion result show that any element f in the Sobolev space W_2^s , $0 < s \leq q+1$, can be approximated to within $O(h^s)$ by elements in S_q^h. We refine this result and obtain, for $0 < s < q+1$, a theorem which is very similar to a theorem on the best approximation by entire functions of exponential type. This result is closely related to work on nonuniform error estimates for finite element methods by Stephens [23] and Babuška and Kellogg [1].

Rather than striving for a maximum in generality we have confined our study to the real n-dimensional Euclidean space R^n. An extension of our results to periodic cases is immediate. Our techniques impose few restrictions on the meshes and we therefore believe that our results could be extended to more general regions. See further a remark in Section 2.

It is well known that the full power of approximation with piecewise polynomial functions can only be realized by varying the relative size of the elements of the mesh with the function f. It is also known that polynomial splines, see de Boor [6], and finite element methods, see Strang [24], permit estimates of the error at a point in terms of the smoothness of the function in a small neighborhood. It would be interesting to develop local versions of our theorems. Such results could serve as a guide to the choice of local refinement of the mesh in cases when the smoothness of the function varies between different parts of the region. We note that closely related questions on nonlinear spline approximation are considered in Burchard and Hale [3], McClure [12], Peetre [17] and Rice [19]. Their results show that for certain families of functions a much more rapid convergence can be obtained if the spline knots are chosen in an optimal way.

2. A CATALOG OF BANACH SPACES.

In this section we introduce the Banach spaces in terms of which our results are formulated. The real n-dimensional Euclidean space is denoted by R^n. For $1 \leq p < \infty$, we denote by $W_p(R^n)$ the $L_p(R^n)$-spaces of equivalence classes of measurable functions which have integrable p-th power. These spaces can also be characterized as the completion of $C_o^\infty(R^n)$, the space of infinitely differentiable functions with compact support, with respect to the norm

$$\|f\|_p = \left(\int_{R^n} |f(x)|^p dx\right)^{1/p} .$$

By $W_\infty(R^n)$ we denote the completion of $C_o^\infty(R^n)$ with respect to the norm

$$\|f\|_\infty = \sup_{x \in R^n} |f(x)| .$$

The space W_∞ can also be characterized as the linear subspace of continuous fonctions which go to zero as $|x| \to \infty$. For $x \in R^n$ we always use the Euclidean norm

$$|x| = (\Sigma |x_i|^2)^{1/2}.$$

We introduce the norms of the Sobolev spaces W_p^ℓ by

$$\|f\|_{p,\ell} = \|f\|_p + |f|_{p,\ell}.$$

Here the semi-norm $|f|_{p,\ell}$ is defined by

$$|f|_{p,\ell} = \sup_{|\alpha|=\ell} \|\partial^\alpha f\|_p \quad ,$$

where $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a multi-index, the α_i are non-negative integers and $|\alpha| = \Sigma\alpha_i$. The derivatives ∂^α are defined by $(\partial/\partial x_1)^{\alpha_1}\ldots(\partial/\partial x_n)^{\alpha_n}$ and similarly $x^\alpha = x_1^{\alpha_1}\ldots\, x_n^{\alpha_n}$.

For positive, non integer values of s, we define the space W_p^s, in a standard way, by using Fourier transforms, see for example Peetre [18].

Translation operators $T_{k,h}$ are introduced by

$$T_{k,h}\phi(x) = \phi(x+h\ e_k)$$

where e_k is the unit vector in the direction of the positive k-th coordinate axis. W_p-moduli of continuity are defined by

$$\omega_p^{(s)}(t,f) = \sup_k \sup_{0<h\le t} \|(T_{k,h}-I)^s f\|_p$$

where s=1,2,... and I is the identity operator. For a nonnegative integer ℓ, $0<\gamma\le 1$ and $1\le q\le\infty$ we introduce the Besov spaces $B_p^{\ell+\gamma,q}$ as the subspaces of W_p such that

$$\|f\|_{B_p^{\ell+\gamma,q}} = \|f\|_p + |f|_{B_p^{\ell+\gamma,q}} \quad .$$

Here the semi-norm $|f|_{B_p^{\ell+\gamma,q}}$ is defined by

$$|f|_{B_p^{\ell+\gamma,q}} = \left(\int_0^\infty \left(\frac{\omega_p^{(\ell+1)}(t,f)}{t^{\ell+\gamma}}\right)^q \frac{dt}{t}\right)^{1/q}$$

for $1\le q<\infty$ and $0<\gamma<1$.

For $\gamma=1$ and $1\le q<\infty$, the modulus of continuity $\omega_p^{(\ell+1)}(t,f)$ must be replaced by $\omega_p^{(\ell+2)}(t,f)$ in the definition of the semi-norm $|f|_{B_p^{\ell+\gamma,q}}$ while for $q=\infty$ the semi-norm is defined by

$$|f|_{B_p^{\ell+\gamma,\infty}} = \sup_{t>0} \frac{\omega_p^{(\ell+1)}(t,f)}{t^{\ell+1}}$$

for $0 < \gamma < 1$ and by

$$|f|_{B_p^{\ell+1,\infty}} = \sup_{t>0} \frac{\omega_p^{(\ell+2)}(t,f)}{t^{\ell+1}}$$

for $\gamma=1$.

In this paper we are concerned mainly with the cases $q=\infty$ and $q=2$. It is easy to show that for $0 < \gamma < 1$, $B_p^{\ell+\gamma,\infty}$ are spaces of functions with W_p-Hölder continuous ℓ-th derivatives, see further section 4. For $\gamma=1$ we impose only a so-called Zygmund condition, in terms of second differences, on the ℓ-th derivatives.

It is easy to show that for $q_1 \le q_2$, B_p^{s,q_1} is a subspace continuously embedded in B_p^{s,q_2} and that $B_2^{s,2} = W_2^s$.

We also need certain other Lipschitz spaces which we denote by $Lip_p^{\ell+1}(R^n)$ ℓ a nonnegative integer. They are the subspaces of W_p such that

$$\|f\|_{Lip_p^{\ell+1}} = \|f\|_p + |f|_{Lip_p^{\ell+1}} < \infty .$$

Here

$$|f|_{Lip_p^{\ell+1}} = \sup_{t>0} \frac{\omega_p^{(\ell+1)}(t,f)}{t^{\ell+1}}$$

It is easy to show that $Lip_p^{\ell+1} \subset B_p^{\ell+1,\infty}$, i.e., $Lip_p^{\ell+1}$ is a subspace continuously embedded in $B_p^{\ell+1,\infty}$. The reverse inclusion does not hold in general, see Peetre [14,18] or Stein [22].

The Lipschitz spaces $Lip_p^{\ell+1}$ are identical to the Sobolev spaces $W_p^{\ell+1}$ for $1 < q \le \infty$, see Stein [22], p. 135, 139 and 159. The case p=1 is different. Let f be identically zero except on the interval [0,1] where it takes on the value one. It is then easy to show that $f \in Lip_1^1(R^1)$ but that it fails to belong to $W_1^1(R^1)$.

Remark : The main results of the classical theory on best approximation by trigonometric polynomials can be formulated in terms of Besov spaces. Let us denote by T_n the class of n-th degree trigonometric polynomials with period 2π and by C, throughout this paper, a generic constant. Then

$$\inf_{t_n \in T_n} \|t_n - f\|_\infty \le C\, n^{-(\ell+\gamma)}$$

if and only if $f \in B_\infty^{\ell+\gamma,\infty}$ on the unit circle, see Shapiro [20] or Timam [26]. For extensions to several variables and entire functions of exponential type, see Section 4, Nikolskii [13] and Peetre [18].

Remark : Another classical result, due to Bernstein, see Zygmund [28], p. 241, illustrates that results on Fourier series can often be formulated in terms of Besov spaces : if $f \in B_\infty^{1/2,1}$ on the unit circle its Fourier series converges absolutely.

This result has been extended by Peetre [16] to eigenfunction expansions for a quite general family of self-adjoint, elliptic operators. His paper illustrates that the Fourier transform, used in our work, can be replaced by eigen-function expansions and it therefore suggests possible tools for the extension of our results to general smooth manifolds.

Remark :The Besov spaces play an important role in the theory of elliptic equations. They are trace classes of the Sobolev spaces, W_p^ℓ. Denote by $\mathcal{R}f$ the restriction to R^{n-1} of $f \in W_p^\ell(R^n)$, $\ell \ge 1$, $1 < p < \infty$. Then $\mathcal{R}f \in B_p^{\ell-1/p,p}(R^{n-1})$. This result was first given, in a special case, by Gagliardo [7]. For the general case see Stein [21] or Taibleson [25].

Remark : Extending results by Campanato [4,5], Grevholm [8] has shown that $f \in B_p^{\ell+\gamma,\infty}$ if and only if it can be approximated in W_p to within $O(r^{\ell+\gamma+n/p})$ by a polynomial of degree $q > \ell+\gamma-1$ on any ball of radius r.

The Besov spaces are systematically studied in Taibleson [25]. For very fine introductions to the subject see Brenner, Thomée and Wahlbin [2], Nikolskii [13], Peetre [14,18], and Stein [22]. An alternative characterization of the Besov spaces is given in Section 4.

3. STATEMENT OF THE DIRECT AND INVERSE THEOREMS.

In this section we formulate a number of assumptions, which could also be regarded as axioms, and state our results. Proofs of these results are given in Widlund [27].

We denote by S the linear space of functions for which a certain approximation procedure is defined. The space S typically contains W_p or a dense subspace of W_p as well as all polynomials. Similarly we denote by S_q^h a linear space of approximating functions. In applications S_q^h will typically, but not necessarily, consist of piecewise polynomial functions.

Assumption 1. The approximation method assigns to each $f \in S$ a unique $f^h \in S_q^h$.

Assumption 2. The method is stable in W_p, i.e.

$$\|(f_1+f_2)^h - f_2^h\|_p \leq C\,\|f_1\|_p \text{ for all } f_1 \text{ and } f_2 \text{ in } S \cap W_p.$$

Here C may depend on f_2 but not on h.

Assumption 3. The method is quasi-linear in W_p, i.e.

$$\|(f_1+f_2)^h - (f_1+f_2)\|_p \leq C(\|f_1^h-f_1\|_p + \|f_2^h-f_2\|_p)$$

for all f_1 and f_2 in S.

Assumption 4. The method is optimally accurate, i.e.

$$\|f^h-f\|_p \leq C\, h^{q+1}|f|_{p,q+1}$$

for all sufficiently smooth f.

We note that uniqueness is assumed primarily for the sake of convenience. We could have assumed instead that the bounds in assumptions 2-4 hold for all approximating elements of a function $f \in S$.

We now state our,

<u>Direct theorem</u>. Let an approximation procedure satisfy assumptions 1-4. Then

$$\| f^h - f \|_p \le C\, h^r |f|_{B_p^{r,\infty}}$$

for $0 < r < q+1$. Furthermore

$$\| f^h - f \|_p \le C\, h^{q+1} |f|_{Lip_p^{q+1}}$$

In order to formulate a correct inverse theorem we must introduce an additional assumption. We have already noted, in Section A, that a very small error can be obtained by using many spline knots close to a point where a function f(x) is not very smooth, see in particular examples by Rice [19]. This leads to a requirement that the mesh should be quasi-uniform. In one dimension this means that the ratio between the largest and smallest mesh size is uniformly bounded. Furthermore, if we obtain our liner meshes solely by refinement of a coarser mesh M^h an element of the corresponding space S_q^h will be approximated without error by elements in any subspace with a finer mesh. The elements of S_q^h are normally not smooth enough to belong to $Lip_p^{(q+1)}$. We must therefore further restrict our admissible class of meshes.

<u>Assumption 5</u>. Let h denote the maximum diameter of the elements of a mesh M^h. The restriction of $f^h \in S_q^h$ to an element of M^h is a polynomial of degree at most q in each variable separately. There exists constants N and $\lambda > \nu > 0$ and for every h meshes M^{h_i}, $i=1,2,\dots,N$, with $h_i \le h$, such that the elements of M^{h_i} contain spheres of radius λh with concentric spheres of radius νh the union of which cover the whole of R^n.

<u>Inverse Theorem</u>. Let an approximation method satisfy assumption 5. Then, if

$$\| f^h - f \|_p = O(h^r)\ ,\ 0 < r < q+1\ ,$$

then

$$|f|_{B_p^{r,\infty}} < \infty\ ,$$

if

$$\| f^h - f \|_p = O(h^{q+1})\ ,$$

then

$$|f|_{Lip_p^{q+1}} < \infty ,$$

and if

$$\|f^h - f\|_p = o(h^{q+1})$$

then f is a polynomial of a degree at most q.

4. RATES OF CONVERGENCE FOR ELEMENTS IN W_2^s.

We now specialize to the case of p=2 and adopt, throughout this section, assumptions 1-4. Since according to Section 2, $W_2^s = B_2^{s,2} \subset B_2^{s,\infty}$, our direct theorem shows that $\|f^h - f\|_2 = O(h^s)$ for any $f \in W_2^s$, $0 < s < q+1$.

We now show that this error is $o(h^s)$,more precisely,

Theorem : Let $f \in W_2^s$, $0 < s < q+1$, and consider an approximation procedure which satisfies assumptions 1-4. Then,

$$\int_0^1 (h^{-s} \|f^h - f\|_2)^2 \, dh/h < \infty .$$

We note that an error bound of this type has been given by Babuška and Kellogg[1]. Our proof is built on a comparison with the best approximation by entire functions of exponential type.

Let f_h^* denote the best approximation in W_p of f by entire functions of exponential type of order 1/h. These are the functions with Fourier transforms with support in a ball of radius 1/h. Peetre [18] has shown the following alternative characterization of the Besov spaces :

$$f \in B_p^{s,q} \text{ if and only if } f \in W_p \text{ and}$$

$$\int_0^1 (h^{-s} \|f_h^* - f\|_p)^q \, dh/h < \infty .$$

This result provides a natural extension of the classical result on the torus to the whole of R^n and to a larger family of spaces. For p=q=2 this condition has a form identical with that of the theorem formulated above.

To prove the theorem, we estimate $\|f^h-f\|_2$ in terms of $\|f_h^*-f\|_2$.
By the triangle inequality

$$\|f^h-f\|_2 \leq \|f^h-(f_h^*)^h\|_2 + \|(f_h^*)^h-f_h^*\|_2 + \|f_h^*-f\|_2 .$$

By the stability assumption the first term can be estimated by $C\,\|f_h^*-f\|_2$. To prove the theorem, there only remains to estimate the second term. By assumption 4

$$\|(f_h^*)^h-f_h^*\|_2 \leq C\, h^{q+1}\, |f_h^*|_{2,q+1} .$$

For each h we find $h_o \in (1/4,1/2]$ and an integer K such that $h = h_o\, 2^{-K}$.
We write

$$\partial^\alpha f_h^* = \partial^\alpha f_{h_o}^* + \sum_{k=1}^{K} \partial^\alpha (f^*_{h_o 2^{-k}} - f^*_{h_o 2^{-k+1}})$$

It is easy to see that the different terms of this series are orthogonal and that therefore,

$$\|\partial^\alpha f_h^*\|_2^2 = \|\partial^\alpha f_{h_o}^*\|_2^2 + \sum_{k=1}^{K} \|\partial^\alpha (f^*_{h_o 2^{-k}} - f^*_{h_o 2^{-k+1}})\|_2^2 .$$

The first term will be ignored since it is easy to estimate. For $|\alpha| = q+1$ we obtain, by using simple arguments,

$$\|\partial^\alpha (f^*_{h_o 2^{-k}} - f^*_{h_o 2^{-k+1}})\|_2 \leq C\; 2^{k(q+1)}\, \|f^*_{h_o 2^{-k}} - f^*_{h_o 2^{-k+1}}\|_2$$

$$\leq C\, 2^{k(q+1)}\, \|f^*_{h_o 2^{-k+1}} - f\|_2 .$$

The sum above can therefore be estimated by

$$\sum_{k=0}^{K-1} 2^{2k(q+1)}\, \|f^*_{h_o 2^{-k}} - f\|_2^2 ,$$

which in turn can be estimated by

$$\int_h^1 (t^{-(q+1)}\, \|f_t^*-f\|_2)^2\, dt/t.$$

To conclude the proof of the theorem, we multiply this integral by $h^{2(q-s)+1}$ and integrate by parts with respect to h. The two resulting terms can be estimated easily by the semi-norm given by Peetre.

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