Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **22:1** (2008), 97–106

ON δ–SETS IN γ–SPACES

V. Renuka Devi and D. Sivaraj

Abstract

We consider a collection of subsets of a set X defined in terms of a function on $\wp(X)$, called the $\gamma-$open sets, which is not a topology but we show that some of the results established for topologies are valid for this collection. In particular, we define $\delta_\gamma-$open sets in a $\gamma-$space and characterize its properties. Also, we discuss the properties of $\gamma-$rare sets and characterize $\delta_\gamma-$open sets in terms of $\gamma-$rare sets.

1. Introduction and Preliminaries.

Let X be a nonempty set and $\Gamma = \{\gamma : \wp(X) \to \wp(X) \mid \gamma(A) \subset \gamma(B)$ whenever $A \subset B\}$. Also, the subcollections, $\Gamma_1 = \{\gamma \in \Gamma \mid \gamma(X) = X\}$ and $\Gamma_2 = \{\gamma \in \Gamma \mid \gamma(\gamma(A)) = \gamma(A)$ for every subset A of $X\}$ of Γ are defined in [3]. If $\gamma \in \Gamma$, a subset A of X is said to be $\gamma-$*open* if $A \subset \gamma(A)$ [3]. The complement of a $\gamma-$open set is $\gamma-$*closed*. The family of all $\gamma-$open sets is denoted by μ_γ. In [3, Proposition 1.1], it is established that $\emptyset \in \mu_\gamma$ and arbitrary union of members of μ_γ is again in μ_γ. Collection of subsets of X satisfying these two conditions is called a *generalized topology* in [4]. X need not be $\gamma-$open [3] and so $\emptyset$ need not be $\gamma-$closed. X is $\gamma-$open if $\gamma \in \Gamma_1$ [3]. The intersection of two $\gamma-$open sets need not be $\gamma-$open [3]. The $\gamma-$*interior* of A is the largest $\gamma-$open set contained in A and is denoted by $i_\gamma(A)$. Therefore, A is $\gamma-$open if and only if $A = i_\gamma(A)$. The smallest $\gamma-$closed set containing A is called the $\gamma-$*closure* of A and is denoted by $c_\gamma(A)$. Therefore, A is $\gamma-$closed if and only if $A = c_\gamma(A)$. In [3], it is established that $c_\gamma \in \Gamma_2$, $i_\gamma \in \Gamma_2$, $i_\gamma \circ c_\gamma = i_\gamma c_\gamma \in \Gamma_2$, $c_\gamma i_\gamma \in \Gamma_2$ and $X - i_\gamma(A) = c_\gamma(X - A)$. A subset A of X is said to be $\gamma-$*semiopen* [5] if there exists a $\gamma-$open set G such that $G \subset A \subset c_\gamma(G)$. The complement of a $\gamma-$semiopen set is said to be $\gamma-$*semiclosed.* It is easy to verify that A is $\gamma-$semiopen if and only if $A \subset c_\gamma i_\gamma(A)$ and A is $\gamma-$semiclosed if and only if $i_\gamma(A) = i_\gamma c_\gamma(A) \subset A$. Recall that, a subset A of X is said to be $\gamma-$*dense* if $X = c_\gamma(A)$. $\sigma(\gamma)$ is the family of all $\gamma-$*semiopen* sets, $\pi(\gamma) = \{A \subset X \mid A \subset$

2000 *Mathematics Subject Classification.* 54 A 05, 54 A 10.
Keywords and Phrases. $\gamma-$open, $\gamma-$interior, $\gamma-$closure, $\gamma-$semiopen, $\gamma-$preopen, $\gamma\alpha-$open, $\gamma\beta-$open, $\gamma b-$open sets and generalized topology.
Received: November 16, 2007
Communicated by Dragan S. Djordjević

$i_\gamma c_\gamma(A)\}$ is the family of all $\gamma-$*preopen* sets [4], $\alpha(\gamma) = \{A \subset X \mid A \subset i_\gamma c_\gamma i_\gamma(A)\}$ is the family of all $\gamma\alpha-$*open* sets [4], $\beta(\gamma) = \{A \subset X \mid A \subset c_\gamma i_\gamma c_\gamma(A)\}$ is the family of all $\gamma\beta-$*open* sets [4] and $b(\gamma) = \{A \subset X \mid A \subset c_\gamma i_\gamma(A) \cup i_\gamma c_\gamma(A)\}$ is the family of all $\gamma b-$*open* sets [7]. The *interior* and *closure* operators of these generalized topologies are respectively denoted by, i_σ and c_σ, i_π and c_π, i_α and c_α, i_β and c_β and i_b and c_b. It is clear that $\mu_\gamma \subset \alpha(\gamma) \subset \sigma(\gamma) \cup \pi(\gamma) \subset b(\gamma) \subset \beta(\gamma)$. In [7], a new family of functions defined on $\wp(X)$, denoted by Γ_4, is introduced. $\Gamma_4 = \{\gamma \in \Gamma \mid G \cap \gamma(A) \subset \gamma(G \cap A)$ for every $\gamma-$open set G and $A \subset X\}$. If $\gamma \in \Gamma_4$, then the pair (X, μ_γ) is called a $\gamma-$space. In [7, Example 2.2], it is established that μ_γ is not a topology on X even if $\gamma \in \Gamma_4$ but the intersection of two $\gamma-$open sets is $\gamma-$open. *It is interesting to note that in a topological space (X, τ), if i is the interior operator, then $i \in \Gamma_4$ and the $i-$space is nothing but the topological space* (X, τ). The following lemma will be useful in the sequel.

Lemma 1.1. *If (X, μ_γ) is a $\gamma-$space, then the following hold.*
(a) *If A and B are $\gamma-$open sets, then $A \cap B$ is a $\gamma-$open set* [7, Theorem 2.1].
(b)$i_\gamma(A \cap B) = i_\gamma(A) \cap i_\gamma(B)$ *for every subsets A and B of X* [7, Theorem 2.3(a)].
(c) $c_\gamma(A \cup B) = c_\gamma(A) \cup c_\gamma(B)$ *for every subsets A and B of X* [7, Theorem 2.3(b)].
(d) $c_\gamma(c_\sigma(A)) = c_\gamma(A)$ *for every subset A of X* [7, Theorem 2.5(f)].
(e) $i_\gamma c_\gamma(i_\pi(A)) = i_\gamma c_\gamma(c_\pi(A)) = i_\gamma c_\gamma(A) = i_\pi(c_\gamma(A))$ *for every subset A of X* [7, Theorem 2.7(f)].
(f)$c_\gamma(i_\pi(A)) = c_\gamma i_\gamma c_\gamma(A)$ *for every subset A of X* [7, Theorem 2.7(v)].
(g) *If X is a nonempty set, A is a subset of X and $\gamma \in \Gamma$, then $i_\gamma(c_\sigma(A)) = i_\gamma c_\gamma(A)$* [7, Theorem 2.4(e)].

2. More results in $\gamma-$spaces

In this section, we establish some of the properties of i_γ and c_γ in a $\gamma-$space and also we prove that $i_\gamma \in \Gamma_4$. Also, we characterize $\gamma\beta-$open sets, $\gamma-$locally closed sets and $\gamma-$preopen sets.

Theorem 2.1. *If (X, μ_γ) is a $\gamma-$space, then the following hold.*
(a) *If G is $\gamma-$open and $A \subset X$, then $G \cap i_\gamma(A) = i_\gamma(G \cap A)$ and so $i_\gamma \in \Gamma_4$.*
(b) *If G is $\gamma-$open and $A \subset X$, then $G \cap c_\gamma(A) \subset c_\gamma(G \cap A)$.*
(c) *$i_\gamma(A \cup F) \subset i_\gamma(A) \cup F$ where F is $\gamma-$closed and $A \subset X$.*
(d) *$c_\gamma(A \cup F) = c_\gamma(A) \cup F$ where F is $\gamma-$closed and $A \subset X$.*
(e) *If G is $\gamma-$open and D is $\gamma-$dense, then $c_\gamma(G \cap D) = c_\gamma(G)$.*

Proof. (a) Let G be $\gamma-$open and A be any subset of X. Then $G \cap i_\gamma(A)$ is a $\gamma-$open set by Lemma 1.1(a), such that $G \cap i_\gamma(A) \subset G \cap A$. Therefore, $G \cap i_\gamma(A) \subset i_\gamma(G \cap A) = i_\gamma(G) \cap i_\gamma(A) = G \cap i_\gamma(A)$, by Lemma 1.1(b). Therefore, $G \cap i_\gamma(A) = i_\gamma(G \cap A)$. Since the set of all $i_\gamma-$open sets coincides with the set of all $\gamma-$open sets, it follows that $i_\gamma \in \Gamma_4$.
(b) Let $x \in G \cap c_\gamma(A)$ and U be an arbitrary $\gamma-$open set containing x. Since $U \cap G$ is a $\gamma-$open set containing x and $x \in c_\gamma(A)$, $(U \cap G) \cap A \neq \emptyset$ and so $U \cap (G \cap A) \neq \emptyset$ which implies that $x \in c_\gamma(G \cap A)$. Therefore, $G \cap c_\gamma(A) \subset c_\gamma(G \cap A)$.
(c) Now $X - i_\gamma(A \cup F) = c_\gamma(X - (A \cup F)) = c_\gamma((X - A) \cap (X - F)) \supset c_\gamma(X - A) \cap$

$(X-F)$, by (b). Therefore, $X-i_\gamma(A\cup F)\supset(X-i_\gamma(A))\cap(X-F)=X-(i_\gamma(A)\cup F)$ and so $i_\gamma(A\cup F)\subset i_\gamma(A)\cup F$.
(d) Now $X-c_\gamma(A\cup F)=i_\gamma(X-(A\cup F))=i_\gamma((X-A)\cap(X-F))=i_\gamma(X-A)\cap(X-F)=(X-c_\gamma(A))\cap(X-F)=X-(c_\gamma(A)\cup F)$ and so $c_\gamma(A\cup F)=c_\gamma(A)\cup F$.
(e) Since $G\cap D\subset G$, $c_\gamma(G\cap D)\subset c_\gamma(G)$. By (b), $c_\gamma(G\cap D)\supset c_\gamma(D)\cap G=G$ which implies that $c_\gamma(G\cap D)\supset c_\gamma(G)$ and so $c_\gamma(G\cap D)=c_\gamma(G)$.

The following Theorem 2.2 shows that the intersection of two $\gamma\alpha-$open sets is a $\gamma\alpha-$open set and the intersection of a $\gamma-$semiopen (resp. $\gamma-$preopen, $\gamma\beta-$open, $\gamma b-$open) set with a $\gamma\alpha-$open set is a $\gamma-$semiopen (resp. $\gamma-$preopen, $\gamma\beta-$open, $\gamma b-$open) set. We will use Lemma 1.1(a), Lemma 1.1(b) and Lemma 1.1(c) in the following Theorem without mentioning them explicitly.

Theorem 2.2. *If (X,μ_γ) is a $\gamma-$space, then the following hold.*
(a)$G\cap A$ *is $\gamma-$semiopen (resp.$\gamma-$preopen, $\gamma\beta-$open, $\gamma b-$open) whenever G is $\gamma\alpha-$open and A is $\gamma-$semiopen (resp.$\gamma-$preopen, $\gamma\beta-$open, $\gamma b-$open).*
(b)$G\cap A$ *is $\gamma\alpha-$open whenever G and A are $\gamma\alpha-$open.*

Proof. (a) Suppose G is $\gamma\alpha-$open and A is $\gamma-$semiopen. Then $G\cap A\subset i_\gamma c_\gamma i_\gamma(G)\cap c_\gamma i_\gamma(A)\subset c_\gamma(i_\gamma c_\gamma i_\gamma(G)\cap i_\gamma(A))=c_\gamma i_\gamma(c_\gamma i_\gamma(G)\cap i_\gamma(A))\subset c_\gamma i_\gamma c_\gamma(i_\gamma(G)\cap i_\gamma(A))=c_\gamma i_\gamma c_\gamma i_\gamma(G\cap A)=c_\gamma i_\gamma(G\cap A)$. Therefore, $G\cap A$ is $\gamma-$semiopen.

Suppose G is $\gamma\alpha-$open and A is $\gamma-$preopen. Then $G\cap A\subset i_\gamma c_\gamma i_\gamma(G)\cap i_\gamma c_\gamma(A)=i_\gamma(c_\gamma i_\gamma(G)\cap i_\gamma c_\gamma(A))\subset i_\gamma c_\gamma(i_\gamma(G)\cap i_\gamma c_\gamma(A))=i_\gamma c_\gamma i_\gamma(i_\gamma(G)\cap c_\gamma(A))\subset i_\gamma c_\gamma i_\gamma c_\gamma(i_\gamma(G)\cap A)\subset i_\gamma c_\gamma(G\cap A)$ and so $G\cap A$ is $\gamma-$preopen.

Suppose G is $\gamma\alpha-$open and A is $\gamma\beta-$open. Then $G\cap A\subset i_\gamma c_\gamma i_\gamma(G)\cap c_\gamma i_\gamma c_\gamma(A)\subset c_\gamma(i_\gamma c_\gamma i_\gamma(G)\cap i_\gamma c_\gamma(A))=c_\gamma i_\gamma(c_\gamma i_\gamma(G)\cap i_\gamma c_\gamma(A))\subset c_\gamma i_\gamma c_\gamma(i_\gamma(G)\cap i_\gamma c_\gamma(A))=c_\gamma i_\gamma c_\gamma i_\gamma(i_\gamma(G)\cap c_\gamma(A))\subset c_\gamma i_\gamma c_\gamma i_\gamma c_\gamma(G\cap A)=c_\gamma i_\gamma c_\gamma(G\cap A)$ and so $G\cap A$ is $\gamma\beta-$open.

Suppose G is $\gamma\alpha-$open and A is $\gamma b-$open. Then $G\cap A\subset G\cap(c_\gamma i_\gamma(A)\cup i_\gamma c_\gamma(A))=(G\cap c_\gamma i_\gamma(A))\cup(G\cap i_\gamma c_\gamma(A))\subset c_\gamma i_\gamma(G\cap A)\cup i_\gamma c_\gamma(G\cap A)$ and so $G\cap A$ is $\gamma b-$open.

(b) Suppose G and A are $\gamma\alpha-$open. Then $G\cap A\subset i_\gamma c_\gamma i_\gamma(G)\cap i_\gamma c_\gamma i_\gamma(A)\subset i_\gamma(c_\gamma i_\gamma(G)\cap i_\gamma c_\gamma i_\gamma(A))\subset i_\gamma c_\gamma(i_\gamma(G)\cap i_\gamma c_\gamma i_\gamma(A))=i_\gamma c_\gamma i_\gamma(i_\gamma(G)\cap c_\gamma i_\gamma(A))\subset i_\gamma c_\gamma i_\gamma c_\gamma(i_\gamma(G)\cap i_\gamma(A))\subset i_\gamma c_\gamma i_\gamma c_\gamma i_\gamma(G\cap A)=i_\gamma c_\gamma i_\gamma(G\cap A)$ and so $G\cap A$ is $\gamma\alpha-$open.

Theorem 2.3. *If (X,μ_γ) is a $\gamma-$space, G is $\gamma-$open and $A\subset X$, then the following hold.*
(a)$G\cap i_\sigma(A)\subset i_\sigma(G\cap A)$.
(b)$G\cap i_\alpha(A)\subset i_\alpha(G\cap A)$.
(c)$G\cap i_\pi(A)\subset i_\pi(G\cap A)$.
(d)$G\cap i_\beta(A)\subset i_\beta(G\cap A)$.
(e)$G\cap i_b(A)\subset i_b(G\cap A)$.
(f)$G\cap c_\sigma(A)\subset c_\sigma(G\cap A)$.
(g)$G\cap c_\alpha(A)\subset c_\alpha(G\cap A)$.
(h)$G\cap c_\pi(A)\subset c_\pi(G\cap A)$.
(i)$G\cap c_\beta(A)\subset c_\beta(G\cap A)$.
(j)$G\cap c_b(A)\subset c_b(G\cap A)$.

Proof. (a) Let G be $\gamma-$open and A be a subset of X. Then $G \cap i_\sigma(A)$ is a $\gamma-$semiopen set by Theorem 2.2(a), such that $G \cap i_\sigma(A) \subset G \cap A$. Therefore, $G \cap i_\sigma(A) \subset i_\sigma(G \cap A)$.
Similarly, we can prove (b), (c) (d) and (e).

(f) Let $x \in G \cap c_\sigma(A)$ and U be an arbitrary $\sigma-$open set containing x. Since $U \cap G$ is a $\sigma-$open set containing x and $x \in c_\sigma(A)$, $(U \cap G) \cap A \neq \emptyset$ and so $U \cap (G \cap A) \neq \emptyset$ which implies that $x \in c_\sigma(G \cap A)$. Therefore, $G \cap c_\sigma(A) \subset c_\sigma(G \cap A)$.
Similarly, we can prove (g), (h), (i) and (j).

The following Corollary 2.4 shows that if $\gamma \in \Gamma_4$, then $i_\alpha \in \Gamma_4$ and Theorem 2.3(b) above is also true for $\gamma\alpha-$open sets. The proof follows from Theorem 2.2(b) and the fact that the set of all $\gamma\alpha-$open sets coincides with the set of all $i_\alpha-$open sets.

Corollary 2.4. If (X, μ_γ) is a $\gamma-$space, G and A are subsets of X, then the following hold.
(a) $i_\alpha(G \cap A) = i_\alpha(G) \cap i_\alpha(A)$.
(b) *If G is $\gamma\alpha-$open, then* $G \cap i_\alpha(A) = i_\alpha(G \cap A)$.
(c) $i_\alpha \in \Gamma_4$.

The following Corollary 2.5 follows from Theorem 2.3.

Corollary 2.5. *If (X, μ_γ) is a $\gamma-$space, $A \subset X$ and G is $\gamma-$open, then the following hold.*
(a) $c_\sigma(G \cap c_\sigma(A)) = c_\sigma(G \cap A)$.
(b) $c_\alpha(G \cap c_\alpha(A)) = c_\alpha(G \cap A)$.
(c) $c_\pi(G \cap c_\pi(A)) = c_\pi(G \cap A)$.
(d) $c_\beta(G \cap c_\beta(A)) = c_\beta(G \cap A)$.
(e) $c_b(G \cap c_b(A)) = c_b(G \cap A)$.

Let X be any nonempty set and $\gamma \in \Gamma$. A subset A of X is said to be $\gamma-$*regular* [3] if $A = \gamma(A)$. The following Theorem 2.6 shows that the intersection of two $i_\gamma c_\gamma-$regular sets is again a $i_\gamma c_\gamma-$regular set and Theorem 2.7 below gives characterizations of $\gamma\beta-$open sets in $\gamma-$spaces.

Theorem 2.6. *If (X, μ_γ) is a $\gamma-$space, and A and B are $i_\gamma c_\gamma-$regular sets, then $A \cap B$ is a $i_\gamma c_\gamma-$regular set.*

Proof. Suppose A and B are $i_\gamma c_\gamma-$regular sets. Now $A \cap B = i_\gamma c_\gamma(A) \cap i_\gamma c_\gamma(B) = i_\gamma(c_\gamma(A) \cap c_\gamma(B))$ by Lemma 1.1(b) and so $i_\gamma c_\gamma(A) \cap i_\gamma c_\gamma(B) \supset i_\gamma c_\gamma(A \cap B)$. Since the intersection of two $\gamma-$open set is a $\gamma-$open set, by Lemma 1.1(a), $A \cap B = i_\gamma(A \cap B) \subset i_\gamma c_\gamma(A \cap B)$. Therefore, $A \cap B = i_\gamma c_\gamma(A \cap B)$ which implies that $A \cap B$ is $i_\gamma c_\gamma-$regular.

Theorem 2.7. *If (X, μ_γ) is a $\gamma-$space and A is a subset of X, then the following statements are equivalent.*
(a) *A is $\gamma\beta-$open.*
(b) $c_\gamma(A) = c_\gamma i_\gamma c_\gamma(A)$.
(c) *$c_\gamma(A)$ is $c_\gamma i_\gamma-$regular.*
(d) *There is a $\gamma-$preopen set U such that $U \subset A \subset c_\gamma(U)$.*
(e) *$c_\gamma(A)$ is $\gamma-$semiopen.*
(f) *$c_\sigma(A)$ is $\gamma-$semiopen.*

(g) $c_\pi(A)$ *is* $\gamma\beta-$*open.*

Proof. The equivalence of (a) and (b) is clear.

(a)$\Rightarrow$(c). If A is $\gamma\beta-$open, then $c_\gamma(A) = c_\gamma i_\gamma c_\gamma(A)$ and so $c_\gamma(A)$ is $c_\gamma i_\gamma-$regular.

(c)$\Rightarrow$(d). Let $U = i_\pi(A)$. Then U is a $\gamma-$preopen set such that $U \subset A$. Now $c_\gamma(U) = c_\gamma(i_\pi(A)) = c_\gamma i_\gamma c_\gamma(A)$, by Lemma 1.1(f). Therefore, $c_\gamma(U) = c_\gamma(A)$ and so $U \subset A \subset c_\gamma(U)$.

(d)$\Rightarrow$(a). Suppose U is a $\gamma-$preopen set such that $U \subset A \subset c_\gamma(U)$. Then $c_\gamma(U) = c_\gamma(A)$. Since U is $\gamma-$preopen, $U \subset i_\gamma c_\gamma(U)$ and so $A \subset c_\gamma(A) = c_\gamma(U) \subset c_\gamma i_\gamma c_\gamma(U) \subset c_\gamma i_\gamma c_\gamma(A)$ and so A is $\gamma\beta-$open.

(c) implies (e) is clear.

(e)$\Rightarrow$(f). Suppose $c_\gamma(A)$ is $\gamma-$semiopen. Now $i_\gamma c_\gamma(A) = i_\gamma c_\sigma(A)$, by Lemma 1.1(g) and so $i_\gamma c_\gamma(A) \subset c_\sigma(A) \subset c_\gamma(c_\sigma(A)) = c_\gamma(A)$, by Lemma 1.1(d). Therefore, $i_\gamma c_\gamma(A) \subset c_\sigma(A) \subset c_\gamma(A) \subset c_\gamma i_\gamma c_\gamma(A)$. Since $i_\gamma c_\gamma(A)$ is $\gamma-$open, $c_\sigma(A)$ is $\gamma-$semiopen.

(f)$\Rightarrow$(a). Suppose $c_\sigma(A)$ is $\gamma-$semiopen. Then, $A \subset c_\sigma(A) \subset c_\gamma i_\gamma(c_\sigma(A)) = c_\gamma i_\gamma c_\gamma(A)$, by Lemma 1.1(g) and so A is $\gamma\beta-$open.

(a)$\Rightarrow$(g). Suppose A is $\gamma\beta-$open. Since every $\gamma-$open set is a $\gamma-$preopen set, $c_\pi(A) \subset c_\gamma(A) \subset c_\gamma i_\gamma c_\gamma(A) = c_\gamma i_\gamma c_\gamma(c_\pi(A))$, by Lemma 1.1(e) and so (g) follows.

(g)$\Rightarrow$(a). Suppose $c_\pi(A)$ is $\gamma\beta-$open. Then $A \subset c_\pi(A) \subset c_\gamma i_\gamma c_\gamma(c_\pi(A)) = c_\gamma i_\gamma c_\gamma(A)$, by Lemma 1.1(e). Therefore, A is $\gamma\beta-$open.

Let X be a nonempty set and $\gamma \in \Gamma$. A subset A of X is said to be $\gamma-$*locally closed* if $A = G \cap F$ where G is $\gamma-$open and F is $\gamma-$closed. Since X is $\gamma-$closed, every $\gamma-$open set is a $\gamma-$locally closed set. The following Theorem 2.8 gives a characterization of $\gamma-$locally closed sets, the proof is similar to the proof of the characterizations of locally closed sets [1] in any topological space and hence is omitted. Theorem 2.9 shows that for $\gamma-$dense sets, the concepts $\gamma-$open and $\gamma-$locally closed on the subsets of X are equivalent.

Theorem 2.8. *Let X be a nonempty set, $\gamma \in \Gamma$ and A be a subset of X. Then the following statements are equivalent.*

(a) A *is* $\gamma-$*locally closed.*

(b) $A = G \cap c_\gamma(A)$ *for some* $\gamma-$*open set* G.

(c) $c_\gamma(A) - A$ *is* $\gamma-$*closed.*

(d) $A \cup (X - c_\gamma(A))$ *is* $\gamma-$*open.*

(e) $A \subset i_\gamma(A \cup (X - c_\gamma(A)))$.

Theorem 2.9. *Let X be a nonempty set, $\gamma \in \Gamma$ and A be a $\gamma-$dense subset of X. Then the following statements are equivalent.*

(a) A *is* $\gamma-$*open.*

(b) A *is* $\gamma-$*locally closed.*

Proof. Enough to prove (b) implies (a). Suppose A is $\gamma-$dense and $\gamma-$locally closed. Then $A = G \cap c_\gamma(A)$ for some $\gamma-$open set G. Therefore, $A = G \cap X = G$ and so A is $\gamma-$open.

The following Theorem 2.10 gives decompositions of $\gamma-$open sets in $\gamma-$spaces.

Theorem 2.10. *Let (X, μ_γ) be a $\gamma-$space and A be a subset of X. Then the following statements are equivalent.*

(a) A *is* $\gamma-$*open.*

(b) *A is $\gamma\alpha-$open and $\gamma-$locally closed.*
(c) *A is $\gamma-$preopen and $\gamma-$locally closed.*
Proof. It is enough to prove that (c) implies (a).
(c)⇒(a). Suppose A is $\gamma-$preopen and $\gamma-$locally closed. Since A is $\gamma-$preopen, $A \subset i_\gamma c_\gamma(A)$. Since A is $\gamma-$locally closed, $A = G \cap c_\gamma(A)$ for some $\gamma-$open set G. Now $A = A \cap i_\gamma c_\gamma(A) = (G \cap c_\gamma(A)) \cap i_\gamma c_\gamma(A) = G \cap i_\gamma c_\gamma(A) = i_\gamma(G \cap c_\gamma(A))$, by Lemma 1.1(b). Therefore, $A = i_\gamma(A)$ which implies that A is $\gamma-$open.

The following Theorem 2.11 gives characterizations of $\gamma-$preopen sets in a $\gamma-$space.

Theorem 2.11. *Let (X, μ_γ) be a $\gamma-$space and $A \subset X$. Then the following statements are equivalent.*
(a) $A \in \pi(\gamma)$.
(b) *There is an $i_\gamma c_\gamma-$regular set G such that $A \subset G$ and $c_\gamma(A) = c_\gamma(G)$.*
(c) *$A = G \cap D$ where G is a $i_\gamma c_\gamma-$regular set and D is a $\gamma-$dense set.*
(d) *$A = G \cap D$ where G is a $\gamma-$open set and D is a $\gamma-$dense set.*
Proof. (a)⇒(b). If $A \in \pi(\gamma)$, then $A \subset i_\gamma c_\gamma(A) \subset c_\gamma(A)$ which implies that $c_\gamma(A) \subset c_\gamma i_\gamma c_\gamma(A) \subset c_\gamma(A)$ and so $c_\gamma i_\gamma c_\gamma(A) = c_\gamma(A)$. Let $G = i_\gamma c_\gamma(A)$. Then $A \subset G$ and $i_\gamma c_\gamma(G) = i_\gamma c_\gamma i_\gamma c_\gamma(A) = i_\gamma c_\gamma(A) = G$ which implies that G is $i_\gamma c_\gamma-$regular. Also $c_\gamma(G) = c_\gamma i_\gamma c_\gamma(G) = c_\gamma(A)$.
(b) ⇒(c). Let G be an $i_\gamma c_\gamma-$regular set such that $A \subset G$ and $c_\gamma(A) = c_\gamma(G)$. Let $D = A \cup (X - G)$. Then $A = G \cap D$ where G is $i_\gamma c_\gamma-$regular. Now $c_\gamma(D) = c_\gamma(A \cup (X - G)) = c_\gamma(A) \cup c_\gamma(X - G) = c_\gamma(G) \cup c_\gamma(X - G) = c_\gamma(G \cup (X - G)) = c_\gamma(X) = X$. Hence D is $\gamma-$dense.
(c)⇒(d). The proof follows from the fact that every $i_\gamma c_\gamma-$regular set is a $\gamma-$open set.
(d)⇒(a). Suppose $A = G \cap D$ where G is $\gamma-$open and D is $\gamma-$dense. Now $G = G \cap X = G \cap c_\gamma(D) \subset c_\gamma(G \cap D)$ and so $G = i_\gamma(G) \subset i_\gamma c_\gamma(G \cap D) = i_\gamma c_\gamma(A)$ which implies that $A \subset i_\gamma c_\gamma(A)$. Hence $A \in \pi(\gamma)$.

3. $\delta_\gamma-$open Sets

Let X be a nonempty set, $\gamma \in \Gamma$ and $A \subset X$. A is said to be *$\delta_\gamma-$open* or $A \in \delta_\gamma$ if and only if $i_\gamma c_\gamma(A) \subset c_\gamma i_\gamma(A)$. *In topological spaces, the set of all δ_i-open sets coincides with the set of all $\delta-$sets* [2]. The *$\gamma-$boundary* of a subset A of X, denoted by $bd_\gamma(A)$, is given by $bd_\gamma(A) = c_\gamma(A) - i_\gamma(A) = c_\gamma(A) \cap c_\gamma(X - A)$. A subset A of X is said to be *$\mu_\gamma - rare$* if $i_\gamma c_\gamma(A) = \emptyset$. *In topological spaces, the set of all μ_i-rare sets coincides with the set of all nowhere dense sets.* Every $\mu_\gamma-$rare set is a $\delta_\gamma-$open set, since $i_\gamma c_\gamma(A) = \emptyset \subset c_\gamma i_\gamma(A)$. It is easy to show that every $\gamma-$closed set is a $\delta_\gamma-$open set. The following Theorem 3.1 gives some properties of $\mu_\gamma-$rare sets.

Theorem 3.1. *Let X be a nonempty set and $\gamma \in \Gamma$. Then the following hold.*
(a) *$\emptyset$ is $\mu_\gamma-$rare.*
(b) *Subset of a $\mu_\gamma-$rare set is a $\mu_\gamma-$rare set.*
(c) *If A is a $\mu_\gamma-$rare set, then $bd_\gamma(A)$ is a $\mu_\gamma-$rare set.*

Proof. (a) If $M_\gamma = \cup\{A \mid A \in \mu_\gamma\}$, then $c_\gamma i_\gamma(X) = c_\gamma(M_\gamma) = X$ and so $X - c_\gamma i_\gamma(X) = \emptyset$ which implies that $i_\gamma c_\gamma(\emptyset) = \emptyset$.
(b) The proof is clear.
(c) Since A is $\mu_\gamma-$rare, $i_\gamma c_\gamma(A) = \emptyset$. Now $i_\gamma c_\gamma(bd_\gamma(A)) = i_\gamma c_\gamma(c_\gamma(A) - i_\gamma(A)) = i_\gamma c_\gamma(c_\gamma(A) \cap (X - i_\gamma(A))) \subset i_\gamma(c_\gamma(A) \cap c_\gamma(X - i_\gamma(A))) \subset i_\gamma c_\gamma(A) = \emptyset$. Therefore, $bd_\gamma(A)$ is a $\mu_\gamma-$rare set.

The following Theorems 3.2, 3.3 and 3.4 deal with $\mu_\gamma-$rare sets and $\gamma-$boundary of subsets of X in a $\gamma-$space, which are essential to characterize $\delta_\gamma-$open sets in Theorem 3.9. Also, in a $\gamma-$space, one can easily prove the formulas 1 to 15 in [6, Page 56].

Theorem 3.2. *Let (X, μ_γ) be a $\gamma-$space and A and B be subsets X. Then the following hold.*
(a) *If A is $\gamma-$open, then $bd_\gamma(A) = c_\gamma(A) - A$ is $\mu_\gamma-$rare.*
(b)*$bd_\gamma(A \cup B) \subset bd_\gamma(A) \cup bd_\gamma(B)$.*

Proof. (a) $i_\gamma c_\gamma(c_\gamma(A) - A) = i_\gamma c_\gamma(c_\gamma(A) \cap (X - A)) \subset i_\gamma(c_\gamma(A) \cap c_\gamma(X - A)) = i_\gamma c_\gamma(A) \cap i_\gamma c_\gamma(X - A) = i_\gamma c_\gamma(A) \cap i_\gamma(X - A) = i_\gamma c_\gamma(A) \cap (X - c_\gamma(A)) = \emptyset$.
(b) $bd_\gamma(A\cup B) = c_\gamma(A\cup B)\cap c_\gamma(X-(A\cup B)) = c_\gamma(A\cup B)\cap(c_\gamma(X-A)\cap c_\gamma(X-B)) \subset (c_\gamma(A) \cup c_\gamma(B)) \cap (c_\gamma(X - A) \cap c_\gamma(X - B)) = (c_\gamma(A) \cap (c_\gamma(X - A) \cap c_\gamma(X - B))) \cup (c_\gamma(B) \cap (c_\gamma(X - A) \cap c_\gamma(X - B))) \subset (c_\gamma(A) \cap c_\gamma(X - A)) \cup (c_\gamma(B) \cap c_\gamma(X - B)) = bd_\gamma(A) \cup bd_\gamma(B)$.

Theorem 3.3. *Let (X, μ_γ) be a $\gamma-$space. If A and B are $\mu_\gamma-$rare subsets of X, then $A \cup B$ is also a $\mu_\gamma-$rare set.*

Proof. $i_\gamma c_\gamma(A \cup B) = i_\gamma(c_\gamma(A) \cup c_\gamma(B))$, by Lemma 1.1(c) and so $i_\gamma c_\gamma(A \cup B) \subset i_\gamma c_\gamma(A)\cup c_\gamma(B) = \emptyset\cup c_\gamma(B)$ by Theorem 2.1(c). Therefore, $i_\gamma c_\gamma(A\cup B) \subset i_\gamma c_\gamma(B) = \emptyset$ and so $A \cup B$ is $\mu_\gamma-$rare.

Theorem 3.4. *If (X, μ_γ) is a $\gamma-$space, G is $\gamma-$open and both $A - G$and $G - A$ are $\mu_\gamma-$rare, then $B - H$ and $H - B$ are $\mu_\gamma-$rare, where $H = X - c_\gamma(G)$ and $B = X - A$.*

Proof. Since $A - c_\gamma(G) \subset A - G$ and $A - G$ is $\mu_\gamma-$rare, $A - c_\gamma(G)$ is $\mu_\gamma-$rare. Since $c_\gamma(G) - A = (G - A) \cup ((c_\gamma(G) - G) - A)$, by Theorem 3.1(b) and Theorem 3.3, $c_\gamma(G) - A$ is $\mu_\gamma-$rare. Now $B - H = B - (X - c_\gamma(G)) = (X - A) \cap c_\gamma(G) = c_\gamma(G) - A$ and $H - B = (X - c_\gamma(G)) - B = (X - c_\gamma(G)) - (X - A) = A - c_\gamma(G)$. Therefore, $B - H$ and $H - B$ are $\mu_\gamma-$rare.

The following Theorem 3.5 shows that every $\gamma-$semiopen is a $\delta_\gamma-$open set and the complement of a $\delta_\gamma-$open set is a $\delta_\gamma-$open set. Theorems 3.6 and 3.8 give more properties of $\delta_\gamma-$open sets.

Theorem 3.5. *Let X be a nonempty set and $\gamma \in \Gamma$. Then the following hold.*
(a) *If A is $\gamma-$semiopen, then $A \in \delta_\gamma$.*
(b) *If $A \in \delta_\gamma$, then $X - A \in \delta_\gamma$.*

Proof. (a) If A is $\gamma-$semiopen, then $A \subset c_\gamma i_\gamma(A)$. Now, $i_\gamma c_\gamma(A) \subset i_\gamma c_\gamma c_\gamma i_\gamma(A) \subset c_\gamma i_\gamma(A)$ and so $A \in \delta_\gamma$.
(b) $A \in \delta_\gamma$ implies that $i_\gamma c_\gamma(A) \subset c_\gamma i_\gamma(A)$ and so $X - c_\gamma i_\gamma(A) \subset X - i_\gamma c_\gamma(A)$ which in turn implies that $i_\gamma(X - i_\gamma(A)) \subset c_\gamma(X - c_\gamma(A))$ and so $i_\gamma c_\gamma(X - A) \subset c_\gamma i_\gamma(X - A)$. Hence $X - A \in \delta_\gamma$.

Theorem 3.6. *Let (X, μ_γ) be a $\gamma-$space. If $A \in \delta_\gamma$ and $B \in \delta_\gamma$, then $A \cap B \in \delta_\gamma$.*

Proof. $A, B \in \delta_\gamma$ implies that $i_\gamma c_\gamma(A) \subset c_\gamma i_\gamma(A)$ and $i_\gamma c_\gamma(B) \subset c_\gamma i_\gamma(B)$. Now $i_\gamma c_\gamma(A \cap B) \subset i_\gamma(c_\gamma(A) \cap c_\gamma(B)) = i_\gamma c_\gamma(A) \cap i_\gamma c_\gamma(B)$ by Lemma 1.1(b). Since $A \in \delta_\gamma$, it follows that $i_\gamma c_\gamma(A \cap B) \subset c_\gamma i_\gamma(A) \cap i_\gamma c_\gamma(B) \subset c_\gamma(i_\gamma(A) \cap i_\gamma c_\gamma(B))$, by Theorem 2.1(b). Since $B \in \delta_\gamma$, $i_\gamma c_\gamma(A \cap B) \subset c_\gamma(i_\gamma(A) \cap c_\gamma i_\gamma(B)) \subset c_\gamma c_\gamma(i_\gamma(A) \cap i_\gamma(B)) = c_\gamma(i_\gamma(A) \cap i_\gamma(B)) = c_\gamma i_\gamma(A \cap B)$. Hence $i_\gamma c_\gamma(A \cap B) \subset c_\gamma i_\gamma(A \cap B)$ and so $A \cap B \in \delta_\gamma$.

Corollary 3.7. *Let (X, μ_γ) be a $\gamma-$space. If $A \in \delta_\gamma$ and $B \in \delta_\gamma$, then $A \cup B \in \delta_\gamma$.*

Proof. The proof follows from Theorem 3.5(b) and Theorem 3.6.

Theorem 3.8. *Let (X, μ_γ) be a $\gamma-$space and A and B be subsets of X such that $A \in \delta_\gamma$. Then $i_\gamma c_\gamma(A \cap B) = i_\gamma c_\gamma(A) \cap i_\gamma c_\gamma(B)$.*

Proof. Since $i_\gamma c_\gamma(A)$and $i_\gamma c_\gamma(B)$ are $\gamma-$open sets, $i_\gamma c_\gamma(A) \cap i_\gamma c_\gamma(B)$ is also $\gamma-$open by Lemma 1.1(a) and so $i_\gamma c_\gamma(A) \cap i_\gamma c_\gamma(B) = i_\gamma(i_\gamma c_\gamma(A) \cap i_\gamma c_\gamma(B)) \subset i_\gamma(c_\gamma i_\gamma(A) \cap i_\gamma c_\gamma(B))$, since $A \in \delta_\gamma$. Therefore, $i_\gamma c_\gamma(A) \cap i_\gamma c_\gamma(B) \subset i_\gamma c_\gamma(i_\gamma(A) \cap i_\gamma c_\gamma(B)) \subset i_\gamma c_\gamma(i_\gamma(A) \cap c_\gamma(B)) \subset i_\gamma c_\gamma c_\gamma(i_\gamma(A) \cap B) \subset i_\gamma c_\gamma(A \cap B)$. Also, $i_\gamma c_\gamma(A \cap B) \subset i_\gamma(c_\gamma(A) \cap c_\gamma(B)) = i_\gamma c_\gamma(A) \cap i_\gamma c_\gamma(B)$. Hence $i_\gamma c_\gamma(A \cap B) = i_\gamma c_\gamma(A) \cap i_\gamma c_\gamma(B)$.

Theorem 3.9. *Let (X, μ_γ) be a $\gamma-$space and $A \subset X$. Then the following are equivalent.*

(a) *$A \in \delta_\gamma$.*
(b) *A is the union of a $\gamma-$semiopen set and a $\mu_\gamma-$rare set.*
(c) *A is the union of a $\gamma-$open set and a $\mu_\gamma-$rare set.*
(d) *$bd_\gamma(A)$ is $\mu_\gamma-$rare.*
(e) *There is a $\gamma-$open set G such that $A - G$ and $G - A$ are $\mu_\gamma-$rare.*
(f) *$A = B \cap C$ where B is $\gamma-$semiopen and C is $\gamma-$closed.*
(g) *$A = B \cap C$ where B is $\gamma-$semiopen and C is $\gamma\alpha-$closed.*
(h) *$A = B \cap C$ where B is $\gamma-$semiopen and C is $\gamma-$semiclosed.*

Proof. (a)⇒(b). $A = (A \cap c_\gamma i_\gamma(A)) \cup (A - c_\gamma i_\gamma(A))$. Let $B = A \cap c_\gamma i_\gamma(A)$ and $C = A - c_\gamma i_\gamma(A)$. Then $i_\gamma(A) \subset B$ and $B \subset c_\gamma i_\gamma(A)$ which implies that $B \subset c_\gamma i_\gamma(B)$ and so B is $\gamma-$semiopen. Now $C \cap i_\gamma(A) = (A - c_\gamma i_\gamma(A)) \cap i_\gamma(A) = \emptyset$ and $c_\gamma(C) \cap i_\gamma(A) = c_\gamma(A - c_\gamma i_\gamma(A)) \cap i_\gamma(A) \subset (c_\gamma(A) - i_\gamma c_\gamma i_\gamma(A)) \cap i_\gamma(A) = \emptyset$. Again, by Lemma 1.1(b), $i_\gamma c_\gamma(C) = i_\gamma c_\gamma(A - c_\gamma i_\gamma(A)) \subset i_\gamma(c_\gamma(A) - i_\gamma c_\gamma i_\gamma(A)) = i_\gamma c_\gamma(A) - c_\gamma i_\gamma c_\gamma i_\gamma(A) = i_\gamma c_\gamma(A) - c_\gamma i_\gamma(A)$, since $c_\gamma i_\gamma \in \Gamma_2$. Since $A \in \delta_\gamma, i_\gamma c_\gamma(C) \subset c_\gamma i_\gamma(A) - c_\gamma i_\gamma(A) = \emptyset$ and so C is $\mu_\gamma-$rare.

(b)⇒(c). Suppose $A = B \cup C$ where B is $\gamma-$semiopen and C is $\mu_\gamma-$rare. Since B is $\gamma-$semiopen, there exists a $\gamma-$open set G such that $G \subset B \subset c_\gamma(G)$ and so $B = G \cup (B - G)$. Since $B - G \subset c_\gamma(G) - G$ and $c_\gamma(G) - G$ is $\mu_\gamma-$rare by Theorem 3.2(a), $B - G$ is $\mu_\gamma-$rare. Therefore, $A = G \cup (B - G) \cup C$ and so (c) follows from Theorem 3.3.

(c)⇒(d). Suppose $A = G \cup B$ where G is $\gamma-$open and B is $\mu_\gamma-$rare. Now $bd_\gamma(A) = bd_\gamma(G \cup B) \subset bd_\gamma(G) \cup bd_\gamma(B)$, by Theorem 3.2(b). By Theorem 3.2(a), $bd_\gamma(G)$ is $\mu_\gamma-$rare and by Theorem 3.1(c), $bd_\gamma(B)$ is $\mu_\gamma-$rare. By Theorem 3.3, $bd_\gamma(G) \cup bd_\gamma(B)$ is $\mu_\gamma-$rare and so $bd_\gamma(A)$ is $\mu_\gamma-$rare.

(d)⇒(e). Suppose $G = i_\gamma(A)$. Then $G - A = \emptyset$ and $A - G = A - i_\gamma(A) \subset c_\gamma(A) - i_\gamma(A) = bd_\gamma(A)$. G is the required $\gamma-$open set such that $G - A$ and $A - G$ are $\mu_\gamma-$rare.

(e)⇒(f). Suppose G is a $\gamma-$open set such that $G - A$ and $A - G$ are $\mu_\gamma-$rare sets. If $H = G - c_\gamma(G - A)$, then H is a $\gamma-$open set such that $H \subset A$ and so $H - A$ is

$\mu_\gamma-$rare. Moreover, $A-H = A-(G-c_\gamma(G-A)) = (A-G)\cup c_\gamma(G-A)$. Since $G-A$ and $A-G$ are $\mu_\gamma-$rare, it follows that $A-H$ is $\mu_\gamma-$rare. Thus $A = H\cup(A-H)$, union of a $\gamma-$open set and a $\mu_\gamma-$rare set which is nothing but (c). If $B = X-A$ and $K = X-c_\gamma(H)$, then $B-K$ and $K-B$ are $\mu_\gamma-$rare by Theorem 3.4. Thus K is a $\gamma-$open set such that $B-K$ and $K-B$ are $\mu_\gamma-$rare. Therefore, by (c), $B = U\cup R$ where U is $\gamma-$open and R is $\mu_\gamma-$rare. Hence $A = (X-U)\cap(X-R)$ where $X-U$ is $\gamma-$closed. Now, $c_\gamma i_\gamma(X-R) = X-i_\gamma c_\gamma(R) = X$ and so $X-R$ is $\gamma-$semiopen. Therefore, A is the intersection of a $\gamma-$closed set and a $\gamma-$semiopen set.

(f)⇒(g). The proof follows from the fact that every $\gamma-$closed set is a $\gamma\alpha-$closed set.

(g)⇒(h). The proof follows from the fact that every $\gamma\alpha-$closed set is a $\gamma-$semiclosed set.

(h)⇒(a). Suppose $A = B\cap C$ where B is $\gamma-$semiopen and C is $\gamma-$semiclosed. Now $i_\gamma c_\gamma(A) = i_\gamma c_\gamma(B\cap C) \subset i_\gamma c_\gamma(c_\gamma i_\gamma(B)\cap C) \subset i_\gamma(c_\gamma i_\gamma(B)\cap c_\gamma(C)) = i_\gamma c_\gamma i_\gamma(B)\cap i_\gamma c_\gamma(C) \subset c_\gamma i_\gamma(B)\cap i_\gamma c_\gamma(C) \subset c_\gamma(i_\gamma(B)\cap i_\gamma c_\gamma(C)) = c_\gamma(i_\gamma(B)\cap i_\gamma(C))$, since C is $\gamma-$semiclosed. Therefore, $i_\gamma c_\gamma(A) \subset c_\gamma i_\gamma(B\cap C) = c_\gamma i_\gamma(A)$. Hence A is $\delta_\gamma-$open.

REFERENCES

[1] N. Bourbaki, *General Topology (Part I)*, Addison-Wesley Publishing Company, Inc., 1966.

[2] C. Chattopadhyay and C. Bandyopadhyay, *On structure of $\delta-$sets*, Bull. Calcutta Math. Soc., 83(1991), 281 - 290.

[3] Á. Császár, *Generalized Open Sets*, Acta Math. Hungar., 75(1-2)(1997), 65 - 87.

[4] Á. Császár, *Generalized topology, generalized continuity*, Acta Math. Hungar., 96(2002), 351 - 357.

[5] A. Güldürdek and O. B. Özbakir, *On $\gamma-$semiopen sets*, Acta Math. Hungar., 109(4)(2005), 347 -355.

[6] K. Kuratowski, *Topology I*, Academic Press, New York, 1966.

[7] P. Sivagami, *Remark on $\gamma-$interior*, Acta Math. Hungar., To appear.

Address

V. Renuka Devi:

Department of Mathematics, A. J. College, Sivakasi, Tamil Nadu, INDIA
E-mail: renu_siva2003@yahoo.com

D. Sivaraj:

Department of Computer Applications, D. J. Academy for Managerial Excellence, Coimbatore - 641 032, Tamil Nadu, INDIA
E-mail: ttn_sivaraj@yahoo.com