ON BIORTHOGONAL SYSTEMS

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1. INTRODUCTION

Let H be a separable Hilbert space. A sequence of pairs $\{x_n, y_n\}$ of elements of H is said to be a *biorthonormal system* if

$$(x_n, y_m) = \delta_{nm} = \begin{cases} 1 & (n = m), \\ 0 & (n \neq m). \end{cases}$$

A biorthonormal system $\{x_n, y_n\}$ is said to be *complete* if every $f \in H$ can be written in the form

$$f = \sum_{n=1}^{\infty} (f, y_n) x_n = \sum_{n=1}^{\infty} (f, x_n) y_n.$$

A sequence $\{\phi_n\}$ of elements of H is said to be *orthonormal* if the system $\{\phi_n, \phi_n\}$ is a biorthonormal system. An orthonormal sequence $\{\phi_n\}$ is said to be *complete* if the system $\{\phi_n, \phi_n\}$ is complete.

Concerning the relation between the completeness of biorthonormal systems and the asymptotic estimates for the eigenfunctions of a Sturm-Liouville problem, F. Brauer [2], [3] proved the following theorem.

THEOREM 1. *Let $\{\phi_n\}$ be a complete orthonormal sequence, and let $\{x_n, y_n\}$ be a biorthonormal system such that*

$$\sum_{n=1}^{\infty} \|\phi_n - x_n\|^2 < +\infty, \qquad \sum_{n=1}^{\infty} \|\phi_n - y_n\|^2 < +\infty.$$

Then the system $\{x_n, y_n\}$ is complete.

Let us define a linear transformation K of H into itself by $x_n - \phi_n = K\phi_n$ $(n = 1, 2, \cdots)$. Under the hypotheses of Theorem 1, it can be shown that K is a bounded transformation (F. Brauer [2, p. 380]). In order to prove Theorem 1, it is sufficient to prove the existence of the bounded inverse transformation of $I + K$, where I is the identity transformation. In particular, if the norm of K is less than 1, the bounded inverse transformation of $I + K$ is given by the Neumann series $\sum_{m=0}^{\infty}(-K)^m$. This is the essential part of the Riesz-Nagy proof of the Paley-Wiener theorem on perturbations of orthonormal sequences (F. Riesz and B. Sz.-Nagy [4, pp. 208-209]). Notice that, if the norm of K is less than 1, we do not need to assume anything on $\{y_n\}$. Actually, the Paley-Wiener theorem can be stated as follows:

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THEOREM 2. *Let* $\{\phi_n\}$ *be a complete orthonormal sequence, and let* $\{x_n\}$ *be a sequence of elements of* H *such that*

$$\sum_{n=1}^{\infty} \|\phi_n - x_n\|^2 < 1.$$

Then there is a sequence $\{y_n\}$ *of elements of* H *such that* $\{x_n, y_n\}$ *is a complete biorthonormal system.*

In the case where the norm of K is not less than 1, we need to assume some conditions on $\{y_n\}$ in order to prove that $\{x_n, y_n\}$ is complete. Under the hypotheses of Theorem 1, Brauer proved the existence of the bounded inverse transformation of $I + K$. In particular, he made essential use of the condition

$$\sum_{n=1}^{\infty} \|\phi_n - y_n\|^2 < +\infty. \tag{1.1}$$

On the other hand, in his proof of the uniqueness of solutions of the inverse Sturm-Liouville problem, G. Borg also proved the completeness of biorthonormal systems in L^2-space [1, pp. 32-60]. In Borg's paper, a sequence $\{x_n\}$ is said to be complete if the condition that $(f, x_n) = 0$ for every n implies $f = 0$. However, Brauer and Borg deal with essentially the same problem. Instead of using the condition (1.1) in any essential manner, Borg derived the existence of the bounded inverse transformation of $I + K$ from the compactness of K.

In this paper, we shall eliminate the condition (1.1) from the hypotheses of Theorem 1 by using a method similar to that of Borg. This refinement of Theorem 1 allows us to reduce the computations required in the asymptotic estimates for eigenfunctions. Actually, if $\{x_n\}$ is the sequence of eigenfunctions of the given boundary value problem, then the adjoint problem will be used only in determining $\{y_n\}$ for large n. We shall prove the following theorem.

THEOREM 3. *Let* $\{u_n, v_n\}$ *be a complete biorthonormal system. Assume that* $\{x_n\}$ $(n = 1, 2, \cdots)$ *is a sequence of elements of* H *such that any finite number of them are linearly independent. Let* $\{y_m\}$ $(m = N, N+1, N+2, \cdots)$ *be a sequence of elements of* H *such that*

$$(x_n, y_m) = \delta_{nm}, \tag{1.2}$$

where N *is a positive integer. Assume that there exists a compact linear transformation* K *of* H *into itself such that*

$$x_n - u_n = Ku_n. \tag{1.3}$$

Then there exist $N - 1$ *elements* $y_1, \cdots, y_{N-1}$ *of* H *such that* $\{x_n, y_n\}$ *is a complete biorthonormal system.*

For the case $N = 1$, Theorem 3 can be stated as follows.

THEOREM 4. *Let* $\{u_n, v_n\}$ *be a complete biorthonormal system, and let* $\{x_n, y_n\}$ *be a biorthonormal system. Assume that there exists a compact linear transformation* K *of* H *into itself such that*

$$x_n - u_n = Ku_n. \tag{1.4}$$

Then the system $\{x_n, y_n\}$ *is complete.*

In Section 3, we shall show that the following is a corollary of Theorem 4.

THEOREM 5. *Let* $\{\phi_n\}$ *be a complete orthonormal sequence, and let* $\{x_n, y_n\}$ *be a biorthonormal system such that*

$$\sum_{n=1}^{\infty} \|\phi_n - x_n\|^2 < +\infty .$$

Then the system $\{x_n, y_n\}$ *is complete.*

Remark. Eigenvalue-problem A in the paper of G. Borg [1, Chapter 2, pp. 32-53] gives an application of Theorem 4, while eigenvalue-problem B in the same paper [Chapter 2, pp. 53-60] gives an application of Theorem 3.

2. PROOF OF THEOREM 3

First of all, we shall prove

$$(2.1) \qquad (I + K^*) y_m = v_m \qquad (m \geq N),$$

where * indicates the adjoint transformation. In fact, the relations (2.1) can be derived from the relations

$$\delta_{nm} = (x_n, y_m) = ((I + K)u_n, y_m) = (u_n, (I + K^*)y_m) = (u_n, v_m)$$

and the completeness of $\{u_n, v_n\}$.

Assume that

$$(2.2) \qquad (I + K)f = 0 .$$

Then we deduce from (2.1) that $(f, v_m) = 0$ $(m \geq N)$. The system $\{u_n, v_n\}$ being complete, $f = \sum_{n<N} (f, v_n)u_n$. Hence the assumption (2.2) implies that

$$0 = \sum_{n<N} (f, v_n)(I + K)u_n .$$

From (1.3) we see that

$$0 = \sum_{n<N} (f, v_n)x_n .$$

Since $x_1, \cdots, x_{N-1}$ are linearly independent, the quantities (f, v_n) $(n < N)$ must be zero. Thus $(f, v_n) = 0$ for every n. Hence $f = 0$.

Now, K being compact, the inverse transformation $(I + K)^{-1}$ exists and is bounded. Define $y_1, \cdots, y_{N-1}$ by

$$y_n = (I + K^*)^{-1} v_n .$$

Then it is evident that $\{x_n, y_n\}$ is complete.

3. PROOF OF THEOREM 5

Let us define a linear transformation K by

$$Kf = \sum_{n=1}^{\infty} (f, \phi_n)(x_n - \phi_n). \tag{3.1}$$

This is bounded (F. Brauer [2, p. 380]). Put

$$K_N f = \sum_{n=1}^{N} (f, \phi_n)(x_n - \phi_n).$$

Then K_N is compact and $\|K - K_N\| \to 0$ as $N \to \infty$. Hence K is compact. On the other hand, $K\phi_n = x_n - \phi_n$. Thus Theorem 5 is a corollary of Theorem 4.

REFERENCES

1. G. Borg, *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte*, Acta Math. 78 (1946), 1-96.

2. F. Brauer, *On the completeness of biorthogonal systems*, Michigan Math. J. 11 (1964), 379-383.

3. ———, *Correction to "On the completeness of biorthogonal systems,"* Michigan Math. J. 12 (1965), 127-128.

4. F. Riesz and B. Sz.-Nagy, *Functional analysis*, F. Ungar Publ. Co., New York, 1955.

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