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ON BIFURCATION FROM INFINITY FOR POTENTIAL OPERATORS

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1. Introduction. Let X be a Hilbert space and $\mathcal{U}$ a neighborhood of infinity; i.e., $X \setminus B_R \subset \mathcal{U}$ for some $R > 0$, where B_R is the ball of radius R centered at the origin in X. We consider the following operator equation with a parameter $\lambda \in \mathbb{R}$:

$$Lu + G(u) = \lambda u, \tag{1}$$

where L is a bounded linear selfadjoint operator on X and $G \in C(\Omega, X)$, with $G(u) = o(\|u\|)$ as $\|u\| \longrightarrow \infty$. Furthermore, we assume that G is of potential type; i.e., there exists $g \in C^1(\Omega, \mathbb{R})$ such that $dg = G$.

We are concerned with solutions of equation (1) which bifurcate from infinity; i.e., solutions (λ, u) of (1) with norm $\|u\|$ large and λ close to some eigenvalue of L. We say that (μ, ∞) is a bifurcation point from infinity for (1) if, for any $\epsilon > 0$ and $M > 0$, there exists a solution (λ, u) such that

$$|\lambda - \mu| < \epsilon \quad \text{and} \quad \|u\| > M.$$

There have been extensive studies on bifurcation from trivial solutions (such as θ) (see [3–6], [9], [12–14], [16] etc). And for bifurcation from infinity, Rabinowitz in [14] proved that (μ, ∞) is a bifurcation point of (1) from infinity provided L is compact and μ is an eigenvalue of L of odd multiplicity. Note that in [14] it is not necessary to assume that L and G are potentials. The result in [14] was obtained by combining a rescaling of the variable and results on bifurcation from the trivial solution due to Krasnosel'skii and Rabinowitz (also see [3–4], [9], [13]). Generally, for an eigenvalue μ of L of even multiplicity, (μ, ∞) may not be a bifurcation point for (1). A result by Toland [16], later improved by Diaz and Hernandez in [6], shows that for a selfadjoint operator L and a potential operator G , (μ, ∞) is always a bifurcation point from infinity for (1) provided μ is an isolated eigenvalue with finite multiplicity. Note that generally there is no connected branch bifurcating from (μ, ∞) (see [3]).

The goal of this paper is to prove that under some further conditions the parameter values of λ of solutions bifurcating from (μ, ∞) cover at least a one-sided neighborhood of μ provided there are no solutions of $(1)_\mu$ tending towards infinity ($(1)_\mu$ is equation (1) for $\lambda = \mu$). This result is a slightly weaker analogue of the results in [4], [5], [13] on bifurcations from trivial solutions for potential operators. Precisely, our results read as follows.

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Theorem 1.1. *Let L and G be as above. Suppose that $J_\lambda \in C^2(\Omega, \mathbb{R})$, with*

$$dJ_\lambda(u) = Lu + G(u) - \lambda u.$$

If μ is an isolated eigenvalue of L of finite multiplicity, then (μ, ∞) is a bifurcation point from infinity for (1). Moreover, if J_μ satisfies the Palais-Smale condition, then at least one of the following alternatives occurs:

(a) *There are infinitely many solutions for equation (1) with $\lambda = \mu$, say (μ, u_n), with $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$.*

(b) *There is a one-sided neighborhood Λ of μ such that for all $\lambda \in \Lambda \setminus \{\mu\}$, equation (1) possesses at least one solution u_λ with $\|u_\lambda\| \longrightarrow \infty$ as $\lambda \longrightarrow \mu$.*

Remark 1.1. We may weaken the smoothness condition in Theorem 1.1 at the expense of placing more restrictions on the operator L. We have:

Theorem 1.2. *Let L be a bounded linear selfadjoint operator with an isolated eigenvalue μ of finite multiplicity. Assume that there are only finitely many eigenvalues of L larger than (or less than) μ and that all these eigenvalues are of finite multiplicity. Suppose that $J_\lambda \in C^1(\Omega, \mathbb{R})$, with*

$$dJ_\lambda(u) = Lu + G(u) - \lambda u,$$

and that J_μ satisfies the Palais-Smale condition. Then (μ, ∞) is a bifurcation point from infinity of (1). Moreover, at least one of the two alternatives in Theorem 1.1 occurs.

This paper is further organized in three sections. In §2 we give the proofs of the above theorems and in §3, besides presenting a few further remarks, we shall give an application to Landesman-Lazer type problems. By combining some a priori estimates and our bifurcation results we obtain extensions of some known results in [11], due to Mawhin and the first author, where only eigenvalues of L of odd multiplicity were considered. Precisely, we shall consider

$$\begin{cases} -\lambda \Delta u = u + h(u), & \text{in } B \\ u = 0, & \text{on } \partial B, \end{cases} \tag{2}$$

where $B \subset \mathbb{R}^n$ is a bounded domain with regular boundary. Let μ be an eigenvalue of $-\Delta$. Under some assumptions on h, we can prove that there exists a neighborhood of μ, $[\mu - \eta, \mu + \eta]$, such that for any $\lambda \in [\mu - \eta, \mu + \eta]$ equation (2) has a solution u_1^λ and for $\lambda \in (\mu, \mu + \eta]$ at least two solutions u_1^λ and u_2^λ, with u_1^λ uniformly bounded for $\lambda \in [\mu - \eta, \mu + \eta]$ and $\|u_2^\lambda\| \longrightarrow \infty$ as $\lambda \longrightarrow \mu$. Similar results were obtained in [11] for the case of μ being of odd multiplicity.

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2. Proofs of the main theorems. Bifurcation from infinity was studied in [6] and [16] for potential operators. For completeness we include a short proof here. And from the proof we shall see that generally we can not get the alternatives in our theorems simply by rescaling the results on bifurcation from zero in [4] and [13]. First, we need the following well known result (see [9], [13]).

Lemma 2.1. *Let L be a bounded linear selfadjoint operator and O be a a neighborhood of the origin θ . Assume that $H \in C^1(O, X)$ is a potential operator; i.e., $\exists h \in C^2(O, \mathbb{R})$ such that $dh = H$. If μ is an isolated eigenvalue of L of finite multiplicity, then (μ, θ) is a bifurcation point of the equation*

$$Lu + H(u) = \lambda u;$$

i.e., there exist solutions (λ_n, u_n) such that $u_n \neq \theta$, $\|u_n\| \longrightarrow 0$ and $\lambda_n \longrightarrow \mu$.

Proof of Theorem 1.1 (part 1): Consider the functional

$$h(w) = \|w\|^4 g(\frac{w}{\|w\|^2})$$

in a neighbourhood O of θ.

By using the properties of G, h is well defined on $O \setminus \{\theta\}$. Define $h(\theta) = 0$. Then it is easy to check that $h \in C^2(O, \mathbb{R})$ and

$$H(w) \stackrel{\text{def}}{=} dh(w) = \circ(\|w\|), \quad \text{as } \|w\| \longrightarrow 0.$$

By Lemma 1.1, there exists (λ_n, w_n) solving

$$Lw_n + H(w_n) = \lambda_n w_n,$$

with $\lambda_n \longrightarrow \mu$ and $w_n \neq \theta$, $\|w_n\| \longrightarrow 0$. Calculating directly, we find that for any $v \in X$,

$$\begin{aligned}\langle H(w), v\rangle &= \langle dh(w), v\rangle \\ &= \langle \|w\|^2 G(\frac{w}{\|w\|^2}) + 4\|w\|^2 g(\frac{w}{\|w\|^2})w - 2\langle G(\frac{w}{\|w\|^2}), w\rangle w, v\rangle,\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in X. By using this and the substitution $u_n = \frac{w_n}{\|w_n\|^2}$, one has

$$\frac{1}{\|u_n\|^2} Lu_n + \frac{1}{\|u_n\|^2} G(u_n) + \frac{1}{\|u_n\|^4}(4g(u_n) - 2\langle G(u_n), u_n\rangle)u_n = \frac{\lambda_n}{\|u_n\|^2} u_n.$$

Eliminating $\|u_n\|^2$, we get

$$Lu_n + G(u_n) + \frac{4g(u_n) - 2\langle G(u_n), u_n\rangle}{\|u_n\|^2} u_n = \lambda_n u_n.$$

Writing

$$\rho_n = \frac{4g(u_n) - 2\langle G(u_n), u_n\rangle}{\|u_n\|^2},$$

then, by the assumption on G, $\rho_n \longrightarrow 0$ as $n \longrightarrow \infty$. Define $\bar{\lambda}_n = \lambda_n - \rho_n$. Then

$$Lu_n + G(u_n) = \bar{\lambda}_n u_n \tag{3}$$

with $\bar{\lambda}_n \longrightarrow \mu$ and $\|u_n\| \longrightarrow \infty$.

In order to prove the second part of Theorem 1.1, we need some preparation of results in critical point theory, especially Morse theory for isolated critical sets as developed by Chang in [4]. Let $f \in C^1(X, \mathbb{R})$ satisfy the Palais-Smale condition, and let

$$K(f) = \{u \in X : f'(u) = \theta\},$$
$$f_a = \{u \in X : f(u) \le a\},$$
$$f^{-1}(\alpha, \beta) = \{u \in X : \alpha < f(u) < \beta\}.$$

A subset $S \subset K$ is called an *isolated critical set* of f if there exist an open set O and an interval $(\alpha, \beta) \subset \mathbb{R}$ such that

$$S = K \cap f^{-1}(\alpha, \beta) \cap \tilde{O}, \quad \text{and} \quad K \cap (f^{-1}(\alpha) \cup f^{-1}(\beta)) \cap \tilde{O} = \emptyset,$$

where

$$\tilde{O} = \bigcup_{t \in \mathbb{R}} \eta(t, O)$$

and η is the negative gradient flow (or negative pseudo gradient flow).

Definition 2.1. (See [5]) Let S be an isolated critical set of f; the critical groups of f at S are defined as

$$C_q(f, S) = H_q(f_\beta \cap \tilde{O}, f_\alpha \cap \tilde{O}; \mathcal{F}), \qquad q = 0, 1, 2, \dots ,$$

where $(\alpha, \beta) \subset \mathbb{R}$ and O satisfy the properties above. $H_*(\cdot, \cdot)$ is the singular relative homology group and $\mathcal{F}$ is the coefficient group.

By the excision property, the critical groups are independent of the special choice of the pseudo gradient vector field. In [5], Chang studied properties of critical groups for isolated critical points and isolated critical sets. Among these properties the most important one we are going to use is the homotopy invariance. To state it clearly we have to expand the theory further.

Definition 2.2. (See [5]) Let S be an isolated critical set of f. A pair of topological spaces (W, W_-) is called a Gromoll-Meyer pair (or G-M pair, for short) with respect to a pesudo gradient vector field V of f, if:

(1) W is a closed neighborhood of S possessing the mean value property, i.e., if for $t_1 < t_2$, $\eta(t_1)$, $\eta(t_2) \in W$, then $\forall t \in [t_1, t_2]$, $\eta(t) \in W$; and

$$W \cap f_\alpha = W_+ \cap f^{-1}(\alpha) \cap K = \emptyset, \quad W \cap K = S,$$

where

$$W_+ = \bigcup_{t \ge 0} \eta(t, W).$$

(2) $W_- = \{x \in W : \eta(t, x) \notin W, \ \forall t > 0\}$ is closed in W.

(3) W_- is a piecewise submanifold and the flow η is transversal to W_-.

It is easy to check that we may simply take

$$W = \tilde{O} \cap f^{-1}[\alpha', \beta'], \quad W_- = W \cap f^{-1}(\alpha') \tag{4}$$

as a G-M pair of S, where α', β' satisfy

$$\alpha < \alpha' < \inf_{x \in S} f(x), \quad \beta > \beta' > \sup_{x \in S} f(x). \tag{5}$$

And we have that W is bounded if O is bounded. The following theorem was given in [5].

Theorem 2.1. *Suppose that (W, W_-) is a G-M pair of S, then*

$$C_*(f, S) = H_*(W, W_-).$$

Now, let f and g be C^1- functions satisfying the Palais-Smale condition. Suppose that there is an open set O in X such that $S_f = K(f) \cap \tilde{O}_{df} \cap f^{-1}(\alpha, \beta)$, $S_g = K(g) \cap \tilde{O}_{dg} \cap g^{-1}(\alpha', \beta')$ are isolated critical sets of f and g respectively, where

$$\tilde{O}_{dg} = \bigcup_{t \in R} \eta_1(t, O), \qquad \tilde{O}_{df} = \bigcup_{t \in R} \eta_2(t, O),$$

and η_1 and η_2 are negative gradient flows of dg and df respectively. Then we have the following theorem from [5] on the homotopy invariance of critical groups.

Theorem 2.2 (See [5]). *Let (W, W_-) be a G-M pair of S_f such that $O \subset W$. Then there exists $\epsilon > 0$ such that (W, W_-) is also a G-M pair of S_g with respect to a certain pseudo gradient vector field of g, provided $\|g - f\|_{C^1(W)} < \epsilon$.*

Proof of Theorem 1.1 (Continuation)**:** Because we are only interested in solutions of (1.1) bifurcating from infinity, i.e., solutions with large norms, we may assume, without loss of generality, that G is defined on the whole space X and satisfies

$$\frac{\|G(u)\|}{\|u\|} < \operatorname{dist}(\mu, \sigma(L) \setminus \{\mu\}). \tag{6}$$

Since μ is an isolated eigenvalue of L, $\operatorname{dist}(\mu, \sigma(L) \setminus \{\mu\}) > 0$. Denote the projection of H onto $\operatorname{Ker}(L - \mu I)$ by P; then (1) is equivalent to

$$\begin{cases} \mu x + PG(x + x^\perp) = \lambda x \\ Lx^\perp + (I - P)G(x + x^\perp) = \lambda x^\perp, \end{cases} \tag{7}$$

where $u = (x, x^\perp) \in \operatorname{Ker}(L - \mu I) \oplus [\operatorname{Ker}(L - \mu I)]^\perp$. By using the Lyapunov-Schmidt reduction method and (6) we know that for each $x \in \operatorname{Ker}(L - \mu I)$ and λ in a neighborhood of μ, there exists a unique $x^\perp \overset{\text{def}}{=} \phi(\lambda, x)$ such that

$$L\phi(\lambda, x) + (I - P)G(x + \phi(\lambda, x)) = \lambda\phi(\lambda, x), \tag{8}$$

where ϕ is a C^1-map with respect to x and satisfies $\phi(\lambda, x) = o(\|x\|)$ as $\|x\| \longrightarrow \infty$ uniformly for λ in a neighborhood of μ. Substituting $\phi(\lambda, x)$ into the first equation in (7), we have

$$\mu x + PG(x + \phi(\lambda, x)) = \lambda x. \tag{9}$$

And this is still an operator equation in potential form. The Euler functional corresponding to (9) is

$$f_\lambda(x) = \tfrac{1}{2}(\mu - \lambda)\|x\|^2 + \tfrac{1}{2}\langle L\phi(\lambda, x), \phi(\lambda, x)\rangle - \frac{\lambda}{2}\|\phi(\lambda, x)\|^2 + g(x + \phi(\lambda, x)),$$

for $x \in \operatorname{Ker}(L - \mu I)$. If x is a critical point of $f_\lambda(x)$, $(x, \phi(\lambda, x))$ is a solution of (1). By our assumption, J_μ satisfies the Palais-Smale condition, and so does f_μ. For $\lambda \neq \mu$ in a neighborhood of μ, f_λ also satisfies the Palais-Smale condition.

Now, if the alternative (a) in Theorem 1.1 is not true, there exists $R > 0$ such that

$$K_\mu \overset{\text{def}}{=} K(f_\mu) \subset \bar{B}_R. \tag{10}$$

If the alternative (b) is also not true, we shall deduce a contradition as follows.

First, there are $\lambda_n^\pm$ with $\lambda_n^- < \mu < \lambda_n^+$ and $\lambda_n^\pm \longrightarrow \mu$ as $n \longrightarrow \infty$ such that

$$K_{\lambda_n^\pm} = K(f_{\lambda_n^\pm}) \subset \bar{B}_R, \tag{11}$$

where, without loss of generality, we have used the same R as above. Take $O = \bar{B}_R$; then $\exists \alpha < \beta$ such that

$$S_\mu = K_\mu \subset f_\mu^{-1}(\alpha, \beta) \cap \tilde{O}_\mu, \qquad S_\mu \cap (f_\mu^{-1}(\alpha) \cup f_\mu^{-1}(\beta)) \cap \tilde{O}_\mu = \emptyset,$$

and

$$S_{\lambda_n^\pm} = K_{\lambda_n^\pm} \subset f_{\lambda_n^\pm}^{-1}(\alpha, \beta) \cap \tilde{O}_{\lambda_n^\pm}, \quad S_{\lambda_n^\pm} \cap (f_{\lambda_n^\pm}^{-1}(\alpha) \cup f_{\lambda_n^\pm}^{-1}(\beta)) \cap \tilde{O}_{\lambda_n^\pm} = \emptyset,$$

where

$$\tilde{O}_{\lambda_n^\pm} = \bigcup_{t \in R} \eta_{\lambda_n^\pm}(t, O)$$

and $\eta_{\lambda_n^\pm}$ is the negative gradient vector field of $df_{\lambda_n^\pm}$. Now the critical groups of f_μ at S_μ, $C_q(f_\mu, S_\mu)$, are well defined. Set

$$W_\mu = \tilde{O}_\mu \cap f^{-1}[\alpha', \beta'], \qquad W_{\mu_-} = W \cap f_\mu^{-1}(\alpha'),$$

with

$$\alpha < \alpha' < \inf_{\lambda_n^\pm} \inf_{x \in S_{\lambda_n^\pm}} f_{\lambda_n^\pm}(x)$$

$$\beta > \beta' > \sup_{\lambda_n^\pm} \sup_{x \in S_{\lambda_n^\pm}} f_{\lambda_n^\pm}(x).$$

Then (W_μ, W_{μ^-}) is a G-M pair of S_μ for f_μ. By Theorem 2.1,

$$C_q(J_\mu, S_\mu) = H_q(W_\mu, W_{\mu^-}). \tag{12}$$

Suppose

$$\alpha < \inf_{x \in \bar{B}_R} f_\mu(x) \le \sup_{x \in \bar{B}_R} f_\mu(x) < \beta$$

(we may choose $\alpha < \beta$ satisfying this), so $\bar{B}_R \subset W_\mu$. By Theorem 2.2, $\exists \epsilon > 0$ such that (W_μ, W_{μ^-}) is also a G-M pair of $S_{\lambda_n^\pm}$ with respect to a certain pseudo gradient vector field of $f_{\lambda_n^\pm}$ provided $\|f_{\lambda_n^\pm} - f_\mu\|_{C^1(W_\mu)} < \epsilon$. Since W_μ is bounded, for this $\epsilon > 0$, $\exists n_0 > 0$ such that

$$\|f_{\lambda_n^\pm} - f_\mu\|_{C^1(W_\mu)} < \epsilon, \qquad \text{for } n \ge n_0.$$

Hence (W_μ, W_{μ^-}) is also a G-M pair of $S_{\lambda_n^\pm}$ with respect to a certain pseudo gradient vector field of $f_{\lambda_n^\pm}$ for $n \ge n_0$. By Theorem 2.1,

$$C_q(f_{\lambda_n^\pm}, S_{\lambda_n^\pm}) \cong H_q(W_\mu, W_{\mu^-}), \quad n \ge n_0. \tag{13}$$

Now let us calculate $C_q(f_{\lambda_n^\pm}, S_{\lambda_n^\pm})$ for $n \ge n_0$.

Lemma 2.2.

$$f_{\lambda_n^\pm}(x) \longrightarrow \mp\infty, \quad \textit{as } \|x\| \longrightarrow \infty. \tag{14}$$

Proof: Let us consider λ_n^- only:

$$f_{\lambda_n^-}(x) =$$

$$\tfrac{1}{2}\|x\|\{(\mu-\lambda_n^-)+\tfrac{1}{2}\langle L\frac{\phi(\lambda_n^-,x)}{\|x\|},\frac{\phi(\lambda_n^-,x)}{\|x\|}\rangle-\frac{\lambda_n^-}{2}\frac{\|\phi(\lambda_n^-,x)\|^2}{\|x\|^2}+\frac{g(x+\phi(\lambda_n^-,x))}{\|x\|^2}\}.$$

By using the fact that $\frac{\phi(\lambda_n^-,x)}{\|x\|} \longrightarrow 0$ as $\|x\| \longrightarrow \infty$, the conclusion follows.

Now it is easy to check that

$$C_q(f_{\lambda_n^-}, S_{\lambda_n^-}) = \begin{cases} \mathcal{F}, & q = 0 \\ 0, & q \neq 0, \end{cases} \tag{15}$$

and

$$C_q(f_{\lambda_n^+}, S_{\lambda_n^+}) = \begin{cases} \mathcal{F}, & q = m = \dim \operatorname{Ker}(L-\mu I) \\ 0, & q \neq m. \end{cases} \tag{16}$$

In fact, say for λ_n^-, we have, by definition,

$$C_q(f_{\lambda_n^-}, S_{\lambda_n^-}) = H_q((f_{\lambda_n^-})_\beta \cap \tilde{O}_{\lambda_n^-}, (f_{\lambda_n^-})_\alpha \cap \tilde{O}_{\lambda_n^-}, \mathcal{F}). \tag{17}$$

Because there are no critical values outside the interval (α,β), then by using the deformation property, (17) also holds when we replace α and β by α',β' satisfying $\alpha' \leq \alpha$ and $\beta' \geq \beta$. Take $\alpha' < 0$ and $|\alpha'|$ large enough, then it follows from Lemma 2.2 that

$$(f_{\lambda_n^-})_{\alpha'} = \{x : f_{\lambda_n^-}(x) \leq \alpha' \} = \emptyset.$$

So

$$C_q(f_{\lambda_n^-}, S_{\lambda_n^-}) = H_q((f_{\lambda_n^-})_\beta \cap \tilde{O}_{\lambda_n^-}).$$

Since $(f_{\lambda_n^-})_\beta \cap \tilde{O}_{\lambda_n^-}$ is contractible, (15) follows immediately. Similarly, we may get (16).

Now (15) and (16) contradict with (13). So the proof of Theorem 1.1 is complete.

The proof of Theorem 1.2 is essentially the same as the proof of Theorem 1.1 except that we do not need to use the Lyapnov-Schmidt reduction. We work in the whole space X with the functional

$$J_\lambda(u) = \tfrac{1}{2}\langle Lu, u\rangle + g(u) - \frac{\lambda}{2}\|x\|^2.$$

If we assume that neither alternative (a) nor (b) in Theorem 1.2 is true, we may finally obtain

$$C_q(J_{\lambda_n^\pm}, S_{\lambda_n^\pm}) = H_q(W_\mu, W_{\mu^-}), \quad \text{for } n \text{ large}. \tag{18}$$

Now, we need to calculate $C_q(J_{\lambda_n^-}, S_{\lambda_n^-})$ and $C_q(J_{\lambda_n^+}, S_{\lambda_n^+})$.

Lemma 2.3. *Assume that there are only a finite number of eigenvalues of L less than μ, say, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$, and that every λ_i $(i = 1, 2, \cdots, k)$ is of finite multiplicity, then*

$$C_q(J_{\lambda_n^-}, S_{\lambda_n^-}) \cong \begin{cases} \mathcal{F}, & q = \sum_{i=1}^k d(\lambda_i) \\ 0, & otherwise \end{cases} \tag{19}$$

and

$$C_q(J_{\lambda_n^+}, S_{\lambda_n^+}) \cong \begin{cases} \mathcal{F}, & q = \sum_{i=1}^k d(\lambda_i) + d(\mu) \\ 0, & otherwise, \end{cases} \tag{20}$$

where $d(\lambda_i)$ equals the algebraic multiplicity of λ_i.

Proof: See [18].

Now (19) and (20) contradict with (18).

3. Bounded perturbations of elliptic problems. In this section, we shall apply the results established above to extend results obtained by Mawhin and the first author [11] for Dirichlet boundary value problems for semilinear elliptic problems on a bounded domain. In fact, we shall show that the results obtained there are valid for any eigenvalue of the linear problem, not only those of odd algebraic multiplicity. Specifically, we shall be interested in the following boundary value problem on the bounded domain B, having a smooth boundary ∂B :

$$\begin{cases} -\lambda \Delta u = u + h(u), & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases} \tag{21}$$

where

$$h : \mathbb{R} \to \mathbb{R},$$

is a smooth and bounded function.

By choosing $X = H_0^1(B)$, it is well known that this equation may be rewritten as an equivalent equation in X of the form (1), where in fact L is a compact selfadjoint operator.

Let us now assume that μ is an eigenvalue of L; that is, in this context,

$$\begin{cases} -\mu \Delta u = u, & \text{in } B, \\ u = 0, & \text{on } \partial B. \end{cases} \tag{22}$$

We shall now give conditions on h, which are motivated by those in [11], which will imply that the associated functional J_λ satisfies the Palais-Smale condition for the eigenvalue μ; for values of λ which are not eigenvalues the Palais-Smale condition is easiliy verified. This then will allow for an application of Theorem 1.1.

Thus let λ be fixed and let $\{u_n\} \subset X$ be a sequence such that $\{J_\lambda(u_n)\}$ is bounded and $\{dJ_\lambda(u_n)\} \to 0$. Then there exists a sequence $\{\epsilon_n\} \subset X$ which converges to θ in that space and such that

$$Lu_n + G(u_n) = \lambda u_n + \epsilon_n, \tag{23}$$

where L is compact and G is completely continuous. If now $\lambda \neq \mu$, then the boundedness of h immediately implies that the sequence $\{u_n\} \subset X$ is bounded, and hence because of the complete continuity of L and G, a convergent subsequence will exist.

We next impose a condition on h to obtain uniform bounds for solutions as well as for the Palais-Smale sequences of our nonlinear problem for values of λ in intervals of the form $[\delta, \mu]$, which besides μ contain no other eigenvalues.

To this end let us write

$$u_n = v_n + w_n,$$

where

$$v_n \in Ker(L - \mu I),$$

and

$$w_n \in Ker(L - \mu I)^{\perp},$$

and u_n satisfies the equation

$$\begin{cases} -\lambda_n \Delta u_n = u_n + h(u_n) + \delta_n, & \text{in } B, \\ u_n = 0, \text{on } \partial B, \end{cases} \tag{24}$$

with $\delta_n \to \theta$ in $L^2(B)$.

Substituting the expression $u_n = v_n + w_n$ into the above equation, we immediately obtain, because of the boundedness of h, a uniform bound (on compact λ intervals) for the sequence $\{w_n\}$. Thus, to obtain a uniform bound on the sequence $\{u_n\}$, we must obtain a bound on the sequence $\{v_n\}$. Using arguments similar to the above, we then obtain a convergent subsequence of $\{u_n\}$.

If $\{u_n\}$ is unbounded, we may write

$$u_n = t_n y_n + w_n,$$

where $\{t_n\} \subset \mathbb{R}$ is an unbounded sequence of positive numbers and $\{y_n\} \subset \mathrm{Ker}(L - \mu I)$ is a sequence, with $\|y_n\| = 1$. Since $\mathrm{Ker}(L - \mu I)$ is finite dimensional, we may assume that $y_n \to y \in \mathrm{Ker}(L - \mu I)$.

We now obtain the following equation:

$$\begin{cases} -\lambda_n \Delta t_n y_n + w_n = t_n y_n + w_n + h(t_n y_n + w_n) + \delta_n, & \text{in } B, \\ t_n y_n + w_n = 0, & \text{on } \partial B, \end{cases} \tag{25}$$

which may be rewritten as

$$\begin{cases} -\lambda_n \Delta w_n = (1 - \frac{\lambda_n}{\mu}) t_n y_n + w_n + h(t_n y_n + w_n) + \delta_n, & \text{in } B, \\ t_n y_n + w_n = 0, & \text{on } \partial B. \end{cases} \tag{26}$$

Multiplying the above equation by y and integrating the result over the domain B, we obtain

$$\int_B (1 - \frac{\lambda_n}{\mu}) t_n y_n y \, dx + \int_B h(t_n y_n + w_n) y \, dx + \int_B \delta_n y \, dx = 0. \tag{27}$$

Since $y_n \to y$ and $\delta_n \to \theta$, it follows that the first term in the above sum will be nonnegative and that the third term tends to zero; hence, we obtain a contradiction to the assumption of unboundedness, once we stipulate the following:

Assumption: For all convergent sequences $\{y_n\} \subset \mathrm{Ker}(L-\mu I)$, $y_n \to y$, $\|y_n\| = 1$, all bounded sequences $\{w_n\} \subset \mathrm{Ker}(L-\mu I)^{\perp}$, all unbounded sequences of positive numbers $\{t_n\} \subset \mathbb{R}$ the following holds:

$$\liminf_{n\to\infty} \int_B h(t_n y_n + w_n) y \, dx > 0.$$

Hence using arguments as in [11] we obtain a continuum of solutions of equation (21) for $\lambda \in [\delta, \mu + \eta]$ as long as this interval only contains μ as an eigenvalue and $\eta > 0$ is small. This gives one solution u_1^λ which is uniformly bounded for $\lambda \in [\delta, \mu + \eta]$. Furthermore, Theorem 1.2 also lets us conclude a bifurcation from (μ, ∞), say u_2^λ, and the parameter values λ of bifurcating solutions cover at least a one-sided neighborhood Λ of μ. By the a priori estimates above, Λ must be a right hand neighborhood of μ. Since $\|u_2^\lambda\| \to \infty$ as $\lambda \to \mu$, we may find a $\eta_1 > 0$ such that for $\lambda \in (\mu, \mu + \eta_1]$ equation (21) has at least two solutions. We summarize this in:

Theorem 3.1. *Under the assumption on h, there exists a $\eta_1 > 0$ such that for any $\lambda \in (\mu, \mu + \eta_1]$ equation (21) has at least two solutions and for $\lambda \in [\mu - \eta_1, \mu]$ at least one.*

Remark 3.1. We have a "dual" version of the above Theorem 3.1. Assume that in the assumption on h, we instead require

$$\limsup_{n\to\infty} \int_B h(t_n y_n + w_n) y \, dx < 0;$$

then there is a $\eta_1 > 0$ such that for any $\lambda \in [\mu - \eta_1, \mu)$ equation (21) has at least two solutions and at least one for any $\lambda \in [\mu, \mu + \eta_1]$.

Remark 3.2. We may also allow sublinear perturbations in equation (21); i.e., we may assume that that $|h(u)| \le c|u|^\alpha$, for $|u|$ large, where $0 < \alpha < 1$ is a constant. Details on the a priori bounds in this case can be found in [11].

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