M. IMADA
KODAI MATH. J.
33 (2010), 135–163

ON BIACCESSIBLE POINTS IN THE JULIA SETS OF SOME RATIONAL FUNCTIONS

Mitsuhiko Imada

Abstract

We are interested in biaccessible points in the Julia sets of rational functions. D. Schleicher and S. Zakeri studied which points can be biaccessible in the Julia sets of quadratic polynomials with irrationally indifferent fixed points [**SZ**, **Za**]. In this paper, we consider the two polynomial families $f_c(z) = z^d + c$, $g_\theta(z) = e^{2\pi i\theta}z + z^d$ and the cubic rational family $h_{\theta,a}(z) = e^{2\pi i\theta}z^2 \dfrac{z-a}{1-\bar{a}z}$.

1. Introduction and results

Let $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ be the Riemann sphere, let $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be a rational function of degree $d \geq 2$. We define the *Fatou set* of f as the union of all open sets $U \subset \hat{\mathbf{C}}$ such that the family of iterates $\{f^{\circ n}|_U\}_{n\geq 0}$ forms a normal family, and the *Julia set* of f as the complement of the Fatou set of f. We denote the Julia set of f by J_f and the Fatou set of f by F_f. Clearly, the Fatou set F_f is open and the Julia set J_f is closed. A connected component of the Fatou set is called a *Fatou component*. Their fundamental properties can be found in [**Mi**].

For each fixed point z_0, the *multiplier* at z_0 is defined as $\lambda = f'(z_0)$ when $z_0 \neq \infty$ and is defined as $\lambda = \lim_{z\to\infty} 1/f'(z)$ when $z_0 = \infty$.

A fixed point z_0 is called *superattracting* if the multiplier λ is equal to zero, or equivalently z_0 is a critical point. Then the point z_0 is contained in the Fatou set F_f. The Fatou component containing the superattracting fixed point z_0 is called the *immediate basin* of z_0, and we denote by $\mathscr{A}_{z_0}$.

A fixed point z_0 is called *irrationally indifferent* if the multiplier λ satisfies $|\lambda| = 1$ but λ is not a root of unity, or equivalently there exists an irrational number θ such that $\lambda = e^{2\pi i\theta}$. So we distinguish between two possibilities.

2000 *Mathematics Subject Classification*. Primary 37F10; Secondary 37F20, 37F50.

Key words and phrases. Julia set, linearizable, biaccessible.

This research was supported in part by JSPS Global COE program "Computationism as a Foundation for the Sciences".

Received December 24, 2008; revised August 27, 2009.

If an irrationally indifferent fixed point z_0 lies in the Fatou set, the point z_0 is called a *Siegel point*. The Fatou component containing a Siegel point z_0 is called the *Siegel disk* with *center* z_0, and we denote by $\mathscr{S}_{z_0}$.

If an irrationally indifferent fixed point z_0 belongs to the Julia set, the point z_0 is called a *Cremer point*. We say that a Cremer point z_0 has the *small cycles property* if every neighborhood of z_0 contains infinitely many periodic orbits. For quadratic polynomials, every Cremer point has the small cycles property [**Yo1**]. However, it is not known whether this is true for arbitrary rational functions.

An invariant Fatou component $\mathscr{H}$ is called a *Herman ring* if $\mathscr{H}$ is conformally isomorphic to some annulus. Then the dynamics of f on $\mathscr{H}$ corresponds to the dynamics of an irrational rotation on this annulus.

Let $\Omega \subset \hat{\mathbf{C}}$ be a simply connected domain. Assume that the boundary $\partial\Omega$ contains at least two points. For the sake of convenience, we assume that Ω contains infinity ∞, and consider a conformal isomorphism $\Phi : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \Omega$ such that $\Phi(\infty) = \infty$. For each *angle* $t \in \mathbf{R}/\mathbf{Z}$, the *external ray* is defined as

$$R_t = \{\Phi(re^{2\pi it}) : r > 1\}.$$

For each *radius* $r > 1$, the *equipotential curve* is defined as

$$E_r = \{\Phi(re^{2\pi it}) : t \in \mathbf{R}/\mathbf{Z}\}.$$

If there exists a point $z \in \partial\Omega$ such that $\lim_{r \searrow 1} \Phi(re^{2\pi it}) = z$, then we say that the external ray R_t *lands* at the point z. A point $z \in \partial\Omega$ is called *accessible* from Ω if there exists a continuous curve $\gamma : [0,1) \to \Omega$ such that $\lim_{s \nearrow 1} \gamma(s) = z$. Then there exists an external ray landing at z (see for example [**Mc**, Corollary 6.4]).

Definition 1.1. We say that a point $z \in \partial\Omega$ is *biaccessible* from Ω if there exist at least two distinct external rays landing at z (see Figure 1).

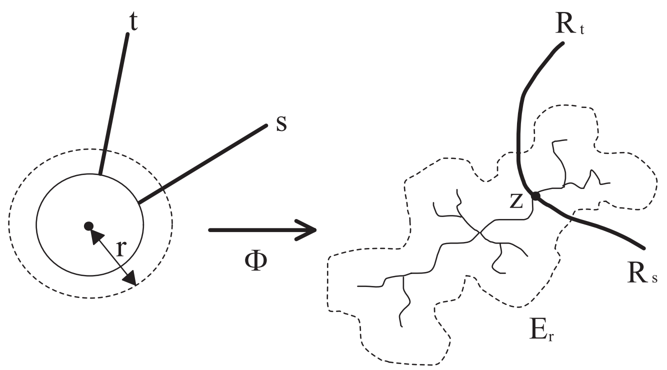

Figure 1

In the above definition, the biaccessibility from Ω does not depend on the choice of the Riemann maps Φ. In fact, it depends only the topology of the boundary $\partial\Omega$. By a theorem of F. and M. Riesz (see [**Mi**]), $\partial\Omega - \{z\}$ is disconnected whenever $z \in \partial\Omega$ is biaccessible from Ω. Moreover, the converse is true (see [**Mc**, Theorem 6.6]). Therefore, $z \in \partial\Omega$ is biaccessible from Ω if and only if $z \in \partial\Omega$ is a *cut point* of $\partial\Omega$, namely $\partial\Omega - \{z\}$ is disconnected.

We are interested in the topological structures of the Julia sets and the boundaries of Fatou components. There are some results about local connectivity (see for example [**Mi**, **P**, **Ra**, **Ro**]) and (bi)accessibility (see for example [**Pe**, **Sch**, **Smi**, **Zd**]). As for Siegel disks, the location of biaccessible points is well known as given in the following proposition.

Proposition 1.1. *Let f be a rational function of degree $d \geq 2$. Assume that infinity ∞ is a Siegel point. Let $\mathscr{S}_\infty$ be the Siegel disk with center ∞. If z is biaccessible from $\mathscr{S}_\infty$, then it is a periodic point of f.*

Proof. We take a conformal isomorphism $\Phi : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \mathscr{S}_\infty$ such that $\Phi(\infty) = \infty$ and $\Phi^{-1} \circ f \circ \Phi(w) = \lambda w$, where λ is the multiplier at ∞. So λ is written as $e^{2\pi i\theta}$ with an irrational number θ. We consider the dynamics of external rays in the Siegel disk $\mathscr{S}_\infty$. It is easy to see $f^{\circ n}(R_t) = R_{t+n\theta}$ for all $n \geq 0$.

If z is biaccessible from $\mathscr{S}_\infty$, then there exist two distinct external rays R_s and R_t landing at z. Since θ is irrational, we may suppose that

$$s < s + N\theta < t < t + N\theta < s + 1,$$

where N is some number. Let U_1 and U_2 be two distinct components of $\mathbf{C} - (R_s \cup \{z\} \cup R_t)$. So we may assume that $f^{\circ N}(R_s) \subset U_1$ and $f^{\circ N}(R_t) \subset U_2$ (see Figure 2).

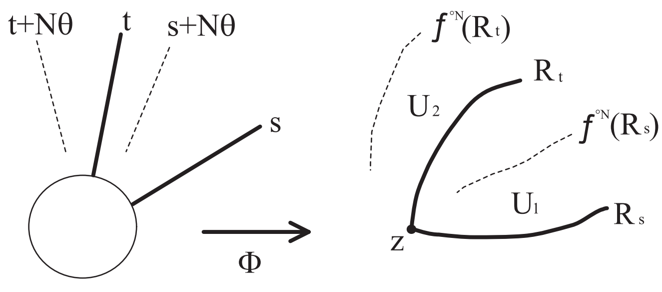

Figure 2

Both $f^{\circ N}(R_s)$ and $f^{\circ N}(R_t)$ land at $f^{\circ N}(z)$ by the continuity of $f^{\circ N}$. Therefore, $f^{\circ N}(z) \subset \overline{U_1} \cap \overline{U_2}$, and thus $f^{\circ N}(z) = z$. $\square$

We consider which points can be biaccessible from the immediate basins of superattracting fixed points. For quadratic polynomials with irrationally indifferent fixed points, S. Zakeri [**Za**] showed the following proposition which is an improvement of [**SZ**, Theorem 3].

Proposition 1.2. *Let $f_c(z) = z^2 + c$ be a quadratic polynomial with an irrationally indifferent fixed point α. Assume that z_0 is biaccessible from the immediate basin $\mathscr{A}_\infty$ of infinity. Then*:

- *if α is a Siegel point, the critical point 0 is contained in the forward orbit $\{f_c^{\circ n}(z_0)\}_{n \geq 0}$ of z_0;*
- *if α is a Cremer point, then the point α is contained in the forward orbit $\{f_c^{\circ n}(z_0)\}_{n \geq 0}$ of z_0.*

In the above proposition, if α is a Cremer point, we are interested in whether the point α is accessible or not. In fact, this is an open problem. If the point α is accessible, then it follows from the Snail Lemma that infinitely many external rays land at the point.

In this paper, we shall extend Proposition 1.2 for more general polynomials and some rational functions of degree 3. In fact, such functions are well known and selected so as to have simple locations of critical points. However, we deal with the biaccessibility of Fatou components of genuine rational functions, which probably has not been studied as yet.

First, we will show the following which is a small extension of the proposition for polynomials with only one critical point in $\mathbf{C}$.

Theorem 1.1. *Let $f_c(z) = z^d + c$ be a polynomial of degree $d \geq 2$ with an irrationally indifferent fixed point α. Assume that z_0 is biaccessible from the immediate basin $\mathscr{A}_\infty$ of infinity. Then*:

- *if α is a Siegel point, the critical point 0 is contained in the forward orbit $\{f_c^{\circ n}(z_0)\}_{n \geq 0}$ of z_0;*
- *if α is a Cremer point, either the point α is contained in the forward orbit $\{f_c^{\circ n}(z_0)\}_{n \geq 0}$ of z_0 or the critical point 0 is contained in the forward orbit $\{f_c^{\circ n}(z_0)\}_{n \geq 0}$ of z_0.*

In the above theorem, if α is a Cremer point which has the small cycles property, then the critical point 0 is not accessible from $\mathscr{A}_\infty$ [**Ki**, Theorem 1.1]. Then $0 \notin \{f_c^{\circ n}(z_0)\}_{n \geq 0}$, and so we can conclude that $\alpha \in \{f_c^{\circ n}(z_0)\}_{n \geq 0}$. According to [**Yo1**], every Cremer point of quadratic polynomials has the small cycles property, so the conclusion of the second part in Proposition 1.2 is just $\alpha \in \{f_c^{\circ n}(z_0)\}_{n \geq 0}$.

The following theorem gives an extension for some polynomials having more than one critical point in $\mathbf{C}$. However, we can make use to the symmetrical locations of critical points.

Theorem 1.2. *Let $g_\theta(z) = e^{2\pi i \theta} z + z^d$ be a polynomial of degree $d \geq 2$ so that the origin is an irrationally indifferent fixed point. Let $c_0, c_1, \ldots, c_{d-2}$ be all* *critical points of* g_θ *in* $\mathbf{C}$. *Assume that* z_0 *is biaccessible from the immediate basin* $\mathscr{A}_\infty$ *of infinity. Then*:

- *if the origin is a Siegel point, there exists a critical point* c_{j_0} *which is contained in the forward orbit* $\{g_\theta^{\circ n}(z_0)\}_{n\geq 0}$ *of* z_0;
- *if the origin is a Cremer point, either the origin is contained in the forward orbit* $\{g_\theta^{\circ n}(z_0)\}_{n\geq 0}$ *of* z_0 *or there exists a critical point* c_{j_0} *which is contained in the forward orbit* $\{g_\theta^{\circ n}(z_0)\}_{n\geq 0}$ *of* z_0.

In the above theorem, if the origin is a Cremer point which has the small cycles property, then there exists a critical point c_{j_0} which is not accessible from $\mathscr{A}_\infty$ [**Ki**, Theorem 1.1]. In addition, the symmetry of the Julia set implies that every critical point c_j is not accessible from $\mathscr{A}_\infty$ (see Section 5). Therefore, $c_j \notin \{g_\theta^{\circ n}(z_0)\}_{n\geq 0}$ for all j, and so we can conclude that $0 \in \{g_\theta^{\circ n}(z_0)\}_{n\geq 0}$.

Finally, we will consider some rational functions of degree 3 which are corresponding to quadratic polynomials with irrationally indifferent fixed points in a sense. Indeed, the dynamics of analytic circle diffeomorphisms with irrational rotation numbers and the local dynamics of irrationally indifferent fixed points are similar in certain respects. So we will suggest a new application of Herman compacta to the proof of the following theorem.

Theorem 1.3. *Let* $h(z) = h_{\theta,a}(z) = e^{2\pi i\theta} z^2 \dfrac{z-a}{1-\bar{a}z}$ *be a rational function so that* $|a| > 3$ *and the rotation number* $\mathrm{Rot}(h|_{\mathbf{S}^1})$ *is irrational. Let* c *be the critical point of* h *such that* $|c| > 1$. *Assume that* z_0 *is biaccessible from the immediate basin* $\mathscr{A}_\infty$ *of infinity. Then the critical point* c *is contained in the forward orbit* $\{h^{\circ n}(z_0)\}_{n\geq 0}$ *of* z_0.

In the above theorem, we fix $|a| > 3$ and consider the one-parameter family $h_{\theta,a}(z) = e^{2\pi i\theta} z^2 \dfrac{z-a}{1-\bar{a}z}$ with θ of rational functions. From the continuity and the monotonous increasing of the rotation function $\theta \mapsto \mathrm{Rot}(h_{\theta,a}|_{\mathbf{S}^1})$, we can adjust the rotation number to be any desired irrational constant (see [**MS**, Section I.4]).

2. Local dynamics

In this section, we suppose that f is a rational function of degree $d \geq 2$ and consider the local dynamics of f. We introduce Siegel compacta and Herman compacta. They are essential for the proofs of the theorems. First, we mention about the linearizability.

Definition 2.1. Let z_0 be an irrationally indifferent fixed point of f. Let λ be the multiplier at z_0, so it is written as $e^{2\pi i\theta}$ with an irrational number θ. If there exists a local holomorphic change of coordinate $z = \Phi(w)$, with $\Phi(0) = z_0$, such that $\Phi^{-1} \circ f \circ \Phi$ is the irrational rotation $w \mapsto e^{2\pi i\theta} w$ near the origin, then we say that f is *linearizable* at the point z_0.

An irrationally indifferent fixed point z_0 of f is either a Siegel point or a Cremer point, according to whether f is linearizable at the point z_0 or not. There are some results about the linearizability of irrationally indifferent fixed points (see for example [**Mi**, Section 11]).

Definition 2.2. Assume that $f|_{\mathbf{S}^1} : \mathbf{S}^1 \to \mathbf{S}^1$ is an analytic circle diffeomorphism whose rotation number $\mathrm{Rot}(f|_{\mathbf{S}^1})$ is irrational. If there exists an analytic circle diffeomorphism $\Phi : \mathbf{S}^1 \to \mathbf{S}^1$ such that $\Phi^{-1} \circ f \circ \Phi$ is the irrational rotation $w \mapsto e^{2\pi i \mathrm{Rot}(f|_{\mathbf{S}^1})} w$, then we say that f is *linearizable* on $\mathbf{S}^1$.

For a general theory on analytic circle diffeomorphisms, we refer to [**MS**]. There are some results about the linearizability for analytic circle diffeomorphisms with irrational rotation numbers (see for example [**Yo2**]). In addition, there are fine theorem correspondences between the linearizability of irrationally indifferent fixed points and the linearizability for analytic circle diffeomorphisms with irrational rotation numbers (see [**PM**, Theorem I.4.1]).

The following two propositions will be used for the proofs of Theorem 1.1 and Theorem 1.2.

Proposition 2.1. *Let z_0 be an irrationally indifferent fixed point of f. Let U be a bounded neighborhood of z_0 so that the boundary ∂U is a Jordan closed curve. Assume that f is univalent on a neighborhood of $\overline{U}$. Then there exists a set S with the following properties*:

- *S is compact, connected, and $\hat{\mathbf{C}} - S$ is connected*;
- *$z_0 \in S \subset \overline{U}$, $S \cap \partial U \neq \emptyset$, and $f(S) = S$.*

Moreover, f is linearizable at z_0 if and only if the interior $\mathrm{Int}\, S$ *of S contains z_0.*

We say that such a set S is a *Siegel compactum* for (f, U). Its applications can be found in [**PM**, Section IV]. The above proposition is described in [**PM**, Theorem 1], however, we do not assume that f^{-1} is defined and univalent on a neighborhood of $\overline{U}$. In fact, the condition leaves no impression on the results.

Proposition 2.2. *Assuming the hypothesis in Proposition* 2.1, *let S be a Siegel compactum for (f, U). Then*:

- *if z_0 is a Siegel point, there are no points which are biaccessible from $\hat{\mathbf{C}} - S$*;
- *if z_0 is a Cremer point, then the point z_0 is the only possible point which is biaccessible from $\hat{\mathbf{C}} - S$.*

Proof. This proof is referred from the explanations of [**Za**, Proposition 1] and [**SZ**, Proposition 2]. We use proof by contradiction.

First, assume that z_0 is a Siegel point and there exists a point z which is biaccessible from $\hat{\mathbf{C}} - S$. Let $\Phi : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \hat{\mathbf{C}} - S$ be a conformal isomorphism such that $\Phi(\infty) = \infty$. So $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on an outer neighborhood of $\mathbf{S}^1$. Then g is extended and univalent on a neighborhood of $\mathbf{S}^1$ by the reflection principle. Furthermore, the rotation number $\mathrm{Rot}(g|_{\mathbf{S}^1})$ corresponds to the irrational number θ which satisfies $\lambda = e^{2\pi i\theta}$, where λ is the multiplier at z_0 [**PM**, Theorem 2].

Let R_s and R_t be two distinct external rays land at z. Let X be the component of Int S which contains the Siegel point z_0. Clearly, $f(X) = X$. Let V be the component of $\mathbf{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain X. We cut off V along an equipotential curve E_r, and thus have the Jordan domain W which is contained in V. Then $D = \Phi^{-1}(W - S)$ has the interval $I \subset \mathbf{S}^1$ as a part of its boundary (see Figure 3).

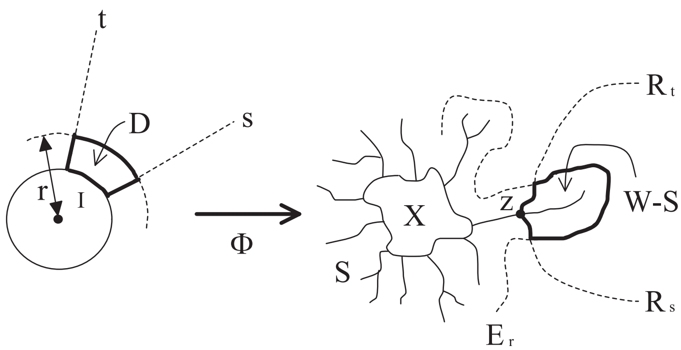

Figure 3

Since the rotation number $\mathrm{Rot}(g|_{\mathbf{S}^1})$ is irrational, there exists N such that $\bigcup_{j=0}^{N} g^{\circ j}(I) = \mathbf{S}^1$. We could take a more smaller $r > 1$, so that $g, g^{\circ 2}, \dots, g^{\circ N}$ are univalent on D, and furthermore, $\bigcup_{j=0}^{N} g^{\circ j}(D)$ is an outer neighborhood of $\mathbf{S}^1$.

Then $f, f^{\circ 2}, \dots, f^{\circ N}$ are univalent on $W - S$, and thus $\bigcup_{j=0}^{N} f^{\circ j}(W - S)$ is an outer neighborhood of S. So any point of the boundary $\partial X \subset \partial S$ can be approximated by some sequence in $\bigcup_{j=0}^{N} f^{\circ j}(W - S)$. Now the injectivity of f implies that each Jordan domain $f^{\circ j}(W)$ does not intersect X, therefore, $\overline{f^{\circ j}(W)} \cap \overline{X}$ contains at most one point $f^{\circ j}(z)$. This contradicts that ∂X has infinitely many points.

Now, assume that z_0 is a Cremer point and there exists a point $z \neq z_0$ which is biaccessible from $\hat{\mathbf{C}} - S$. Let $\Phi : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \hat{\mathbf{C}} - S$ be a conformal isomorphism such that $\Phi(\infty) = \infty$. So $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on an outer neighborhood of $\mathbf{S}^1$. Then g is extended and univalent on a neighborhood of $\mathbf{S}^1$ by the reflection principle. Furthermore, the rotation number $\mathrm{Rot}(g|_{\mathbf{S}^1})$ corresponds to the irrational number θ which satisfies $\lambda = e^{2\pi i\theta}$, where λ is the multiplier at z_0 [**PM**, Theorem 2].

Let R_s and R_t be two distinct external rays land at z. Let V be the component of $\mathbf{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain z_0. We cut off V along an equipotential curve E_r, and thus have the Jordan domain W which is contained in V. Then $D = \Phi^{-1}(W - S)$ has the interval $I \subset \mathbf{S}^1$ as a part of its boundary (see Figure 4).

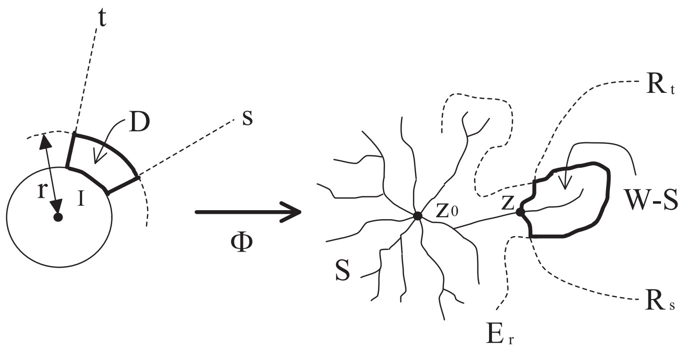

FIGURE 4

Since the rotation number $\mathrm{Rot}(g|_{\mathbf{S}^1})$ is irrational, there exists N such that $\bigcup_{j=0}^{N} g^{\circ j}(I) = \mathbf{S}^1$. We could take a more smaller $r > 1$, so that $g, g^{\circ 2}, \dots, g^{\circ N}$ are univalent on D, and furthermore, $\bigcup_{j=0}^{N} g^{\circ j}(D)$ is an outer neighborhood of $\mathbf{S}^1$.

Then $f, f^{\circ 2}, \dots, f^{\circ N}$ are univalent on $W - S$, and thus $\bigcup_{j=0}^{N} f^{\circ j}(W - S)$ is an outer neighborhood of S. So the Cremer point $z_0 \in \partial S$ can be approximated by some sequence in $\bigcup_{j=0}^{N} f^{\circ j}(W - S)$. However, the injectivity of f implies that each Jordan domain $f^{\circ j}(W)$ does not contain z_0 and each $f^{\circ j}(z)$ is not z_0, therefore, $\overline{f^{\circ j}(W)} \cap \{z_0\} = \emptyset$. □

The following two propositions will be used for the proof of Theorem 1.3.

PROPOSITION 2.3. *Let U be a bounded annular neighborhood of $\mathbf{S}^1$ such that the boundary ∂U consists of two Jordan closed curves $\gamma_1 \subset \mathbf{C} - \overline{\mathbf{D}}$ and $\gamma_2 \subset \mathbf{D}$. Assume that f is univalent on a neighborhood of $\overline{U}$ and $f|_{\mathbf{S}^1} : \mathbf{S}^1 \to \mathbf{S}^1$ is an analytic circle diffeomorphism whose rotation number* $\mathrm{Rot}(f|_{\mathbf{S}^1})$ *is irrational. Assume that $f(U)$ does not contain the bounded component of $\hat{\mathbf{C}} - \gamma_2$. Then there exists a set H with the following properties*:

- *H is compact, connected, and $\hat{\mathbf{C}} - H$ has just two connected components*;
- *$\mathbf{S}^1 \subset H \subset \overline{U}$, $H \cap \gamma_1 \neq \emptyset$, $H \cap \gamma_2 \neq \emptyset$, and $f(H) = H$.*

Moreover, f is linearizable on $\mathbf{S}^1$ if and only if the interior Int H *of H contains $\mathbf{S}^1$.*

We say that such a set H is a *Herman compactum* for (f, U). The above proposition is described in [**PM**, Theorem V.1.1]. We do not assume that f^{-1} is defined and univalent on a neighborhood of $\overline{U}$, however, we add the assumption that $f(U)$ does not contain the bounded component of $\hat{\mathbf{C}} - \gamma_2$.

Proposition 2.4. *Assuming the hypothesis in Proposition* 2.3, *let* H *be a Herman compactum for* (f, U). *Then there are no points which are biaccessible from the unbounded component of* $\hat{\mathbf{C}} - H$.

In the rest of this section, we shall show the above two propositions.

Lemma 2.1. *Let* U *be a bounded annular neighborhood of* $\mathbf{S}^1$ *such that the boundary* ∂U *consists of two Jordan closed curves* $\gamma_1 \subset \mathbf{C} - \overline{\mathbf{D}}$ *and* $\gamma_2 \subset \mathbf{D}$. *Assume that* f *is univalent on a neighborhood of* $\overline{U}$ *and* $f|_{\mathbf{S}^1} : \mathbf{S}^1 \to \mathbf{S}^1$ *is an analytic circle diffeomorphism whose rotation number* $\mathrm{Rot}(f|_{\mathbf{S}^1})$ *is Diophantine. Then the Herman ring* $\mathscr{H}$ *intersects both* γ_1 *and* γ_2.

Proof. This proof is referred from the proof of [**PM**, Theorem II.3.1]. Since the rotation number $\mathrm{Rot}(f|_{\mathbf{S}^1})$ is Diophantine, f is linearizable on $\mathbf{S}^1$ [**Yo2**, Theorem 1.4]. So we have the Herman ring $\mathscr{H}$ such that $\mathbf{S}^1 \subset \mathscr{H}$.

We use proof by contradiction. Assume that $\mathscr{H} \cap \gamma_1 = \emptyset$. Let $\{K_n\}_{n \geq 1}$ be a sequence of closed annuli in the Herman ring $\mathscr{H}$ such that $f(K_n) = K_n$, $K_n \subset \mathrm{Int}\, K_{n+1}$ and $\bigcup_{n=1}^{+\infty} K_n = \mathscr{H}$. So K_n converges to $\overline{\mathscr{H}}$ in the sense of Haus dorff convergence. Let Ω_n be the unbounded component of $\hat{\mathbf{C}} - K_n$, let Ω be the unbounded component of $\hat{\mathbf{C}} - \overline{\mathscr{H}}$. So Ω_n converges to Ω with respect to ∞ in the sense of Carathéodory kernel convergence. We consider the following conformal isomorphisms

$$\Phi_n : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \Omega_n, \quad \Phi : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \Omega$$

so that $\Phi_n(\infty) = \Phi(\infty) = \infty$, $\lim_{z \to \infty} \Phi_n(z)/z > 0$ and $\lim_{z \to \infty} \Phi(z)/z > 0$. So Φ_n converges locally uniformly to Φ by the Carathéodory kernel theorem (see for example [**Po**, Theorem 1.8]).

Since f is univalent on a neighborhood of $\overline{U}$ and $\mathscr{H} \cap \gamma_1 = \emptyset$, there exists $r_0 > 1$ such that $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on $\{z : 1 < |z| < r_0\}$. So $g_n = \Phi_n^{-1} \circ f \circ \Phi_n$ is also univalent on $\{z : 1 < |z| < r_0\}$. By the reflection principle, g_n and g are extended and univalent on $\{z : 1/r_0 < |z| < r_0\}$. We fix r such that $1 < r < r_0$. Since Φ_n converges locally uniformly to Φ, g_n converges uniformly to g on $r\mathbf{S}^1$. So g_n converges uniformly to g on $1/r\mathbf{S}^1$. By the maximum principle, g_n converges uniformly to g on $\{z : 1/r \leq |z| \leq r\}$, particularly on $\mathbf{S}^1$.

Let L_n be the outer boundary of K_n, let L be the outer boundary of the Herman ring $\mathscr{H}$. We notice that the dynamics of g_n on $\mathbf{S}^1$ corresponds to the dynamics of f on L_n. Since L_n is a Jordan closed curve in the Herman ring $\mathscr{H}$ such that $f(L_n) = L_n$, the dynamics of f on L_n corresponds to the dynamics of the irrational rotation $z \mapsto e^{2\pi i\, \mathrm{Rot}(f|_{\mathbf{S}^1})} z$. Therefore, the rotation number $\mathrm{Rot}(g_n|_{\mathbf{S}^1})$ corresponds to $\mathrm{Rot}(f|_{\mathbf{S}^1})$. Then,

$$\mathrm{Rot}(g|_{\mathbf{S}^1}) = \lim_{n \to +\infty} \mathrm{Rot}(g_n|_{\mathbf{S}^1}) = \lim_{n \to +\infty} \mathrm{Rot}(f|_{\mathbf{S}^1}) = \mathrm{Rot}(f|_{\mathbf{S}^1}).$$

Therefore, $\mathrm{Rot}(g|_{\mathbf{S}^1})$ is Diophantine, and thus g is linearizable on $\mathbf{S}^1$. So we can take a Jordan closed curve η in an outer neighborhood of $\mathbf{S}^1$ such that $g(\eta) = \eta$, and thus $\Phi(\eta)$ is a Jordan closed curve such that $f(\Phi(\eta)) = \Phi(\eta)$. Let V be the Jordan annular domain which is surrounded by $\Phi(\eta)$ and $\mathbf{S}^1$ (see Figure 5).

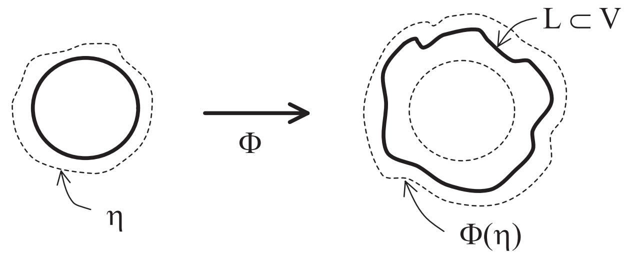

FIGURE 5

We notice $f(V) = V$. Moreover, the dynamics of f on V corresponds to the dynamics of the irrational rotation $z \mapsto e^{2\pi i \mathrm{Rot}(f|_{\mathbf{S}^1})}z$ by the classification theorem of dynamics on hyperbolic surfaces (see for example [**Mi**, Theorem 5.2]). Then $L \subset V \subset F_f$. This contradicts that L is the outer boundary of the Herman ring $\mathscr{H}$. Therefore, we conclude $\mathscr{H} \cap \gamma_1 \neq \emptyset$. It is possible to see $\mathscr{H} \cap \gamma_2 \neq \emptyset$, as in the above argument. □

Proof of Proposition 2.3. This proof is referred from [**PM**, Section III.2]. Since the rotation number $\mathrm{Rot}(f|_{\mathbf{S}^1})$ is irrational, there exists a sequence $\{\alpha_n\}_{n\geq 1}$ such that $\lim_{n\to+\infty} \alpha_n = 0$ and each $f_n(z) = e^{2\pi i\alpha_n} f(z)$ has the rotation number $\mathrm{Rot}(f_n|_{\mathbf{S}^1})$ which is Diophantine (see also [**MS**, Lemma 4.1]). So f_n is univalent on a neighborhood of $\overline{U}$.

From Lemma 2.1, we take the closed annulus H_n in the Herman ring $\mathscr{H}_n$ of f_n with the following properties:

- H_n is compact, connected, and $\hat{\mathbf{C}} - H_n$ has just two connected components;
- $\mathbf{S}^1 \subset H_n \subset \overline{U}$, $H_n \cap \gamma_1 \neq \emptyset$, $H_n \cap \gamma_2 \neq \emptyset$, and $f_n(H_n) = H_n$.

Every H_n is contained in $\overline{U}$, so there exists a subsequence $\{H_{n_i}\}_{i\geq 1}$ and a set H' such that H_{n_i} converges to H' in the sense of Hausdorff convergence. Then H' has the following properties:

- H' is compact and connected;
- $\mathbf{S}^1 \subset H' \subset \overline{U}$, $H' \cap \gamma_1 \neq \emptyset$ and $H' \cap \gamma_2 \neq \emptyset$.

Since f_{n_i} converges uniformly to f on $\overline{U}$, it follows from [**PM**, Lemma III.1.2] that $f_{n_i}(H_{n_i})$ converges to $f(H')$ in the sense of Hausdorff convergence. Then $f_{n_i}(H_{n_i}) = H_{n_i}$ implies $f(H') = H'$. Let H be the union of H' and all the components of $\hat{\mathbf{C}} - H'$ contained in U. So $\hat{\mathbf{C}} - H$ has just two connected components. Since $f(U)$ does not contain the bounded component of $\hat{\mathbf{C}} - \gamma_2$, it is not difficult to see $f(H) = H$, and thus H satisfies the required properties.

Now, we show the last part of Proposition 2.3. If f is linearizable on $\mathbf{S}^1$, it is obvious that $\mathbf{S}^1 \subset \operatorname{Int} H$. Conversely, assume that $\mathbf{S}^1 \subset \operatorname{Int} H$. Let V be the component of $\operatorname{Int} H$ which contains $\mathbf{S}^1$. So V is conformally isomorphic to some annulus, and $f(V) = V$. The dynamics of f on V corresponds to the dynamics of the irrational rotation $z \mapsto e^{2\pi i \operatorname{Rot}(f|_{\mathbf{S}^1})} z$ by the classification theorem of dynamics on hyperbolic surfaces. Therefore, f is linearizable on $\mathbf{S}^1$. □

The following lemma corresponds to [**PM**, Theorem 2].

LEMMA 2.2. *Assuming the hypothesis in Proposition* 2.3, *let H be a Herman compactum for (f, U). Let Ω be the unbounded component of $\hat{\mathbf{C}} - H$, let $\Phi : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \Omega$ be a conformal isomorphism such that $\Phi(\infty) = \infty$. So $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on an outer neighborhood of $\mathbf{S}^1$. Then g is extended and univalent on a neighborhood of $\mathbf{S}^1$ by the reflection principle. Furthermore, the rotation number* $\operatorname{Rot}(g|_{\mathbf{S}^1})$ *corresponds to the rotation number* $\operatorname{Rot}(f|_{\mathbf{S}^1})$.

Proof. First, we show that there exists a Herman compactum H for (f, U) such that $\operatorname{Rot}(g|_{\mathbf{S}^1}) = \operatorname{Rot}(f|_{\mathbf{S}^1})$. It is referred from the proof of [**PM**, Lemma III.3.3]. Since the rotation number $\operatorname{Rot}(f|_{\mathbf{S}^1})$ is irrational, there exists a sequence $\{\alpha_n\}_{n \ge 1}$ such that $\lim_{n \to +\infty} \alpha_n = 0$ and each $f_n(z) = e^{2\pi i \alpha_n} f(z)$ has the rotation number $\operatorname{Rot}(f_n|_{\mathbf{S}^1})$ which is Diophantine. So f_n is univalent on a neighborhood of $\overline{U}$.

From Lemma 2.1, we take the closed annulus H_n in the Herman ring $\mathscr{H}_n$ of f_n as the Herman compactum for (f_n, U). Every H_n is contained in $\overline{U}$, so there exists a subsequence $\{H_{n_i}\}_{i \ge 1}$ and a set H' such that H_{n_i} converges to H' in the sense of Hausdorff convergence.

Since f_{n_i} converges uniformly to f on $\overline{U}$, it follows from [**PM**, Lemma III.1.2] that $f_{n_i}(H_{n_i})$ converges to $f(H')$ in the sense of Hausdorff convergence. Then $f_{n_i}(H_{n_i}) = H_{n_i}$ implies $f(H') = H'$. Let H be the union of H' and all the components of $\hat{\mathbf{C}} - H'$ contained in U. It is not difficult to see that H is a Herman compactum for (f, U).

Let Ω_{n_i} be the unbounded component of $\hat{\mathbf{C}} - H_{n_i}$, let $\Phi_{n_i} : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \Omega_{n_i}$ be a conformal isomorphism so that $\Phi_{n_i}(\infty) = \infty$. For the sake of convenience, we assume that $\lim_{z \to \infty} \Phi_{n_i}(z)/z > 0$ and $\lim_{z \to \infty} \Phi(z)/z > 0$. We notice that Ω is the unbounded component of $\hat{\mathbf{C}} - H$, and is the unbounded component of $\hat{\mathbf{C}} - H'$ as well. So Ω_{n_i} converges to Ω with respect to ∞ in the sense of Carathéodory kernel convergence, and thus Φ_{n_i} converges locally uniformly to Φ by the Carathéodory kernel theorem.

Since f is univalent on a neighborhood of $\overline{U}$, there exists $r_0 > 1$ such that $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on $\{z : 1 < |z| < r_0\}$. So $g_{n_i} = \Phi_{n_i}^{-1} \circ f_{n_i} \circ \Phi_{n_i}$ is also univalent on $\{z : 1 < |z| < r_0\}$. By the reflection principle, g_{n_i} and g are extended and univalent on $\{z : 1/r_0 < |z| < r_0\}$. We fix r such that $1 < r < r_0$. Since Φ_{n_i} converges locally uniformly to Φ, g_{n_i} converges uniformly to g on $r\mathbf{S}^1$. So g_{n_i}

converges uniformly to g on $1/r\mathbf{S}^1$. By the maximum principle, g_{n_i} converges uniformly to g on $\{z : 1/r \le |z| \le r\}$, particularly on $\mathbf{S}^1$.

Let L_{n_i} be the outer boundary of H_{n_i}. We notice that the dynamics of g_{n_i} on $\mathbf{S}^1$ corresponds to the dynamics of f_{n_i} on L_{n_i}. Since L_{n_i} is a Jordan closed curve in the Herman ring $\mathscr{H}_{n_i}$ such that $f_{n_i}(L_{n_i}) = L_{n_i}$, the dynamics of f_{n_i} on L_{n_i} corresponds to the dynamics of the irrational rotation $z \mapsto e^{2\pi i \operatorname{Rot}(f_{n_i}|_{\mathbf{S}^1})} z$. Therefore, the rotation number $\operatorname{Rot}(g_{n_i}|_{\mathbf{S}^1})$ corresponds to the rotation number $\operatorname{Rot}(f_{n_i}|_{\mathbf{S}^1})$. Then,

$$\operatorname{Rot}(g|_{\mathbf{S}^1}) = \lim_{i \to +\infty} \operatorname{Rot}(g_{n_i}|_{\mathbf{S}^1}) = \lim_{i \to +\infty} \operatorname{Rot}(f_{n_i}|_{\mathbf{S}^1}) = \operatorname{Rot}(f|_{\mathbf{S}^1}).$$

Now, we show that such the rotation number $\operatorname{Rot}(g|_{\mathbf{S}^1})$ does not depend on choosing the Herman compactum H for (f, U). It is referred from the proof of [**PM**, Lemma III.3.4]. We fix a Herman compactum H for (f, U). A sequence $\{z_n\}_{n \in \mathbf{Z}}$ is called a *full orbit* of z_0 if $z_{n+1} = f(z_n)$ for all $n \in \mathbf{Z}$, and we denote by $\mathcal{O}(z_0)$. Let H_M be the connected component of the set $\{z \in \overline{U} : \exists \mathcal{O}(z) \subset \overline{U}\}$ which contains $\mathbf{S}^1$. Clearly, $f(H_M) = H_M$ and $H \subset H_M$. It is not difficult to see that H_M is the maximal Herman compactum for (f, U).

Let Ω_M be the unbounded component of $\hat{\mathbf{C}} - H_M$, let $\Phi_M : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \Omega_M$ be a conformal isomorphism such that $\Phi_M(\infty) = \infty$. So $g_M = \Phi_M^{-1} \circ f \circ \Phi_M$ is univalent on an outer neighborhood of $\mathbf{S}^1$. Then g_M is extended and univalent on a neighborhood of $\mathbf{S}^1$ by the reflection principle.

We fix a point $z \in H \cap \gamma_1 \subset H_M \cap \gamma_1$. Since γ_1 is a Jordan closed curve, the point z is accessible from the unbounded component of $\hat{\mathbf{C}} - \overline{U}$, and is accessible from Ω_M as well. Let $\eta \subset \Omega_M \subset \Omega$ be a path converging to z. Then $\Phi^{-1}(\eta)$ converges to some point $w \in \mathbf{S}^1$ and $\Phi_M^{-1}(\eta)$ converges to some point $w_M \in \mathbf{S}^1$ (see [**Mc**, Corollary 6.4]). Now the conformal isomorphism $\Phi^{-1} \circ \Phi_M$ preserves the cyclic ordering between $\{g^{\circ n}(\Phi^{-1}(\eta))\}_{n \ge 0}$ and $\{g_M^{\circ n}(\Phi_M^{-1}(\eta))\}_{n \ge 0}$ (see Figure 6).

Therefore, the cyclic ordering of $\{g^{\circ n}(w)\}_{n \ge 0}$ corresponds to the cyclic ordering of $\{g_M^{\circ n}(w_M)\}_{n \ge 0}$, and thus $\operatorname{Rot}(g|_{\mathbf{S}^1}) = \operatorname{Rot}(g_M|_{\mathbf{S}^1})$. □

Proof of Proposition 2.4. The method of the proof is similar to that of Proposition 2.2. We use proof by contradiction.

First, we consider the case where f is linearizable on $\mathbf{S}^1$. Assume that there exists a point z which is biaccessible from the unbounded component Ω of $\hat{\mathbf{C}} - H$. Let $\Phi : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \Omega$ be a conformal isomorphism such that $\Phi(\infty) = \infty$. So $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on an outer neighborhood of $\mathbf{S}^1$. Then g is extended and univalent on a neighborhood of $\mathbf{S}^1$ by the reflection principle. From Lemma 2.2, the rotation number $\operatorname{Rot}(g|_{\mathbf{S}^1})$ corresponds to the rotation number $\operatorname{Rot}(f|_{\mathbf{S}^1})$.

Let R_s and R_t be two distinct external rays land at z. Let X be the component of $\operatorname{Int} H$ which contains $\mathbf{S}^1$, let L be the outer boundary of X. Clearly, $f(X) = X$ and $f(L) = L$. Let V be the component of $\mathbf{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain L. We cut off V along an equipotential curve E_r, and thus

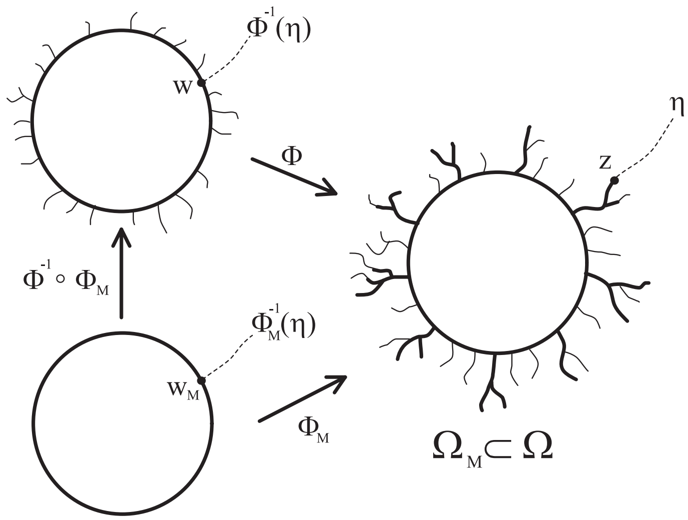

FIGURE 6

have the Jordan domain W which is contained in V. Then $D = \Phi^{-1}(W - H)$ has the interval $I \subset \mathbf{S}^1$ as a part of its boundary (see Figure 7).

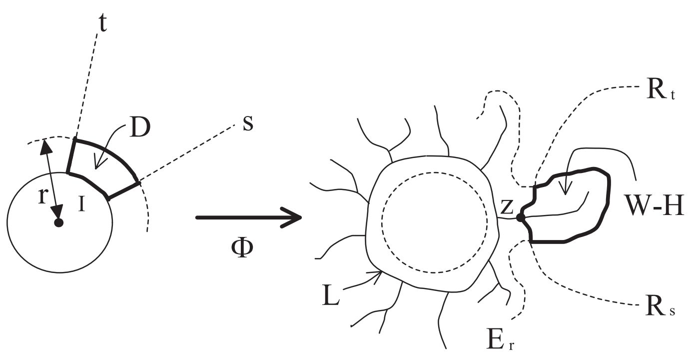

FIGURE 7

Since the rotation number $\mathrm{Rot}(g|_{\mathbf{S}^1})$ is irrational, there exists N such that $\bigcup_{j=0}^{N} g^{\circ j}(I) = \mathbf{S}^1$. We could take a more smaller $r > 1$, so that $g, g^{\circ 2}, \ldots, g^{\circ N}$ are univalent on D, and furthermore, $\bigcup_{j=0}^{N} g^{\circ j}(D)$ is an outer neighborhood of $\mathbf{S}^1$.

Then $f, f^{\circ 2}, \dots, f^{\circ N}$ are univalent on $W - H$, and thus $\bigcup_{j=0}^{N} f^{\circ j}(W - H)$ is an outer neighborhood of H. So any point of $L \subset \partial\Omega$ can be approximated by some sequence in $\bigcup_{j=0}^{N} f^{\circ j}(W - H)$. Now the injectivity of f implies that each Jordan domain $f^{\circ j}(W)$ does not intersect L, therefore, $\overline{f^{\circ j}(W)} \cap L$ contains at most one point $f^{\circ j}(z)$. This contradicts that L has infinitely many points.

Now, we consider the case where f is not linearizable on $\mathbf{S}^1$. Assume that there exists a point z which is biaccessible from the unbounded component Ω of $\hat{\mathbf{C}} - H$. Let $\Phi : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \Omega$ be a conformal isomorphism such that $\Phi(\infty) = \infty$. So $g = \Phi^{-1} \circ f \circ \Phi$ is univalent on an outer neighborhood of $\mathbf{S}^1$. Then g is extended and univalent on a neighborhood of $\mathbf{S}^1$ by the reflection principle. From Lemma 2.2, the rotation number $\mathrm{Rot}(g|_{\mathbf{S}^1})$ corresponds to the rotation number $\mathrm{Rot}(f|_{\mathbf{S}^1})$.

Let R_s and R_t be two distinct external rays land at z. Let V be the component of $\mathbf{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain $\mathbf{S}^1$. We cut off V along an equipotential curve E_r, and thus have the Jordan domain W which is contained in V. Then $D = \Phi^{-1}(W - H)$ has the interval $I \subset \mathbf{S}^1$ as a part of its boundary (see Figure 8).

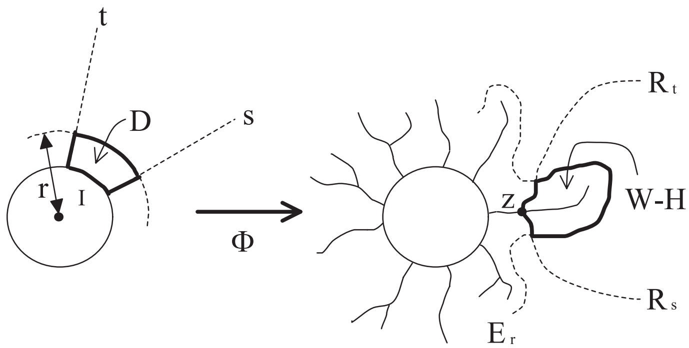

Figure 8

Since the rotation number $\mathrm{Rot}(g|_{\mathbf{S}^1})$ is irrational, there exists N such that $\bigcup_{j=0}^{N} g^{\circ j}(I) = \mathbf{S}^1$. We could take a more smaller $r > 1$, so that $g, g^{\circ 2}, \dots, g^{\circ N}$ are univalent on D, and furthermore, $\bigcup_{j=0}^{N} g^{\circ j}(D)$ is an outer neighborhood of $\mathbf{S}^1$.

Then $f, f^{\circ 2}, \dots, f^{\circ N}$ are univalent on $W - H$, and thus $\bigcup_{j=0}^{N} f^{\circ j}(W - H)$ is an outer neighborhood of H. So any point of $\mathbf{S}^1 \subset \partial\Omega$ can be approximated by some sequence in $\bigcup_{j=0}^{N} f^{\circ j}(W - H)$. Now the injectivity of f implies that each Jordan domain $f^{\circ j}(W)$ does not intersect $\mathbf{S}^1$, therefore, $\overline{f^{\circ j}(W)} \cap \mathbf{S}^1$ contains at most one point $f^{\circ j}(z)$. This contradicts that $\mathbf{S}^1$ has infinitely many points. $\square$

3. Preliminaries for proofs

In this section, we shall see preparations for the proofs of the theorems. The following notion will be often used later.

Definition 3.1. Let $\Omega \subset \hat{\mathbf{C}}$ be a simply connected domain which contains ∞. Assume that the boundary $\partial\Omega$ contains at least two points. Let $\Phi : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \Omega$ be a conformal isomorphism such that $\Phi(\infty) = \infty$. Let R_s and R_t be two distinct external rays land at z. Let U_1 and U_2 be two distinct components of $\mathbf{C} - (R_s \cup \{z\} \cup R_t)$. Then for each $l = 1, 2$, *angle* of U_l is defined as

$$A(U_l) = \frac{\text{length}(\Phi^{-1}(U_l \cap E_r))}{2\pi r}.$$

It does not depend on $r > 1$, so it is well defined. Clearly, $0 < A(U_1), A(U_2) < 1$ and $A(U_1) + A(U_2) = 1$. The *angle* between R_s and R_t is defined as $A(R_s, R_t) = \min\{A(U_1), A(U_2)\}$. Clearly, $A(R_s, R_t) \le 1/2$ (see Figure 9).

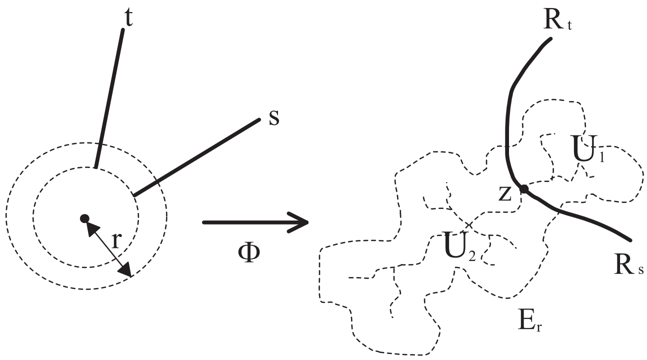

Figure 9

The following two lemmas will be used for the proofs of the theorems.

Lemma 3.1. *Let K be a compact subset of the complex plane $\mathbf{C}$. Assume that f is analytic on a neighborhood of K, there are no critical points of f in K and f is injective on K. Then there exists $\varepsilon > 0$ such that f is univalent on $N_\varepsilon(K)$, where $N_\varepsilon(K) = \{z \in \mathbf{C} : \min_{w \in K} |z - w| < \varepsilon\}$.*

Proof. Assume that f is not univalent on $N_{\frac{1}{n}}(K)$ for all $n \in \mathbf{N}$. Then there exist $x_n \in N_{\frac{1}{n}}(K)$ and $y_n \in N_{\frac{1}{n}}(K)$ such that $x_n \neq y_n$ and $f(x_n) = f(y_n)$. Since $\{x_n\}_{n \ge 1}$ is contained in $\overline{N_1(K)}$, we take a subsequence $\{x_{n_i}\}_{i \ge 1}$ and a point x_0 such that $\lim_{i \to +\infty} x_{n_i} = x_0$. Similarly, we take a subsequence $\{y_{n_{i_j}}\}_{j \ge 1}$ of

$\{y_{n_i}\}_{i\geq 1}$ and a point y_0 such that $\lim_{j\to+\infty} y_{n_{i_j}} = y_0$. Then both x_0 and y_0 are belong to K, and

$$f(x_0) = \lim_{j\to+\infty} f(x_{n_{i_j}}) = \lim_{j\to+\infty} f(y_{n_{i_j}}) = f(y_0).$$

Now f is injective on K, and thus $x_0 = y_0$. So f is not univalent on any neighborhood of x_0, and thus x_0 is a critical point of f. This contradicts that there are no critical points of f in K. □

LEMMA 3.2. *Let Ω be a bounded domain by a cycle $\gamma \subset \mathbf{C}$ which consists of finite Jordan closed curves. Let f be a complex-valued function defined on a neighborhood of $\overline{\Omega}$. Assume that f is analytic on $\overline{\Omega}$ and injective on $\partial\Omega$. Assume that f preserves the orientation on each Jordan closed curve which constructs a part of $\partial\Omega$. Then Ω' is well defined as the bounded domain by the cycle $f(\partial\Omega) \subset \mathbf{C}$, and f maps Ω conformally onto Ω'* (*see Figure* 10).

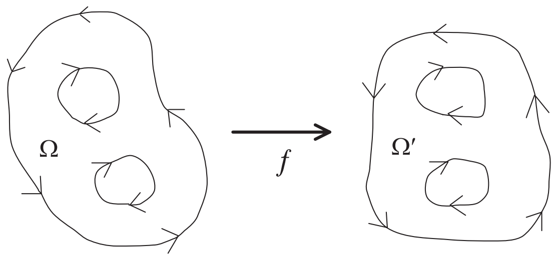

FIGURE 10

Proof. From the open mapping theorem, it is easy to see that Ω' is well defined as the bounded domain by the cycle $f(\partial\Omega) \subset \mathbf{C}$.

Let w_0 be a point in Ω'. Let $\Gamma(z) = f(z) - w_0 = w - w_0$. Then $\Gamma(z)$ is analytic on $\overline{\Omega}$ and does not take the zeros on $\partial\Omega$. From the argument principle,

$$\frac{1}{2\pi}\int_{\partial\Omega} d \arg \Gamma(z) = \frac{1}{2\pi}\int_{f(\partial\Omega)} d \arg(w - w_0) = N,$$

where N is the number of the zeros in Ω. We obtain $N = 1$, so there exists the zero z_0 of Γ in Ω. Therefore, z_0 is the point in Ω satisfies $f(z_0) = w_0$.

Similarly, we can see that there are no points $z \in \Omega$ such that $f(z) = w_0$ when $w_0 \notin \overline{\Omega'}$. □

4. Proof of Theorem 1.1

In this section, we consider a polynomial $f_c(z) = z^d + c$ of degree $d \geq 2$. For each $0 \leq j \leq d-1$, let $\sigma_j(z) = e^{2\pi i \frac{j}{d}} z$ be a j/d-rotation. Then $f_c \circ \sigma_j = f_c$ implies $\sigma_j(J_{f_c}) = J_{f_c}$. The origin is only one critical point of f_c in $\mathbf{C}$.

Assume that α is an irrationally indifferent fixed point of f_c. Then the origin is recurrent (see [**Ma**]), so the superattracting fixed point ∞ is the only critical point in the immediate basin $\mathscr{A}_\infty$. Therefore, there exists a conformal isomorphism $\Phi : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \mathscr{A}_\infty$ such that $\Phi(\infty) = \infty$ and $\Phi^{-1} \circ f_c \circ \Phi(w) = w^d$.

We consider the dynamics of external rays and the equipotential curves in the immediate basin $\mathscr{A}_\infty$. It is easy to see that $f_c(R_t) = R_{dt}$, $f_c^{-1}(R_t) = \bigcup_{j=0}^{d-1} R_{\frac{t+j}{d}}$, $f_c(E_r) = E_{r^d}$ and $f_c^{-1}(E_r) = E_{\sqrt[d]{r}}$. Moreover, $\sigma_j(\mathscr{A}_\infty) = \mathscr{A}_\infty$ implies $\sigma_j \circ \Phi = \Phi \circ \sigma_j$, so that $\sigma_j(R_t) = R_{t+\frac{j}{d}}$ and $\sigma_j(E_r) = E_r$.

Lemma 4.1. *Let R_s and R_t be two distinct external rays land at $z \neq 0$. Let U be the component of $\overline{\mathbf{C} - (R_s \cup \{z\} \cup R_t)}$ such that $A(U) = A(R_s, R_t)$. Then $A(U) < 1/d$ and $\overline{\sigma_j(U)} \cap \overline{\sigma_k(U)} = \emptyset$ for $j \neq k$. Therefore, U does not contain both two σ_j-symmetric points and $\mathbf{C} - \overline{U}$ contains the origin.*

Proof. Assume that $A(U) \geq 1/d$. Then $A(\mathbf{C} - \overline{U}) \geq A(R_s, R_t) = A(U) \geq 1/d$, so we may suppose that

$$s < s + \frac{1}{d} \leq t < t + \frac{1}{d} \leq s + 1,$$

and furthermore, $\sigma_1(R_s) \subset \overline{U}$ and $\sigma_1(R_t) \subset \overline{\mathbf{C} - U}$ (see Figure 11).

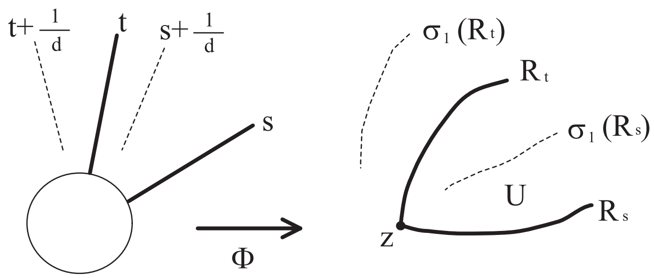

Figure 11

Then both $R_{s+\frac{1}{d}}$ and $R_{t+\frac{1}{d}}$ land at $\sigma_1(z)$, so $\sigma_1(z) \in \overline{U} \cap \overline{\mathbf{C} - U} = \partial U$, and thus $\sigma_1(z) = z$. This implies $z = 0$, which contradicts the assumption $z \neq 0$.

Now assume that there are two distinct numbers j and k such that $\overline{\sigma_j(U)} \cap \overline{\sigma_k(U)} \neq \emptyset$. We have $A(U) < 1/d$, so we may suppose

$$s + \frac{j}{d} < t + \frac{j}{d} < s + \frac{k}{d} < t + \frac{k}{d} < s + \frac{j}{d} + 1.$$

Two distinct external rays does not intersect, so we conclude that $\sigma_j(z) = \sigma_k(z)$. This implies $z = 0$, which contradicts the assumption $z \neq 0$. $\square$

Lemma 4.2. *Let R_s and R_t be two distinct external rays land at $z \neq 0$. Let U be a component of $\mathbf{C} - (R_s \cup \{z\} \cup R_t)$. Then the following three conditions are equivalent to each other*:

(a) $A(U) < 1/d$;

(b) f_c *is univalent on* U;

(c) U *does not contain the origin.*

Proof. (a) $\Rightarrow$ (b): Assume that $A(U) < 1/d$. So we cut off U along an equipotential curve E_r, and thus have the Jordan domain V which is contained in U. Then f_c is injective on ∂V and preserves the orientation, so Lemma 3.2 implies that f_c is univalent on V. We could take a more bigger $r > 1$, so that f_c is univalent on U. Moreover, $f_c(U)$ is the component of $\mathbf{C} - f_c(R_s \cup \{z\} \cup R_t)$ such that $A(f_c(U)) = d \cdot A(U)$.

(b) $\Rightarrow$ (c): It is obvious.

(c) $\Rightarrow$ (a): Assume that U does not contain the origin. If $A(\mathbf{C} - \overline{U}) = A(R_s, R_t)$, then Lemma 4.1 implies that U contains the origin. This contradicts the assumption, and thus $A(\mathbf{C} - \overline{U}) \neq A(R_s, R_t)$. Therefore, $A(U) = A(R_s, R_t)$ and thus Lemma 4.1 implies $A(U) < 1/d$. □

Lemma 4.3. *Assume that z is biaccessible from the immediate basin $\mathscr{A}_\infty$ such that $\alpha \notin \{f_c^{\circ n}(z)\}_{n \geq 0}$ and $0 \notin \{f_c^{\circ n}(z)\}_{n \geq 0}$. Then there exist two distinct external rays R_u and R_v with a common landing point w such that $R_u \cup \{w\} \cup R_v$ separates α from the origin.*

Proof. Let R_s and R_t be two distinct external rays landing at z. Let U be the component of $\mathbf{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain the origin. Then Lemma 4.2 implies that f_c is univalent on U and thus $A(f_c(U)) = d \cdot A(U)$.

If $f_c(U)$ does not contain the origin, then we have that f_c is univalent on $f_c(U)$ and thus $A(f_c^{\circ 2}(U)) = d^2 \cdot A(U)$ as the above argument. Otherwise, $f_c(U)$ contains the origin.

By repeating the above step, we see that there exists $N \geq 0$ such that $f_c^{\circ N}(U)$ does not contain the origin and $f_c^{\circ N+1}(U)$ contains the origin. Then f_c is univalent on $f_c^{\circ N}(U)$ and thus $A(f_c^{\circ N+1}(U)) = d^{N+1} \cdot A(U)$.

If $\alpha \in f_c^{\circ N}(U)$, then put $R_u \cup \{w\} \cup R_v = f_c^{\circ N}(R_s \cup \{z\} \cup R_t)$.

Otherwise, if $\alpha \notin f_c^{\circ N}(U)$, then we may consider the following two cases:

(1) $f_c^{\circ N}(U)$ contains some $\sigma_{j_0}(\alpha)$;

(2) $f_c^{\circ N}(U)$ does not contain any $\sigma_j(\alpha)$.

In the case (1), put $R_u \cup \{w\} \cup R_v = \sigma_{d-j_0}(f_c^{\circ N}(R_s \cup \{z\} \cup R_t))$.

In the case (2), if $f_c^{\circ N+1}(U)$ contains α, then $f_c^{\circ N}(U)$ contains one point of inverse image of α. Since $f_c^{-1}(\alpha) = \{\sigma_j(\alpha) \mid 0 \leq j \leq d-1\}$, it follows that $f_c^{\circ N}(U)$ contains some $\sigma_{j_0}(\alpha)$. However, this contradicts that $f_c^{\circ N}(U)$ does not contain any $\sigma_j(\alpha)$. Therefore, $f_c^{\circ N+1}(U)$ does not contain α, and thus we put $R_u \cup \{w\} \cup R_v = f_c^{\circ N+1}(R_s \cup \{z\} \cup R_t)$. □

Proof of Theorem 1.1. We use proof by contradiction. If α is a Siegel point, assume that $0 \notin \{f_c^{\circ n}(z_0)\}_{n \geq 0}$. If α is a Cremer point, assume that $\alpha \notin \{f_c^{\circ n}(z_0)\}_{n \geq 0}$ and $0 \notin \{f_c^{\circ n}(z_0)\}_{n \geq 0}$. In both cases, it follows that z_0 is biaccessible from $\mathscr{A}_\infty$ such that $\alpha \notin \{f_c^{\circ n}(z_0)\}_{n \geq 0}$ and $0 \notin \{f_c^{\circ n}(z_0)\}_{n \geq 0}$.

Lemma 4.3 implies that there exist two distinct external rays R_u and R_v with a common landing point w such that $R_u \cup \{w\} \cup R_v$ separates α from the origin. Let U be the component of $\mathbf{C} - (R_u \cup \{w\} \cup R_v)$ which contains α. Then f_c is injective on $\overline{U}$. We cut off U along an equipotential curve E_r, and thus have the Jordan domain V which contains α.

Since $\overline{V}$ contains no critical points of f_c, it follows from Lemma 3.1 that there exists a Jordan domain W such that $\overline{V} \subset W$ and f_c is univalent on a neighborhood of $\overline{W}$ (see Figure 12).

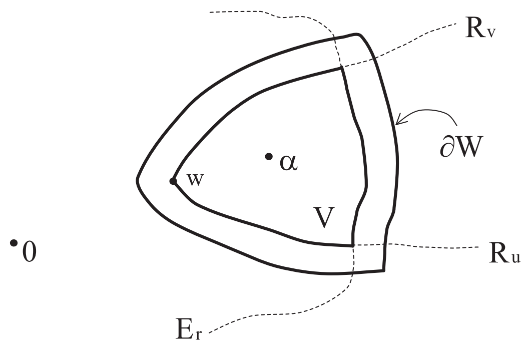

FIGURE 12

Now we take a Siegel compactum S for (f_c, W) by Proposition 2.1. Then S meets the boundary ∂W but not $\partial V - \{w\}$, so S must contain w. Furthermore, $\partial(\hat{\mathbf{C}} - S) - \{w\}$ is disconnected, and thus the point w is biaccessible from $\hat{\mathbf{C}} - S$. However, the biaccessibility of w contradicts Proposition 2.2. $\square$

5. Proof of Theorem 1.2

In this section, we consider a polynomial $g_\theta(z) = e^{2\pi i\theta}z + z^d$ of degree $d \geq 2$. Actually, we may consider the cases of $d \geq 3$ and thus assume that $d \geq 3$ in the following arguments. For each $0 \leq j \leq d-2$, let $\tau_j(z) = e^{2\pi i \frac{j}{d-1}}z$ be a $j/(d-1)$-rotation. Then $g_\theta \circ \tau_j = \tau_j \circ g_\theta$ implies $\tau_j(J_{g_\theta}) = J_{g_\theta}$. So g_θ has $d-1$ symmetric critical points $c_j = \tau_j(c)$, where c is one of the solutions of $e^{2\pi i\theta} + dz^{d-1} = 0$.

Assume that the origin is an irrationally indifferent fixed point of g_θ. Then some critical point c_{j_0} is recurrent (see [**Ma**]), so $g_\theta \circ \tau_j = \tau_j \circ g_\theta$ implies that every critical point c_j is recurrent. Therefore, the superattracting fixed point ∞ is the only critical point in the immediate basin $\mathscr{A}_\infty$. Then there exists a conformal isomorphism $\Phi : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \mathscr{A}_\infty$ such that $\Phi(\infty) = \infty$ and $\Phi^{-1} \circ g_\theta \circ \Phi(w) = w^d$.

We consider the dynamics of external rays and the equipotential curves in the immediate basin $\mathscr{A}_\infty$. It is easy to see that $g_\theta(R_t) = R_{dt}$, $g_\theta^{-1}(R_t) = \bigcup_{j=0}^{d-1} R_{\frac{t+j}{d}}$, $g_\theta(E_r) = E_{r^d}$ and $g_\theta^{-1}(E_r) = E_{\sqrt[d]{r}}$. Moreover, $\tau_j(\mathscr{A}_\infty) = \mathscr{A}_\infty$ implies $\tau_j \circ \Phi = \Phi \circ \tau_j$, so that $\tau_j(R_t) = R_{t+\frac{j}{d-1}}$ and $\tau_j(E_r) = E_r$.

Lemma 5.1. *Let R_s and R_t be two distinct external rays land at $z \neq 0$. Let U be the component of $\mathbf{C} - (R_s \cup \{z\} \cup R_t)$ such that $A(U) = A(R_s, R_t)$. Then $A(U) < 1/(d-1)$ and $\overline{\tau_j(U)} \cap \overline{\tau_k(U)} = \emptyset$ for $j \neq k$. Therefore, U does not contain both two τ_j-symmetric points and $\mathbf{C} - \overline{U}$ contains the origin.*

The method of the proof is similar to that of Lemma 4.1.

Lemma 5.2. *Let R_s and R_t be two distinct external rays land at $z \neq 0$. Let U be a component of $\mathbf{C} - (R_s \cup \{z\} \cup R_t)$. Then the following three conditions are equivalent to each other*:

(a) *$A(U) < 1/d$;*
(b) *g_θ is univalent on U;*
(c) *U does not contain any c_j.*

Proof. The method of the proof of (a) $\Rightarrow$ (b) is similar to that of Lemma 4.2. The proof of (b) $\Rightarrow$ (c) is obvious. We give the proof of (c) $\Rightarrow$ (a) here.

Assume that U does not contain any c_j. If $A(\mathbf{C} - \overline{U}) = A(R_s, R_t)$, then Lemma 5.1 implies that $\mathbf{C} - \overline{U}$ does not contain both two τ_j-symmetric points. Therefore, U contains at least one point of c_j. This contradicts the assumption, and thus we have $A(\mathbf{C} - \overline{U}) \neq A(R_s, R_t)$. Therefore, $A(U) = A(R_s, R_t)$ and we see from Lemma 5.1 that $\overline{\tau_j(U)} \cap \overline{\tau_k(U)} = \emptyset$ for $j \neq k$.

If $A(U) \geq 1/d$, then $A(\tau_j(U)) \geq 1/d$ and thus $g_\theta(\overline{\tau_j(U)}) = \mathbf{C}$. Then each $\overline{\tau_j(U)}$ contains at least one point of inverse image of some critical value v_{j_0}, where $v_{j_0} = g_\theta(c_{j_0})$. Therefore, $\bigcup_{j=0}^{d-2} \overline{\tau_j(U)}$ contains at least $d-1$ points of inverse image of v_{j_0}. However, this contradicts that $\mathbf{C} - \bigcup_{j=0}^{d-2} \overline{\tau_j(U)}$ contains the critical point c_{j_0}. Therefore, we conclude $A(U) < 1/d$. $\square$

Lemma 5.3. *Assume that z is biaccessible from the immediate basin $\mathscr{A}_\infty$ such that $0 \notin \{g_\theta^{\circ n}(z)\}_{n \geq 0}$ and $c_j \notin \{g_\theta^{\circ n}(z)\}_{n \geq 0}$ for all j. Then for each j, there exist two distinct external rays R_{u_j} and R_{v_j} with a common landing point w_j such that $R_{u_j} \cup \{w_j\} \cup R_{v_j}$ separates c_j from the origin.*

Proof. By τ_j-symmetry, it is enough to show Lemma 5.3 for some j_0. Now let R_s and R_t be two distinct external rays landing at z. Let U be the component of $\mathbf{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain the origin. If U contains some c_{j_0}, put $R_{u_{j_0}} \cup \{w_{j_0}\} \cup R_{v_{j_0}} = R_s \cup \{z\} \cup R_t$.

On the other hand, assume that U does not contain any c_j. Then Lemma 5.2 implies g_θ is univalent on U and thus $A(g_\theta(U)) = d \cdot A(U)$.

If $g_\theta(U)$ does not contain any c_j, then we have that g_θ is univalent on $g_\theta(U)$ and thus $A(g_\theta^{\circ 2}(U)) = d^2 \cdot A(U)$ as the above argument. Otherwise, $g_\theta(U)$ contains some c_{j_0}.

By repeating the above step, we see that there exists $N \geq 0$ such that $g_\theta^{\circ N}(U)$ does not contain any c_j and $g_\theta^{\circ N+1}(U)$ contains some c_{j_0}. Then g_θ is univalent on $g_\theta^{\circ N}(U)$ and thus $A(g_\theta^{\circ N+1}(U)) = d^{N+1} \cdot A(U)$.

So we may consider the following three cases:

(1) $\overline{g_\theta^{\circ N+1}(U)}$ contains only some one of c_j;

(2) $\mathbf{C} - g_\theta^{\circ N+1}(U)$ contains only some one of c_j;

(3) $g_\theta^{\circ N+1}(U)$ contains all c_j.

In the case (1) and case (2), put $R_{u_{j_0}} \cup \{w_{j_0}\} \cup R_{v_{j_0}} = g_\theta^{\circ N+1}(R_s \cup \{z\} \cup R_t)$.

Now, we consider the case (3). To simplify the notation, we set as the following:

$$L = g_\theta^{\circ N}(R_s \cup \{z\} \cup R_t), \quad V = g_\theta^{\circ N}(U),$$

$$W = \mathbf{C} - \bigcup_{j=0}^{d-2} \overline{\tau_j(V)}, \quad W' = \bigcap_{j=0}^{d-2} g_\theta(\tau_j(V)) = \bigcap_{j=0}^{d-2} \tau_j(g_\theta(V)).$$

Then both W and W' are τ_j-symmetrical domains, which contain the origin as well as all c_j (see Figure 13).

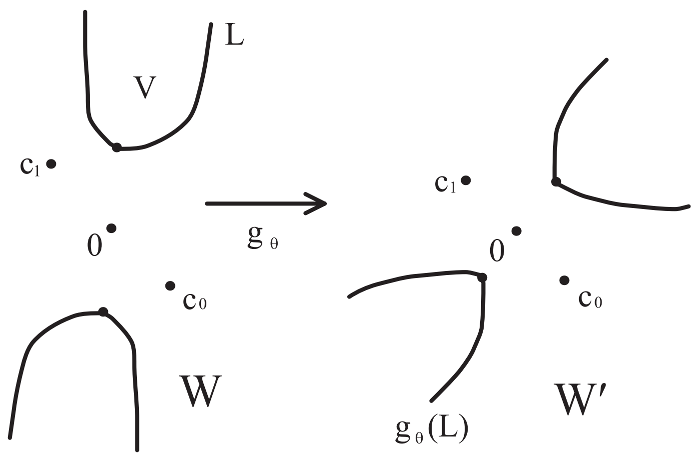

FIGURE 13

If $\overline{W'}$ contains some critical value $v_{j_0} = g_\theta(c_{j_0})$, then each $\overline{\tau_j(V)}$ contains one point of inverse image of v_{j_0}, and thus $\bigcup_{j=0}^{d-2} \overline{\tau_j(V)}$ contains $d-1$ points of inverse image of v_{j_0}. However, this contradicts that $W = \mathbf{C} - \bigcup_{j=0}^{d-2} \overline{\tau_j(V)}$ contains the critical point c_{j_0}. Therefore, $\overline{W'}$ does not contain any $v_j = g_\theta(c_j)$.

Now we may suppose that $\mathbf{C} - \overline{g_\theta(V)}$ contains some $v_{j_0} = g_\theta(c_{j_0})$. For each $0 \le j \le d-2$, we consider the following bijection:

$$g_\theta|_{\overline{\tau_j(V)}} : \overline{\tau_j(V)} \to g_\theta(\overline{\tau_j(V)}).$$

Then the image $g_\theta(\overline{\tau_j(V)})$ contains $g_\theta(L)$. Therefore, $\deg g_\theta = d$ implies $W \cap g_\theta^{-1}(g_\theta(L)) \neq \emptyset$. Now we set $L' = W \cap g_\theta^{-1}(g_\theta(L))$.

If L' does not separate c_{j_0} from the origin, then there exists a continuous curve γ in $W - L'$ between c_{j_0} and the origin. Then $g_\theta(\gamma)$ is a continuous curve between v_{j_0} and the origin. So $g_\theta(\gamma) \cap g_\theta(L) \neq \emptyset$ and thus $\gamma \cap g_\theta^{-1}(g_\theta(L)) \neq \emptyset$. However, this contradicts $\gamma \subset W - L'$.

Therefore, it is concluded that L' separates c_{j_0} from the origin, and thus we put $R_{u_{j_0}} \cup \{w_{j_0}\} \cup R_{v_{j_0}} = L'$. □

Proof of Theorem 1.2. We use proof by contradiction. If the origin is a Siegel point, assume that $c_j \notin \{g_\theta^{\circ n}(z_0)\}_{n \ge 0}$ for all j. If the origin is a Cremer point, assume that $0 \notin \{g_\theta^{\circ n}(z_0)\}_{n \ge 0}$ and $c_j \notin \{g_\theta^{\circ n}(z_0)\}_{n \ge 0}$ for all j.

In both cases, it follows that z_0 is biaccessible from $\mathscr{A}_\infty$ so that $0 \notin \{g_\theta^{\circ n}(z_0)\}_{n \ge 0}$ and $c_j \notin \{g_\theta^{\circ n}(z_0)\}_{n \ge 0}$ for all j. Lemma 5.3 implies that for each j, there exist two distinct external rays R_{u_j} and R_{v_j} with a common landing point w_j such that $R_{u_j} \cup \{w_j\} \cup R_{v_j}$ separates c_j from the origin. Then we may suppose that all $R_{u_j} \cup \{w_j\} \cup R_{v_j}$ are τ_j-symmetrical.

Let U be the component of $\mathbf{C} - \bigcup_{j=0}^{d-2} (R_{u_j} \cup \{w_j\} \cup R_{v_j})$ which contains the origin. We cut off U along an equipotential curve E_r and thus have the τ_j-symmetric Jordan domain V which contains the origin. Then g_θ is injective on ∂V and preserves the orientation, so Lemma 3.2 implies that g_θ is injective on $\overline{V}$.

Since $\overline{V}$ contains no critical points of g_θ, it follows from Lemma 3.1 that there exists a Jordan domain W such that $\overline{V} \subset W$ and g_θ is univalent on a neighborhood of $\overline{W}$ (see Figure 14).

Now we take a Siegel compactum S for (g_θ, W) by Proposition 2.1. Then S meets the boundary ∂W but not $\partial V - \bigcup_{j=0}^{d-2} \{w_j\}$, so S must contain some w_{j_0}. Furthermore, $\partial(\hat{\mathbf{C}} - S) - \{w_{j_0}\}$ is disconnected, and thus the point w_{j_0} is biaccessible from $\hat{\mathbf{C}} - S$. However, the biaccessibility of w_{j_0} contradicts Proposition 2.2. □

6. Proof of Theorem 1.3

In this section, we consider a rational function $h(z) = e^{2\pi i\theta} z^2 \dfrac{z-a}{1-\bar{a}z}$. Let $\upsilon(z) = 1/\bar{z}$ be an inversion. Then $h \circ \upsilon = \upsilon \circ h$ implies $\upsilon(J_h) = J_h$. The zeros are

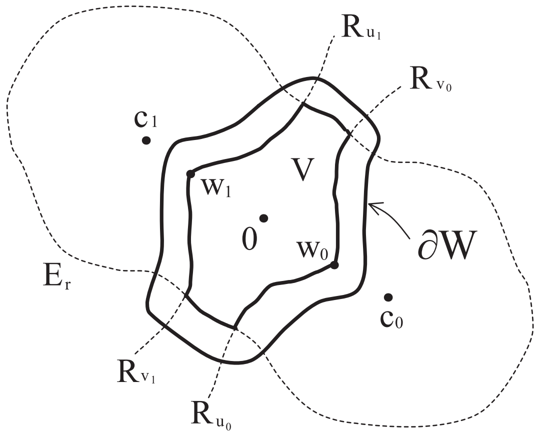

FIGURE 14

the origin and a, and the poles are infinity and $\upsilon(a)$. We suppose $|a| > 3$ such that $h|_{\mathbf{S}^1}$ is an analytic circle diffeomorphism. Then both of infinity and the origin are superattracting fixed points with local degree 2, and thus $h \circ \upsilon = \upsilon \circ h$ implies $\upsilon(\mathscr{A}_\infty) = \mathscr{A}_0$. Let c be the critical point of h such that $|c| > 1$, and thus $\upsilon(c)$ is also a critical point of h.

Assume that the rotation number $\mathrm{Rot}(h|_{\mathbf{S}^1})$ is irrational. If h is linearizable on $\mathbf{S}^1$, then there exists a Herman ring $\mathscr{H}$ and thus $\mathbf{S}^1 \subset \mathscr{H} \subset F_h$. On the other hand, if h is not linearizable on $\mathbf{S}^1$, then $\mathbf{S}^1 \subset J_h$. In either case, some critical point is recurrent (see [**Ma**]), so that both c and $\upsilon(c)$ are recurrent by $h \circ \upsilon = \upsilon \circ h$. Therefore, each of superattracting fixed points infinity and the origin is the only critical point in each immediate basin. We may consider only the immediate basin $\mathscr{A}_\infty$. So there exists a conformal isomorphism $\Phi : \hat{\mathbf{C}} - \overline{\mathbf{D}} \to \mathscr{A}_\infty$ such that $\Phi(\infty) = \infty$ and $\Phi^{-1} \circ h \circ \Phi(w) = w^2$.

We consider the dynamics of external rays and the equipotential curves in the immediate basin $\mathscr{A}_\infty$. It is easy to see that $h(R_t) = R_{2t}$, $h^{-1}(R_t) \cap \mathscr{A}_\infty = R_{\frac{t}{2}} \cup R_{\frac{t+1}{2}}$, $h(E_r) = E_{r^2}$ and $h^{-1}(E_r) \cap \mathscr{A}_\infty = E_{\sqrt{r}}$.

LEMMA 6.1. *There are no points in* $\mathbf{S}^1$ *which are biaccessible from* $\mathscr{A}_\infty$.

Proof. This proof is referred from the last part of the proof of [**Za**, Theorem 5]. We use proof by contradiction. Assume that there exists a point $z_0 \in \mathbf{S}^1$ which is biaccessible from $\mathscr{A}_\infty$. Let R_s and R_t be two distinct external rays landing at z_0, let U_0 be the component of $\mathbf{C} - (R_s \cup \{z_0\} \cup R_t)$ which does not contain $\mathbf{S}^1$. Let $z_n = h^{\circ n}(z_0)$ and U_n be the component of $\mathbf{C} - h^{\circ n}(R_s \cup \{z_0\} \cup R_t)$ which does not contain $\mathbf{S}^1$ (see Figure 15).

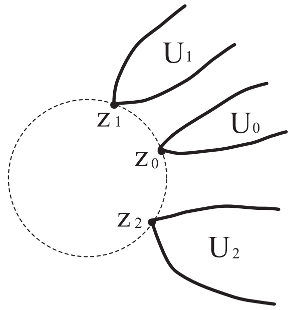

FIGURE 15

There are no critical points in $\mathbf{S}^1$, and so we notice that $A(U_n) \neq 1/2$ for all $n \geq 0$. First, we show that $A(U_n) > 1/2$ for some U_n. Assume that $A(U_0) < 1/2$. By the similar method of the proof of (a) $\Rightarrow$ (b) in Lemma 4.2, we see h is injective on $\overline{U}_0$. Since z_0 is not a critical point, $\mathbf{S}^1 \not\subset h(\overline{U}_0)$, therefore, $h(\overline{U}_0) = \overline{U}_1$ and $A(U_1) = 2 \cdot A(U_0)$. If $A(U_1) < 1/2$, then we similarly have that $h(\overline{U}_1) = \overline{U}_2$ and $A(U_2) = 2 \cdot A(U_1)$. By repeating the above step, we conclude there exists U_N such that $A(U_N) > 1/2$.

We shall see contradiction. Let $V = \mathbf{C} - \overline{U}_N$. Then $A(V) < 1/2$ by $A(U_N) > 1/2$. Since the rotation number $\mathrm{Rot}(h|_{\mathbf{S}^1})$ is irrational, the orbit $\{z_n\}_{n \geq 0}$ is infinite. So $U_n \subset V$ for all $n \geq N+1$ (see Figure 16).

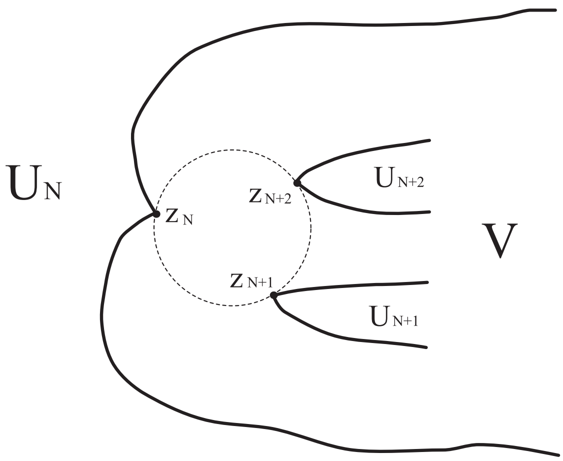

FIGURE 16

By the above argument, we obtain that $h(\overline{U}_n) = \overline{U}_{n+1}$ and $A(U_{n+1}) = 2 \cdot A(U_n)$ for all $n \geq N+1$. This monotonous increasing contradicts $A(U_n) < A(V) < 1/2$ for all $n \geq N+1$. □

In the rest of this section, we shall use the above lemma without any explanation.

LEMMA 6.2. *Let R_s and R_t be two distinct external rays land at $z \neq c$. Let U be a component of $\mathbf{C} - (R_s \cup \{z\} \cup R_t)$. Then the following two conditions are equivalent to each other*:

(a) $A(U) < 1/2$;

(b) *U does not contain c.*

Proof. (a) $\Rightarrow$ (b): Assume that $A(U) < 1/2$. Then we cut off U along an equipotential curve E_r, and thus have the Jordan domain V which is contained in U. Then h is injective on ∂V and preserves the orientation. We may consider the following two cases:

(1) $\mathbf{S}^1 \cap \overline{V} = \emptyset$;

(2) $\mathbf{S}^1 \subset V$.

In the case (1), Lemma 3.2 implies that h is injective on $\overline{V}$. We could take a more bigger $r > 1$, so that h is univalent on U. Therefore, U does not contain c.

In the case (2), we set $W = V - \overline{\mathbf{D}}$ (see Figure 17).

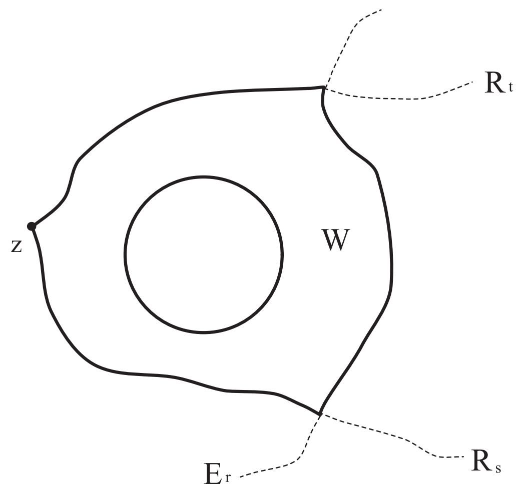

FIGURE 17

Then h is injective on ∂W and preserves the orientation. So Lemma 3.2 implies that h is injective on $\overline{W}$, and thus $c \notin W$. Since $c \notin \overline{\mathbf{D}}$, the domain V

does not contain c. We could take a more bigger $r > 1$, so that U does not contain c.

(b) $\Rightarrow$ (a): Assume that U does not contain c. Then $\mathbf{C} - \overline{U}$ contains c. It follows from the contraposition of (a) $\Rightarrow$ (b) that $A(\mathbf{C} - \overline{U}) > 1/2$, and thus $A(U) < 1/2$. $\square$

LEMMA 6.3. *Assume that z is biaccessible from the immediate basin $\mathcal{A}_\infty$ such that $c \notin \{h^{\circ n}(z)\}_{n\geq 0}$. Then there exist two distinct external rays R_u and R_v with a common landing point w such that $R_u \cup \{w\} \cup R_v$ separates $\mathbf{S}^1$ from c.*

Proof. Let R_s and R_t be two distinct external rays landing at z. Let U be the component of $\mathbf{C} - (R_s \cup \{z\} \cup R_t)$ which does not contain c. Then U satisfies $A(U) < 1/2$ by Lemma 6.2. If $\mathbf{S}^1 \subset U$, we put $R_u \cup \{w\} \cup R_v = R_s \cup \{z\} \cup R_t$. On the other hand, if $\mathbf{S}^1 \cap \overline{U} = \emptyset$, then we see h is univalent on U and thus $A(h(U)) = 2 \cdot A(U)$ by the similar method of the proof of (a) $\Rightarrow$ (b) in Lemma 4.2.

We consider $h(U)$ instead of U. If $h(U)$ contains neither c nor $\mathbf{S}^1$, then we similarly have that h is univalent on $h(U)$ and thus $A(h^{\circ 2}(U)) = 2^2 \cdot A(U)$. Otherwise, $h(U)$ contains c or $\mathbf{S}^1$.

By repeating the above step, we see that there exists $N \geq 0$ such that $h^{\circ N}(U)$ does not contain c nor $\mathbf{S}^1$ and $h^{\circ N+1}(U)$ contains c or $\mathbf{S}^1$. Then h is univalent on $h^{\circ N}(U)$ and thus $A(h^{\circ N+1}(U)) = 2^{N+1} \cdot A(U)$. So we may consider the following three cases:

(1) $h^{\circ N+1}(U)$ contains $\mathbf{S}^1$ but not c;
(2) $h^{\circ N+1}(U)$ contains c but not $\mathbf{S}^1$;
(3) $h^{\circ N+1}(U)$ contains both c and $\mathbf{S}^1$.

In the case (1) and case (2), put $R_u \cup \{w\} \cup R_v = h^{\circ N+1}(R_s \cup \{z\} \cup R_t)$.

Now, we consider the case (3). Since $h|_{\overline{h^{\circ N}(U)}} : \overline{h^{\circ N}(U)} \to \overline{h^{\circ N+1}(U)}$ is bijective, $\overline{h^{\circ N}(U)}$ contains the Jordan closed curve γ such that $h(\gamma) = \mathbf{S}^1$. So $h \circ \upsilon = \upsilon \circ h$ implies that $h^{-1}(\mathbf{S}^1) = \mathbf{S}^1 \cup \gamma \cup \upsilon(\gamma)$ (see Figure 18).

To simplify the notation, we set $h^{\circ N}(R_s) = R_{s'}$ and $h^{\circ N}(R_t) = R_{t'}$. Let $R_u = R_{s'+\frac{1}{2}}$, $R_v = R_{t'+\frac{1}{2}}$, and w be their landing point. Then $h(R_u \cup \{w\} \cup R_v) = h^{\circ N+1}(R_s \cup \{z\} \cup R_t)$. We shall see that $R_u \cup \{w\} \cup R_v$ separates $\mathbf{S}^1$ from c as following.

Assume that $R_u \cup \{w\} \cup R_v$ does not separate $\mathbf{S}^1$ from c. Let V be the component of $\mathbf{C} - (R_u \cup \{w\} \cup R_v)$ which does not contain c, and thus it does not contain $\mathbf{S}^1$. Then $A(V) = A(h^{\circ N}(U))$ by $A(V) < 1/2$. So h is univalent on V and thus $A(h(V)) = 2 \cdot A(V)$. Then $A(h(V)) = 2 \cdot A(V) = 2 \cdot A(h^{\circ N}(U)) = A(h^{\circ N+1}(U))$ implies that $h(V) = h^{\circ N+1}(U) \supset \mathbf{S}^1$. So V contains a preimage of $\mathbf{S}^1$. This is impossible, for $h^{-1}(\mathbf{S}^1) = \mathbf{S}^1 \cup \gamma \cup \upsilon(\gamma)$. $\square$

Proof of Theorem 1.3. We use proof by contradiction, and thus assume that $c \notin \{h^{\circ n}(z_0)\}_{n\geq 0}$. Then there exist two distinct external rays R_u and R_v with a common landing point w such that $R_u \cup \{w\} \cup R_v$ separates $\mathbf{S}^1$ from c by Lemma 6.3. Let U be the component of $\mathbf{C} - (R_u \cup \{w\} \cup R_v)$ which contains $\mathbf{S}^1$. We

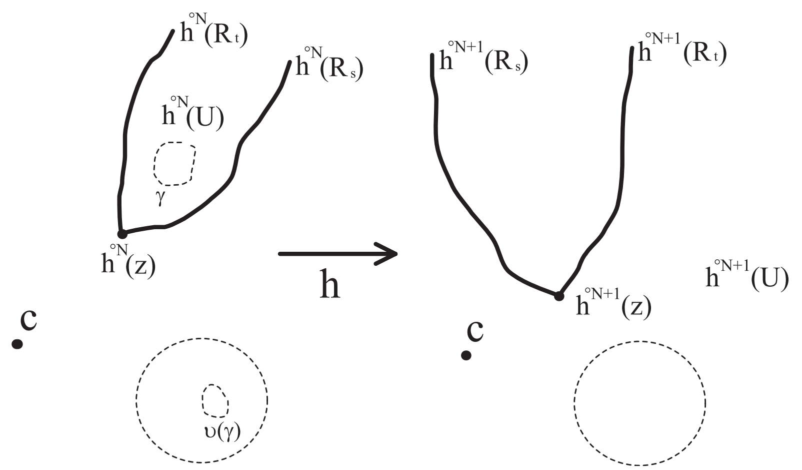

FIGURE 18

cut off U along an equipotential curve E_r, and thus have the Jordan closed curve $\gamma \subset \mathbf{C} - \overline{\mathbf{D}}$. Then h is injective on γ and preserves the orientation. Let V' be the Jordan annular domain which is surrounded by γ and $\mathbf{S}^1$. Since $\overline{V'}$ does not contain the pole $\upsilon(a)$, it follows from Lemma 3.2 that h is injective on $\overline{V'}$. Then $\overline{h(V')} \subset \mathbf{C} - \mathbf{D}$ implies that $\overline{V'}$ does not contain the zero a.

We put $V = V' \cup \mathbf{S}^1 \cup \upsilon(V')$. So $\overline{V}$ does not contain any of the pole $\upsilon(a)$, the zero a, two critical points c and $\upsilon(c)$ (see Figure 19).

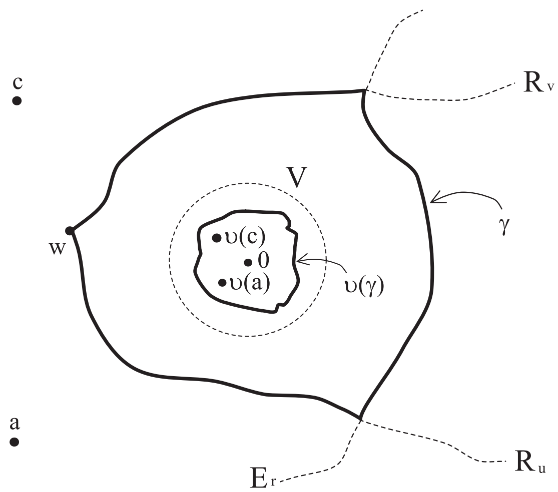

FIGURE 19

Moreover, h is injective on $\overline{V}$ by $h \circ \upsilon = \upsilon \circ h$. It follows from Lemma 3.1 that there exists a Jordan annular domain W such that $\overline{V} \subset W$ and h is univalent on a neighborhood of $\overline{W}$. We may suppose that both W and $h(W)$ do not contain the origin.

Now we take a Herman compactum H for (h, W) by Proposition 2.3. Then H meets the outer component of the boundary ∂W but not $\gamma - \{w\}$, so H must contain w. Let Ω be the unbounded component of $\hat{\mathbf{C}} - H$. Then $\partial\Omega - \{w\}$ is disconnected, and thus the point w is biaccessible from Ω. However, the biaccessibility of w contradicts Proposition 2.4. $\square$

Acknowledgements. I would like to thank the referee for taking the time to read this paper and for making perceptive comments. I am grateful to Hiroshige Shiga and Naoya Sumi for many fruitful discussions.

References

[**Ki**] J. Kiwi, Non-accessible critical points of Cremer polynomials, Ergod. Th. & Dynam. Sys. **20** (2000), 1391–1403.

[**Ma**] R. Mañé, On a theorem of Fatou, Bol. Soc. Bras. Math. **24** (1993), 1–11.

[**Mc**] C. McMullen, Complex dynamics and renormalization, Princeton University Press, 1994.

[**Mi**] J. Milnor, Dynamics in one complex variable, 3rd ed., Princeton University Press, 2006.

[**MS**] W. de Melo and S. van Strien, One-dimensional dynamics, Springer-Verlag, 1993.

[**P**] C. L. Petersen, Local connectivity of some Julia sets containing a circle with an irrational rotation, Acta Math. **177** (1996), 163–224.

[**Pe**] L. Petracovici, Non-accessible critical points of certain rational functions with Cremer points, Ann. Acad. Sci. Fenn. Math. **31** (2006), 3–11.

[**PM**] R. Pérez-Marco, Fixed points and circle maps, Acta Math. **179** (1997), 243–294.

[**Po**] Ch. Pommerenke, Boundary behaviour of conformal maps, Springer-Verlag, 1992.

[**Ra**] M. Rabii, Local connectedness of the Julia set of the family $z^m(z^n - b)$, Ergod. Th. & Dynam. Sys. **18** (1998), 457–470.

[**Ro**] P. Roesch, Some rational maps whose Julia sets are not locally connected, Conf. Geom. and Dynam. **10** (2006), 125–135.

[**S**] D. E. K. Sørensen, Accumulation theorems for quadratic polynomials, Ergod. Th. & Dynam. Sys. **16** (1996), 555–590.

[**Sch**] W. Schmidt, Accessible fixed points on the boundary of stable domains, Result. Math. **32** (1997), 115–120.

[**Smi**] S. K. Smirnov, On supports of dynamical laminations and biaccessible points in polynomial Julia sets, Colloquium Mathematicum. **87**(2) (2001), 287–295.

[**SZ**] D. Schleicher and S. Zakeri, On biaccessible points in the Julia set of a Cremer quadratic polynomial, Proc. Amer. Math. **128** (1999), 933–937.

[**Yo1**] J.-C. Yoccoz, Théorème de Siegel, nombres de Bruno et polynômes quadratiques, Astérisque **231** (1995), 3–88.

[**Yo2**] J.-C. Yoccoz, Analytic linearization of circle diffeomorphisms, Dynamical systems and small divisors, Lecture notes in math. **1784**, Springer-Verlag, 2002, 125–173.

[**Za**] S. Zakeri, Biaccessibility in quadratic Julia sets, Ergod. Th. & Dynam. Sys. **20** (2000), 1859–1883.

[**Zd**] A. Zdunik, On biaccessible points in Julia sets of polynomials, Fund. Math. **163** (2000), 277–286.

Mitsuhiko Imada
Department of Mathematics
Tokyo Institute of Technology
2-12-1 Oh-okayama, Meguro-ku
Tokyo 152-8551
Japan
E-mail: imada.m.aa@m.titech.ac.jp