International Journal of Wavelets, Multiresolution
and Information Processing
Vol. 9, No. 3 (2011) 485–499

DOI: 10.1142/S0219691311004195

ON BIORTHOGONAL WAVELETS RELATED TO THE WALSH FUNCTIONS

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Received 17 September 2009
Revised 10 January 2011

In this paper, we describe an algorithm for computing biorthogonal compactly supported dyadic wavelets related to the Walsh functions on the positive half-line $\mathbb{R}_+$. It is noted that a similar technique can be applied in very general situations, e.g., in the case of Cantor and Vilenkin groups. Using the feedback-based approach, some numerical experiments comparing orthogonal and biorthogonal dyadic wavelets with the Haar, Daubechies, and biorthogonal 9/7 wavelets are prepared.

Keywords: Walsh functions; Walsh–Fourier transform; Haar wavelet; Daubechies wavelets; biorthogonal dyadic wavelets; Riesz basis; image processing.

AMS Subject Classification: 42C10, 42C40, 68U10

1. Introduction

A multiresolution analysis and compactly supported orthogonal wavelets related to the Walsh series were studied in Refs. 1–7 (see also the review article in Ref. 8). Let $\mathbb{R}_+ = [0, \infty)$ be the positive half-line. The algorithm proposed in Ref. 6 can be used for computing new examples of orthogonal compactly supported dyadic wavelets on $\mathbb{R}_+$. In the present paper, by analogy with biorthogonal compactly supported wavelets on the real line $\mathbb{R}$, which are determined by appropriately chosen trigonometric polynomials (e.g., Ref. 9, §8.3.5 and Ref. 10, §1.3), we introduce dyadic biorthogonal wavelets on $\mathbb{R}_+$ related to the Walsh polynomials. In Sec. 2, we give an algorithm for construction of a biorthogonal wavelet basis in $L^2(\mathbb{R}_+)$. Then, in Sec. 3, we discuss several numerical experiments comparing biorthogonal wavelets on $\mathbb{R}_+$ with the standard Haar, Daubechies, and biorthogonal 9/7 wavelets in an image processing scheme.[11,12] Note that analogues of formulated below Theorems 1.1 and 1.2 hold in very general situations, e.g., in the case of Cantor and Vilenkin groups.[13] The results of this paper have been partially announced in Ref. 14.

Let us recall that the *Walsh system* $\{w_n \mid n \in \mathbb{Z}_+\}$ on $\mathbb{R}_+$ is defined by

$$w_0(x) \equiv 1, \quad w_n(x) = \prod_{j=0}^{k} (w_1(2^j x))^{\nu_j}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}_+,$$

where w_1 is defined on $[0, 1)$ by the formula

$$w_1(x) = \begin{cases} 1, & x \in [0, 1/2), \\ -1, & x \in [1/2, 1) \end{cases}$$

and is extended to $\mathbb{R}_+$ by periodicity: $w_1(x + 1) = w_1(x)$ for all $x \in \mathbb{R}_+$, while the ν_j are the digits of the binary expansion of n:

$$n = \sum_{j=0}^{k} \nu_j 2^j, \quad \nu_j \in \{0, 1\}, \quad \nu_k = 1, \quad k = k(n).$$

By a *Walsh polynomial* we shall mean a finite linear combination of Walsh functions.

We shall denote the integer and the fractional parts of a number x by $[x]$ and $\{x\}$, respectively. For $x \in \mathbb{R}_+$ and $j \in \mathbb{N}$ we define $x_j, x_{-j} \in \{0, 1\}$ as follows:

$$x = [x] + \{x\} = \sum_{j<0} x_j 2^{-j-1} + \sum_{j>0} x_j 2^{-j} \tag{1.1}$$

(for a dyadic rational x we obtain an expansion with finitely many non-zero terms). The *dyadic addition* on $\mathbb{R}_+$ is defined by the formula

$$x \oplus y = \sum_{j<0} |x_j - y_j| 2^{-j-1} + \sum_{j>0} |x_j - y_j| 2^{-j}$$

and plays a key role in the Walsh analysis and its applications.[15,16]

For $x, y \in \mathbb{R}_+$ put

$$\chi(x, y) = (-1)^{t(x,y)}, \quad \text{where } t(x, y) = \sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j)$$

and x_j, y_j are calculated as in (1.1). For every integer j we have

$$\chi(x, 2^j l) = \chi(2^j x, l) = w_l(2^j x), \quad l \in \mathbb{Z}_+, \quad x \in [0, 2^{-j}).$$

Further, if $x, y, z \in \mathbb{R}_+$ and $x \oplus y$ is dyadic irrational, then

$$\chi(x, z)\chi(y, z) = \chi(x \oplus y, z) \tag{1.2}$$

(see Ref. 15, §1.5). Thus, for fixed x and z, the last equality holds for all $y \in \mathbb{R}_+$ except for countably many.

The inner product and the norm on $L^2(\mathbb{R}_+)$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. For each function $f \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$, its Walsh–Fourier transform $\widehat{f}$,

$$\widehat{f}(\omega) = \int_{\mathbb{R}_+} f(x)\chi(x, \omega) dx, \quad \omega \in \mathbb{R}_+,$$

belongs to $L^2(\mathbb{R}_+)$. The Walsh–Fourier operator

$$\mathcal{F} : L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+), \quad \mathcal{F}f = \widehat{f},$$

extends uniquely to the whole space $L^2(\mathbb{R}_+)$. The properties of the Walsh–Fourier transform are quite similar to those of the classical Fourier transform. In particular, if $f, g \in L^2(\mathbb{R}_+)$ then $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle$ (Parseval's equality).

We denote the support of a function $f \in L^2(\mathbb{R}_+)$ by supp f; it is defined as the minimal (with respect to inclusion) closed set such that on its complement f vanishes almost everywhere. The set of functions in $L^2(\mathbb{R}_+)$ with compact support is denoted by $L^2_c(\mathbb{R}_+)$.

Definition 1.1. A function $\varphi \in L^2_c(\mathbb{R}_+)$ is called a *refinable function*, if it satisfies an equation of the type

$$\varphi(x) = \sum_{k=0}^{2^n-1} c_k \varphi(2x \oplus k), \quad x \in \mathbb{R}_+, \tag{1.3}$$

where c_k are complex coefficients.

We denote the characteristic function of a set $E \subset \mathbb{R}_+$ by $\mathbf{1}_E$. If $c_0 = c_1 = 1$ and $c_k = 0$ for all $k \geq 2$, then $\varphi = \mathbf{1}_{[0,2^{n-1})}$ is a solution of Eq. (1.3); in particular, when $n = 1$ we obtain the Haar function. Some other examples of refinable functions determining orthogonal wavelets in $L^2(\mathbb{R}_+)$, are given in Refs. 2, 4 and 6.

The functional equation (1.3) is known as the *refinement equation*. Applying the Walsh–Fourier transform, we can write this equation as $\widehat{\varphi}(\omega) = m(\omega/2)\widehat{\varphi}(\omega/2)$, where

$$m(\omega) = \frac{1}{2} \sum_{k=0}^{2^n-1} c_k w_k(\omega)$$

is a Walsh polynomial, which is called the *mask* of the refinable function φ.

By a *dyadic interval of range* n we shall mean an interval of the form $I_s^{(n)} = [s2^{-n}, (s+1)2^{-n})$, $s \in \mathbb{Z}_+$. It follows easily from definition that, for every $0 \leq l \leq 2^n - 1$, the Walsh function $w_l(x)$ is piecewise constant: on each interval $I_s^{(n)}$ it is either equal to 1 or to -1. Besides, $w_l(x) = 1$ for $x \in [0, 2^{-n})$. The coefficients of the refinement equation (1.3) are related to the values $b_s = m(s2^{-n})$, $0 \leq s \leq 2^n - 1$, by the discrete Walsh transform:

$$c_k = \frac{1}{2^n} \sum_{s=0}^{2^n-1} b_s w_s(k2^{-n}), \quad 0 \leq k \leq 2^n - 1, \tag{1.4}$$

which can be realized by a fast algorithm, see Ref. 16, p. 463.

A set $M \subset [0, 1)$ is said to be *blocking set* for a mask m, if it is the union of dyadic intervals of range $n - 1$ or coincides with some of these intervals, does not contain the interval $[0, 2^{-n+1})$, and such that each point of the set $(M/2) \cup (1/2 + M/2)$, which is not contained in M, is a zero for m. It is evident, that m can have only finitely many blocking set.

Theorem 1.1. *Let φ be a function in $L^2_c(\mathbb{R}_+)$ satisfying the refinement equation* (1.3) *with m as its mask, and let $\widehat{\varphi}(0) = 1$. Then*

$$\sum_{\alpha=0}^{2^n-1} c_k = 2, \quad \operatorname{supp} \varphi \subset [0, 2^{n-1}],$$

and

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(2^{-j}\omega).$$

Moreover, the following properties are true:

(i) $\widehat{\varphi}(k) = 0$ *for all* $k \in \mathbb{N}$ (*the modified Strang–Fix condition*);
(ii) $\sum_{k\in\mathbb{Z}_+} \varphi(x \oplus k) = 1$ *for almost every* $x \in \mathbb{R}_+$ (*the partition of unity property*);
(iii) $\{\varphi(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\}$ *is a linearly independent system if and only if m does not have blocking sets.*

Definition 1.2. We say that a function $f \in L^2(\mathbb{R}_+)$ is *stable* if there exists positive constants c_1 and c_2 such that

$$c_1 \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^{1/2} \leq \left\| \sum_{k=0}^{\infty} a_k f(\cdot \oplus k) \right\| \leq c_2 \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^{1/2}$$

for every sequence $\{a_k\}$ from ℓ^2. In other words, a function f is stable in $L^2(\mathbb{R}_+)$ if the family $\{f(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\}$ is a Riesz system in $L^2(\mathbb{R}_+)$.

We say that a function $g : \mathbb{R}_+ \to \mathbb{C}$ has a *periodic zero* at a point $\omega \in \mathbb{R}_+$ if $g(\omega \oplus k) = 0$ for all $k \in \mathbb{Z}_+$. The following theorem characterizes compactly supported stable functions in $L^2(\mathbb{R}_+)$.

Theorem 1.2. *For any $f \in L^2_c(\mathbb{R}_+)$ the following properties are equivalent*:

(i) f *is stable in* $L^2(\mathbb{R}_+)$;
(ii) $\{f(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\}$ *is a linearly independent system*;
(iii) *the Walsh–Fourier transform of f does not have periodic zeros.*

Theorems 1.1 and 1.2 were proved in Ref. 6 (see also their generalizations in Ref. 17).

Definition 1.3. A family of closed subspaces $V_j \subset L^2(\mathbb{R}_+)$, $j \in \mathbb{Z}$, is called a *multiresolution analysis* (or, briefly, an *MRA*) in $L^2(\mathbb{R}_+)$ if the following hold:

(i) $V_j \subset V_{j+1}$ for $j \in \mathbb{Z}$;
(ii) $\overline{\cup V_j} = L^2(\mathbb{R}_+)$ and $\cap V_j = \{0\}$;
(iii) $f(\cdot) \in V_j \Leftrightarrow f(2\,\cdot) \in V_{j+1}$ for $j \in \mathbb{Z}$;
(iv) $f(\cdot) \in V_0 \Rightarrow f(\cdot \oplus k) \in V_0$ for $k \in \mathbb{Z}_+$;
(v) there exists a function $\varphi \in L^2(\mathbb{R}_+)$ such that the system $\{\varphi(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\}$ is a Riesz basis in V_0.

Given a function $f \in L^2(\mathbb{R}_+)$, we set

$$f_{j,k}(x) = 2^{j/2} f(2^j x \oplus k), \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}_+.$$

Definition 1.4. We say that a *function* φ *generates an MRA in* $L^2(\mathbb{R}_+)$ if, first, the family $\{\varphi(\cdot \oplus k) \mid k \in \mathbb{Z}_+\}$ is a Riesz system in $L^2(\mathbb{R}_+)$ and, second, the closed subspaces $V_j = \overline{\operatorname{span}}\{\varphi_{j,k} \mid k \in \mathbb{Z}_+\}$, $j \in \mathbb{Z}$, form an MRA in $L^2(\mathbb{R}_+)$.

For each refinable function φ generating an MRA in $L^2(\mathbb{R}_+)$, we can construct an orthogonal wavelet ψ such that $\{\psi_{j,k}\}$ is an orthonormal basis in $L^2(\mathbb{R}_+)$.[1,6] Moreover, according to Ref. 4, Theorem 2, for any positive integer n there exists the coefficients c_k such that the corresponding refinement equation (1.3) has a solution $\varphi \in L^2_c(\mathbb{R}_+)$ which generates an MRA in $L^2(\mathbb{R}_+)$. In Sec. 2, we study the following problem: given two masks

$$m(\omega) = \frac{1}{2}\sum_{k=0}^{2^n-1} c_k w_k(\omega), \quad \widetilde{m}(\omega) = \sum_{l=0}^{2^{\tilde{n}}-1} \widetilde{c}_l w_l(\omega) \tag{1.5}$$

how can we find biorthogonal wavelet bases in $L^2(\mathbb{R}_+)$? Then, in Sec. 3, we give some examples of masks (1.5) with their application to maximizing the PSNR values for several images. In these numerical experiments we find biorthogonal dyadic wavelets which have some advantages in comparison with the Haar, Daubechies and biorthogonal 9/7 wavelets.

By analogy with Ref. 10, Proposition 1.1.12, we have the following

Lemma 1.1. *Let* $\varphi, \widetilde{\varphi} \in L^2(\mathbb{R}_+)$. *The systems* $\{\varphi(\cdot \oplus k) \mid k \in \mathbb{Z}_+\}$ *and* $\{\widetilde{\varphi}(\cdot \oplus k) \mid k \in \mathbb{Z}_+\}$ *are biorthonormal in* $L^2(\mathbb{R}_+)$ *if and only if*

$$\sum_{l=0}^{\infty} \widehat{\varphi}(\omega \oplus l)\overline{\widehat{\widetilde{\varphi}}(\omega \oplus l)} = 1 \quad \textit{for a.e. } \omega \in \mathbb{R}_+.$$

Let $\mathcal{E}_n$ be the space of functions defined on $\mathbb{R}_+$ and constant on all dyadic intervals of range n.

Lemma 1.2 (see Ref. 15, §6.2). *If* $f \in L^1(\mathbb{R}_+)$ *and* supp $f \subset [0, 2^n]$, *then* $\widehat{f} \in \mathcal{E}_n$. *In a similar way, if* $g \in L^1(\mathbb{R}_+)$ *and* supp $g \subset [0, 2^n]$, *then the inverse Walsh–Fourier transform of* g *belongs to* $\mathcal{E}_n$.

Now, let us represent each $l \in \mathbb{N}$ in the form of a binary expansion

$$l = \sum_{j=0}^{k} \mu_j 2^j, \quad \mu_j \in \{0,1\}, \quad \mu_k = 1, \quad k = k(l) \in \mathbb{Z}_+, \tag{1.6}$$

and denote by $\mathbb{N}_0(n)$ the set of all positive integers $l \geq 2^{n-1}$ for which the ordered sets $(\mu_j, \mu_{j+1}, \ldots, \mu_{j+n-1})$ of the coefficients in (1.6) do not contain $(0, 0, \ldots, 0, 1)$. Further, let

$$\gamma(i_1, i_2, \ldots, i_n) = b_s, \quad s = i_1 2^0 + i_2 2^1 + \cdots + i_n 2^{n-1}, \quad i_j \in \{0,1\},$$

where b_s are defined as in (1.4). Then we write

$$c_l[m] = \gamma(\mu_0, 0, 0, \ldots, 0, 0) \quad \text{if } k(l) = 0;$$

$$c_l[m] = \gamma(\mu_1, 0, 0, \ldots, 0, 0)\gamma(\mu_0, \mu_1, 0, \ldots, 0, 0) \quad \text{if } k(l) = 1;$$

$$\cdots$$

$$c_l[m] = \gamma(\mu_k, 0, 0, \ldots, 0, 0)\gamma(\mu_{k-1}, \mu_k, 0, \ldots, 0, 0) \cdots \gamma(\mu_0, \mu_1, \mu_2, \ldots, \mu_{n-2}, \mu_{n-1})$$

if $k = k(l) \geq n - 1$. Note that in the last product the subscripts of each factor starting with the second are obtained by starting those of the preceding factor by one position to the right and placing one new digit from the binary expansion (1.6) at the vacant first position. If φ generate an MRA in $L^2(\mathbb{R}_+)$, then the following expansion take place

$$\widehat{\varphi}(\omega) = \mathbf{1}_{[0,2^{1-n})}(\omega) + \sum_{l \in \mathbb{N}(n)} c_l[m]\mathbf{1}_{[0,2^{1-n})}(\omega \oplus 2^{1-n}l), \quad \omega \in \mathbb{R}_+, \tag{1.7}$$

where $\mathbb{N}(n) := \{1, 2, \ldots, 2^{n-1} - 1\} \cup \mathbb{N}_0(n)$. The last expansion was obtained in Ref. 4 (for some partial cases see Ref. 3).

Let $r = 2^{n-1}$. Recall that the joint spectral radius of $r \times r$ complex matrices A_0 and A_1 is defined as

$$\widehat{\rho}(A_0, A_1) := \lim_{k \to \infty} \max\{\|A_{d_1}A_{d_2} \cdots A_{d_k}\|^{1/k} : d_j \in \{0, 1\},\ 1 \leq j \leq k\},$$

where $\|\cdot\|$ is an arbitrary norm on $\mathbb{C}^{r \times r}$. If $A_0 = A_1$, then $\widehat{\rho}(A_0, A_1)$ coincides with the spectral radius $\rho(A_0)$. The joint spectral radius of finite-dimensional linear operators L_0, L_1 is defined as the joint spectral radius of their matrices in an arbitrary fixed basis of the corresponding linear space.

Given a refinement equation of the form (1.3), we set $(r \times r)$ matrices T_0, T_1 by

$$(T_0)_{i,j} = c_{(2i-2)\oplus(j-1)}, \quad (T_1)_{i,j} = c_{(2i-1)\oplus(j-1)},$$

where $i, j \in \{1, 2, \ldots, r\}$. Consider the subspace

$$V := \{u = (u_1, \ldots, u_r)^t | u_1 + \cdots + u_r = 0\}$$

and denote by L_0, L_1 the restrictions to V of the linear operators defined on the whole space $\mathbb{C}^r$ by the matrices T_0, T_1, respectively.

Lemma 1.3. *Let the mask m of refinement equation* (1.3) *satisfies* $m(0) = 1$, $m(1/2) = 0$ *and* $\widehat{\rho}[m] := \widehat{\rho}(L_0, L_1, \ldots, L_{p-1}) < 1$. *Then the function φ defined by* (1.7) *satisfies Eq.* (1.3) *and belongs to* $L^2(\mathbb{R}_+)$. *Moreover, we have*

$$\sum_{k=0}^{2^{n-1}-1} c_{2k} = \sum_{k=0}^{2^{n-1}-1} c_{2k+1} = 1.$$

This lemma can be proved by analogy with the similar results in Ref. 5, Secs. 2 and 4; see also Ref. 10, §5.1 and Ref. 18.

2. Construction of Biorthogonal Wavelets on $\mathbb{R}_+$

Let us define two scaling functions φ, $\widetilde{\varphi}$ by the Walsh–Fourier transforms

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(2^{-j}\omega), \quad \widehat{\widetilde{\varphi}}(\omega) = \prod_{j=1}^{\infty} \widetilde{m}(2^{-j}\omega), \tag{2.1}$$

where $\widehat{\varphi}(0) = \widehat{\widetilde{\varphi}}(0) = 1$ and m, $\widetilde{m}$ are as in (1.5). Suppose that

$$m(1/2) = \widetilde{m}(1/2) = 0. \tag{2.2}$$

We set $m^*(\omega) = m(\omega)\overline{\widetilde{m}(\omega)}$, $N = \max\{n, \widetilde{n}\}$ and let $c_k = \widetilde{c}_l = 0$ for $k \geq 2^n, l \geq 2^{\widetilde{n}}$. Using the equality $w_k(\omega)w_l(\omega) = w_{k\oplus l}(\omega)$, we obtain

$$m^*(\omega) = \frac{1}{2}\sum_{k=0}^{2^N-1} c_k^* w_k(\omega), \quad c_k^* = \frac{1}{2}\sum_{l=0}^{2^N-1} c_l \overline{\widetilde{c}}_{k\oplus l}. \tag{2.3}$$

Proposition 2.1. *If the systems* $\{\varphi(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\}$, $\{\widetilde{\varphi}(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\}$ *are biorthonormal in* $L^2(\mathbb{R}_+)$, *then*

$$m^*(\omega) + m^*(\omega + 1/2) = 1 \quad \textit{for all } \omega \in [0, 1/2). \tag{2.4}$$

Proof. The Walsh–Fourier transforms of φ and $\widetilde{\varphi}$ belongs to $\mathcal{E}_{N-1}$ (see Theorem 1.1 and Lemma 1.2). Then, by means of Lemma 1.1, for any $\omega \in [0, 1)$ we have

$$1 = \sum_{l=0}^{\infty} \widehat{\varphi}(\omega \oplus l)\overline{\widehat{\widetilde{\varphi}}(\omega \oplus l)} = \sum_{l=0}^{\infty} m^*(\omega/2 \oplus l/2)\widehat{\varphi}(\omega/2 \oplus l/2)\overline{\widehat{\widetilde{\varphi}}(\omega/2 \oplus l/2)}.$$

Further, using periodicity of m^*, we obtain

$$1 = m^*(\omega/2)\sum_{k=0}^{\infty} \widehat{\varphi}(\omega/2 \oplus k)\overline{\widehat{\widetilde{\varphi}}(\omega/2 \oplus k)}$$

$$+ m^*(\omega/2 + 1/2)\sum_{k=0}^{\infty} \widehat{\varphi}((\omega \oplus (1/2 \oplus k)))\overline{\widehat{\widetilde{\varphi}}(\omega \oplus (1/2 \oplus k))},$$

where both sums equal to 1. $\square$

Let us define φ^* by the formula

$$\varphi^*(x) := \int_{\mathbb{R}_+} \varphi(t \oplus x)\overline{\widetilde{\varphi}(t)}dt.$$

The Walsh–Fourier transform of φ^* can be written as $\widehat{\varphi}^*(\omega) = \widehat{\varphi}(\omega)\overline{\widehat{\widetilde{\varphi}}(\omega)}$ (see Ref. 15, §6.1). Hence, $\widehat{\varphi}^*(\omega) = m^*(\omega/2)\widehat{\varphi}^*(\omega/2)$. Using the inverse Walsh–Fourier transform, we can deduce from (2.3) that φ^* is a scaling function and φ^* satisfies

$$\varphi^*(x) = \sum_{k=0}^{2^N-1} c_k^* \varphi^*(2x \oplus k), \quad x \in \mathbb{R}_+. \tag{2.5}$$

Thus, m^* is the mask of φ^*.

Proposition 2.2. *If one of the masks m, $\widetilde{m}$, m^* has blocking set, then the systems $\{\varphi(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\}$, $\{\widetilde{\varphi}(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\}$ are not biorthonormal in $L^2(\mathbb{R}_+)$.*

Proof. Assume that systems

$$\{\varphi(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\}, \quad \{\widetilde{\varphi}(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\} \tag{2.6}$$

are biorthonormal in $L^2(\mathbb{R}_+)$. It is well known that each biorthonormal system is linearly independent. It follows from Theorem 1.1 that m and $\widetilde{m}$ have no blocking sets. Besides, in view of Lemmas 1.1 and 1.2, for any $\omega \in [0, 1)$ we have $\Phi(\omega) = 1$, where Φ is defined by the formula

$$\Phi(\omega) := \sum_{l=0}^{\infty} \widehat{\varphi}(\omega \oplus l)\overline{\widehat{\widetilde{\varphi}}(\omega \oplus l)}, \quad \omega \in \mathbb{R}_+. \tag{2.7}$$

Now, let m^* have a blocking set. Then, according to Theorem 1.2, $\{\varphi^*(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\}$ is a linearly dependent system. Let us show that there is a dyadic interval of a range $N-1$ such that all points of it are periodic zeros of $\widehat{\varphi}^*$. Indeed, if $k > 2^N$, then support of $\varphi^*(\cdot \oplus k)$ disjoint with $[0, 2^{N-1}]$. Therefore, the system $\{\varphi^*(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\}$ is linearly dependent only in the case when there exists numbers $a_0, \ldots, a_{2^{N-1}-1}$ such that

$$\sum_{k=0}^{2^{N-1}-1} a_k \varphi^*(\cdot \oplus k) = 0 \quad \text{and} \quad \sum_{k=0}^{2^{N-1}-1} |a_k| > 0.$$

Applying the Walsh–Fourier transform, we get

$$\widehat{\varphi}^*(\omega) \sum_{k=0}^{2^{N-1}-1} a_k w_k(\omega) = 0 \quad \text{for a.e. } \omega \in \mathbb{R}_+.$$

The polynomial $w(\omega) = \sum_{k=0}^{2^{N-1}-1} a_k w_k(\omega)$ is not identically equal to zero. Therefore, there is an dyadic interval $I \subset [0, 1)$ with range $N-1$ such that $w(I + r) \neq 0$ for all $r \in \mathbb{Z}_+$. Since $\widehat{\varphi}^* \in \mathcal{E}_{N-1}$, we obtain that $\widehat{\varphi}^*(I + r) = 0$ for all $r \in \mathbb{Z}_+$. That is, on $[0, 1)$ there is a dyadic interval of range $N-1$, entirely consisting of periodic zeros of $\widehat{\varphi}^*$. But then $\Phi(\omega) = 0$ on this interval. This contradiction completes the proof of Proposition 2.2. □

Remark 2.1. From identity

$$m^*(\omega) + m^*(\omega + 1/2) = \frac{1}{2} \sum_k \sum_l c_k \overline{\widetilde{c}}_{k \oplus 2l} w_{k \oplus 2l}(\omega)$$

we see that (2.4) is equivalent to the following

$$\sum_k c_k \overline{\widetilde{c}}_{k \oplus 2l} = 2\delta_{0,l}, \quad l \in \mathbb{Z}_+. \tag{2.8}$$

Besides, it is possible to write (2.4) in the form

$$b_l \overline{\widetilde{b}}_l + b_{l+2^{N-1}} \overline{\widetilde{b}}_{l+\nu 2^{N-1}} = 1, \quad 0 \le l \le 2^{N-1} - 1, \tag{2.9}$$

where $b_l = m(l2^{-N})$, $\widetilde{b}_l = \widetilde{m}(l2^{-N})$.

The family $\{[0,2^{-j})\,|\,j\in\mathbb{Z}\}$ forms a fundamental system of neighborhoods of zero in the dyadic topology on $\mathbb{R}_+$ (e.g., Refs. 2 and 3). A subset E of $\mathbb{R}_+$ that is compact in the dyadic topology is said to be *W-compact.* It is easy to see that the union of finite family of dyadic intervals is W-compact.

A W-compact set is said to be *congruent to* $[0,1)$ *modulo* $\mathbb{Z}_+$, if its Lebesgue measure is equal to 1 and for each $x\in[0,1)$ there exists $k\in\mathbb{Z}_+$ such that $x\oplus k\in E$.

Theorem 2.1. *Let m^* satisfies the condition (2.4). Then systems $\{\varphi(\cdot\oplus k)\,|\,k\in\mathbb{Z}_+\}$, $\{\widetilde{\varphi}(\cdot\oplus k)\,|\,k\in\mathbb{Z}_+\}$ are biorthonormal in $L^2(\mathbb{R}_+)$ if and only if there exists a W-compact set E congruent to $[0,1)$ modulo $\mathbb{Z}_+$, and containing a neighborhood of zero such that*

$$\inf_{j\in\mathbb{N}}\inf_{\omega\in E}|m(2^{-j}\omega)|>0,\qquad \inf_{j\in\mathbb{N}}\inf_{\omega\in E}|\widetilde{m}(2^{-j}\omega)|>0. \tag{2.10}$$

Note that the condition (2.10) is valid for $E=[0,1)$ if $m^*(\omega)\neq 0$ for all $\omega\in[0,1/2)$.

Example 2.1. Let $n=\widetilde{n}=2$. Suppose that the masks of scaling functions $\varphi,\widetilde{\varphi}$ are given by

$$m(\omega)=\begin{cases}1, & \omega\in[0,1/4),\\ a, & \omega\in[1/4,1/2),\\ 0, & \omega\in[1/2,3/4),\\ b, & \omega\in[3/4,1),\end{cases}\qquad \widetilde{m}(\omega)=\begin{cases}1, & \omega\in[0,1/4),\\ \widetilde{a}, & \omega\in[1/4,1/2),\\ 0, & \omega\in[1/2,3/4),\\ \widetilde{b}, & \omega\in[3/4,1),\end{cases}$$

where $a\overline{\widetilde{a}}+b\overline{\widetilde{b}}=1$. If $a=0$ or $\widetilde{a}=0$, then $[1/2,1)$ is blocking set for m or $\widetilde{m}$, otherwise blocking sets for m, $\widetilde{m}$, m^* do not exist. Now let $|b|<1$ and $|\widetilde{b}|<1$. Then $a\widetilde{a}\neq 0$ and blocking sets are absent. Moreover, as above, the condition (2.10) is satisfied for $E=[0,1)$. Besides, both functions φ, $\widetilde{\varphi}$ belong to $L^2(\mathbb{R}_+)$. Really, from Ref. 8, Example 4.3 and Ref. 9, Remark 3 we see that $\widehat{\rho}\,[m]=|b|$, $\widehat{\rho}\,[\widetilde{m}]=|\widetilde{b}|$; hence, Lemma 1.3 is applicable. The following decomposition takes place

$$\varphi(x)=(1/2)\mathbf{1}_{[0,1)}(x/2)\left(1+a\sum_{j=0}^{\infty}b^j w_{2^{j+1}-1}(x/2)\right)$$

and the similar decomposition is true for $\widetilde{\varphi}$. Thus, under the conditions $a\overline{\widetilde{a}}+b\overline{\widetilde{b}}=1$, $|b|<1$, $|\widetilde{b}|<1$, the family of integer shifts of scaling functions φ, $\widetilde{\varphi}$ forms an biorthonormal system in $L^2(\mathbb{R}_+)$.

Example 2.2. Let $n=3,\widetilde{n}=2$. Suppose $\widetilde{m}$ is the same as in Example 2.1 and the mask m is defined by

$$\begin{array}{llll} m(\omega)=1 & \text{for } \omega\in[0,1/8), & m(\omega)=a & \text{for } \omega\in[1/8,1/4),\\ m(\omega)=b & \text{for } \omega\in[1/4,3/8), & m(\omega)=c & \text{for } \omega\in[3/8,1/2),\\ m(\omega)=0 & \text{for } \omega\in[1/2,5/8), & m(\omega)=\alpha & \text{for } \omega\in[5/8,3/4),\\ m(\omega)=\beta & \text{for } \omega\in[3/4,7/8), & m(\omega)=\gamma & \text{for } \omega\in[7/8,1). \end{array}$$

The interval $[1/2, 1)$ is a blocking set for $m, \widetilde{m}, m^*$ in the following cases (1) $b = c = 0$, (2) $\widetilde{a} = 0$, (3) $b\widetilde{a} = c\widetilde{a} = 0$ respectively. Besides, the interval $[3/4, 1)$ is a blocking set for m and m^* in the cases $c = 0$ and $c\widetilde{a} = 0$, respectively. For the other values of parameters the blocking sets for $m, \widetilde{m}, m^*$ do not exist. Note that in the case $\widetilde{a} = 0$, $bc \neq 0$ the interval $[3/4, 1)$ is a blocking set for m^* and is not a blocking set for $m, \widetilde{m}$. Now assume that

$$a = 1, \quad b\overline{\overline{\widetilde{a}}} + \beta\overline{\overline{\widetilde{b}}} = c\overline{\overline{\widetilde{a}}} + \gamma\overline{\overline{\widetilde{b}}} = 1$$

(in particular, for $b = 0$, $c = 1$ we have $\beta = 1/\overline{\overline{\widetilde{b}}}$, $\gamma = (1 - \widetilde{a})/\overline{\overline{\widetilde{b}}})$. The blocking sets for $m, \widetilde{m}, m^*$ exist only if $\widetilde{a} = 0$ or $c = 0$. In the case $\widetilde{a} \neq 0$ the condition (2.10) is valid for $E = [0, 1/2) \cup [3/4, 1) \cup [3/2, 7/4)$. According to Examples 4.3 and 4.4 from Ref. 8 we have $\widehat{\rho}\,[m] = |\gamma|$ and $\widehat{\rho}\,[\widetilde{m}] = |\widetilde{b}|$. Thus, if $\widetilde{a} \neq 0$, $|\gamma| < 1$ and $|\widetilde{b}| < 1$, the families of integer shifts of φ, $\tilde{\varphi}$ are biorthonormal systems in $L^2(\mathbb{R}_+)$.

Suppose that $\{V_j\}$, $\{\widetilde{V}_j\}$ are two MRA's in $L^2(\mathbb{R}_+)$. We say that two functions $\psi \in V_1$ and $\widetilde{\psi} \in \widetilde{V}_1$ form a *biorthogonal wavelet pair* if $\psi \perp \widetilde{V}_0$, $\widetilde{\psi} \perp V_0$ and $(\psi(\cdot \oplus k), \widetilde{\psi}(\cdot \oplus l)) = \delta_{k,l}$, $k, l \in \mathbb{Z}_+$. As usual, let I be the identity matrix and let $\mathcal{M}^*$ denote the conjugate matrix of a matrix $\mathcal{M}$.

Theorem 2.2. *Let $\{V_j\}$, $\{\widetilde{V}_j\}$ be two MRA's generated by scaling functions φ, $\widetilde{\varphi}$, respectively, and let $\{\varphi(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\}$, $\{\widetilde{\varphi}(\cdot \oplus k) \,|\, k \in \mathbb{Z}_+\}$ are biorthonormal systems. Suppose that the matrices*

$$\mathcal{M} = \begin{pmatrix} m(\omega) & m(\omega + 1/2) \\ m_1(\omega) & m_1(\omega + 1/2) \end{pmatrix}, \quad \widetilde{\mathcal{M}} = \begin{pmatrix} \widetilde{m}(\omega) & \widetilde{m}(\omega + 1/2) \\ \widetilde{m}_1(\omega) & \widetilde{m}_1(\omega + 1/2) \end{pmatrix},$$

satisfy the condition $\mathcal{M}\widetilde{\mathcal{M}}^* = I$ for a.e. $\omega \in [0, 1/2)$. Then ψ and $\widetilde{\psi}$ given by the equalities

$$\widehat{\psi}(\omega) = m_1(\omega/2)\widehat{\varphi}(\omega/2), \quad \widehat{\widetilde{\psi}}(\omega) = \widetilde{m}_1(\omega/2)\widehat{\widetilde{\varphi}}(\omega/2),$$

form a biorthogonal wavelet pair. In particular, we can choose

$$m_1(\omega) = -w_1(\omega)\overline{m(\omega \oplus 1/2)}, \quad \widetilde{m}_1(\omega) = -w_1(\omega)\overline{\widetilde{m}(\omega \oplus 1/2)}.$$

The analogues of Theorems 2.1 and 2.2 for the space $L^2(\mathbb{R})$ are well known, their proofs for the case $\varphi = \tilde{\varphi}$ are given in Ref. 9 and can be easily modified for the biorthogonal case. The above formulated results justify the following *algorithm of biorthogonal wavelet pair construction.*

(i) Choose parameters b_l, $\widetilde{b}_s$ with $0 \le l \le 2^{n-1} - 1$, $0 \le s \le 2^{\widetilde{n}-1} - 1$, so that the equalities (2.9) are true.
(ii) Calculate c_k, $\widetilde{c}_k$ by (1.4) and check that the corresponding masks m, $\widetilde{m}$ have no blocking sets.
(iii) Check that these masks m, $\widetilde{m}$ satisfy the condition (2.10) (search a set E as a finite union of dyadic intervals) and check that the corresponding scaling functions φ, $\tilde{\varphi}$ belong to $L^2(\mathbb{R}_+)$.

(iv) Define ψ, $\widetilde{\psi}$ by the formulas

$$\psi(x) = \sum_{k=0}^{2^n-1} (-1)^k \overline{\widetilde{c}}_{k\oplus 1} \varphi(2x \oplus k), \quad \widetilde{\psi}(x) = \sum_{k=0}^{2^{\tilde{n}}-1} (-1)^k \overline{c}_{k\oplus 1} \widetilde{\varphi}(2x \oplus k).$$

Note that Step 3 of this algorithm can be realized by means of Lemma 1.3.

3. Application to Image Processing

In image processing there are several methods of finding the best wavelet for each input image from within a class of wavelets with a fixed number of parameters (e.g., Ref. 19, Sec. 7.3 and Refs. 20 and 21). The so-called "feedback-based" approach to this problem can be broken down into three basic steps: (a) coding the image by encoding the wavelet coefficients of the input image, (b) decoding (reconstruction) the image and calculating the PSNR, (c) updating the parameters to achieve the best PSNR (see Ref. 12). Let us recall that for given $N \times N$ input image $f(x, y)$ and output image $g(x, y)$ the PSNR value is defined by

$$\mathrm{PSNR} = 20 \lg \frac{255N}{\left(\sum\limits_{x,y=1}^{N} (f(x,y) - g(x,y))^2\right)^{1/2}}.$$

As usual, the Daubechies wavelet of order N will be denoted through DN (see, e.g., Ref. 9, §6.4). In Ref. 11, the following method, which we shall call **Method A**, was used:

(i) Apply a direct discrete wavelet transform to the wavelet coefficients of the input image.
(ii) Allocate a certain percentage of the wavelet coefficients with largest absolute value (10%, 5%, 1%) and nullify the remaining coefficients.
(iii) Apply a reverse wavelet transform to the modified array of the wavelet coefficients.
(iv) Calculate the PSNR value.

Using this method, it is shown in Ref. 11 that the D4 wavelet have an advantage over the Haar wavelet for the "lena" image. In the following, we compare the Haar, Daubechies', and biorthogonal 9/7 wavelets for several images with the wavelets from Examples 2.1 and 2.2 (marked DBW4 and DBW8, respectively) and also with the orthogonal wavelets (marked DOW8) from the next example.

Example 3.1. Let φ be the solution of Eq. (1.3) for $n = 3$ with the coefficients

$$c_0 = \frac{1}{4}(1 + a + b + c + \alpha + \beta + \gamma), \quad c_1 = \frac{1}{4}(1 + a + b + c - \alpha - \beta - \gamma),$$

$$c_2 = \frac{1}{4}(1 + a - b - c + \alpha + \beta - \gamma), \quad c_3 = \frac{1}{4}(1 + a - b - c - \alpha - \beta + \gamma),$$

$$c_4 = \frac{1}{4}(1 - a + b - c - \alpha + \beta - \gamma), \quad c_5 = \frac{1}{4}(1 - a + b - c + \alpha - \beta + \gamma),$$
$$c_6 = \frac{1}{4}(1 - a - b + c - \alpha - \beta + \gamma), \quad c_7 = \frac{1}{4}(1 - a - b + c + \alpha + \beta - \gamma),$$

where $|a^2| + |\alpha^2| = |b^2| + |\beta^2| = |a^2| + |\gamma^2| = 1$, $a \neq 0$, $c \neq 0$. The corresponding wavelet (see Refs. 4 and 6) is defined by

$$\psi(t) = \sum_{k \in \mathbb{Z}_+} (-1)^k c_{1 \oplus k} \varphi(2t \oplus k).$$

Let us recall the definitions of the direct and reverse biorthogonal dyadic wavelet transforms. Suppose that $h_k = c_k/\sqrt{2}, g_k = (-1)^k h_{1\oplus k}$, $\widetilde{h}_k = \widetilde{c}_k/\sqrt{2}, \widetilde{g}_k = (-1)^k \widetilde{h}_{1\oplus k}$. Then the approximating and detailing wavelet coefficients are calculated by the formulas

$$a_k = \sum_l h_{l \oplus 2k} x_l, \quad d_k = \sum_l \widetilde{g}_{l \oplus 2k} x_l.$$

The reconstruction formula is

$$x_l = \sum_k \left(\widetilde{h}_{l \oplus 2k} a_k + g_{l \oplus 2k} d_k\right).$$

In numerical experiments, Method A and its two modified versions were tested on three images: “lena”, “rentgen” and “bird”. All images are grayscale and have 256×256 size. To find the best set parameters we used the Nelder-Mead simplex method (see, e.g., Ref. 22) with PSNR as the target function. As free parameters we choose $a, b, \widetilde{\alpha}$ for Example 2.1, b, c, β, γ, for Example 2.2 and a, b, c for Example 3.1. When we used Method A with feedback process, the best PSNR values for the dyadic wavelets are the same as for the Haar wavelet. In **Method B**, we replace Step 2 of Method A on the uniform quantization of wavelet coefficients (that is, a line segment, containing all wavelet coefficients, are divided into several segments of equal length, and then the set of wavelet coefficients that are within

Table 1. PSNR values for Method B with $\Delta = 50$.

Image	Haar	D4	D8	D9/7	DOW8	DBW4	DBW8
Lena	23,139304	23,115207	23,064597	23,094191	23,819995	27,648008	28,555064
Bird	32,326904	21,865437	21,586381	21,35232	32,326904	33,07666	33,450892
Rentgen	21,342584	21,20615	21,192967	21,161501	22,159873	27,819712	31,704022

Table 2. PSNR values for Method C with $\Delta = 50$.

Image	Haar	D4	D8	D9/7	DOW8	DBW4	DBW8
Lena	29,699286	30,226144	30,393501	30,554083	29,699286	29,699286	29,853585
Bird	33,16109	33,655059	33,625577	33,959729	33,16109	33,485331	33,612731
Rentgen	33,085391	34,269849	34,390778	34,546407	33,085391	33,743917	33,592838

Fig. 1. Reconstructed “lena” image for Haar (top left), D8 (top right), DOW8 (bottom left) and DBW8 (bottom right) wavelets.

Fig. 2. Reconstructed “rentgen” image for Haar (top left), D8 (top right), DOW8 (bottom left) and DBW8 (bottom right) wavelets.

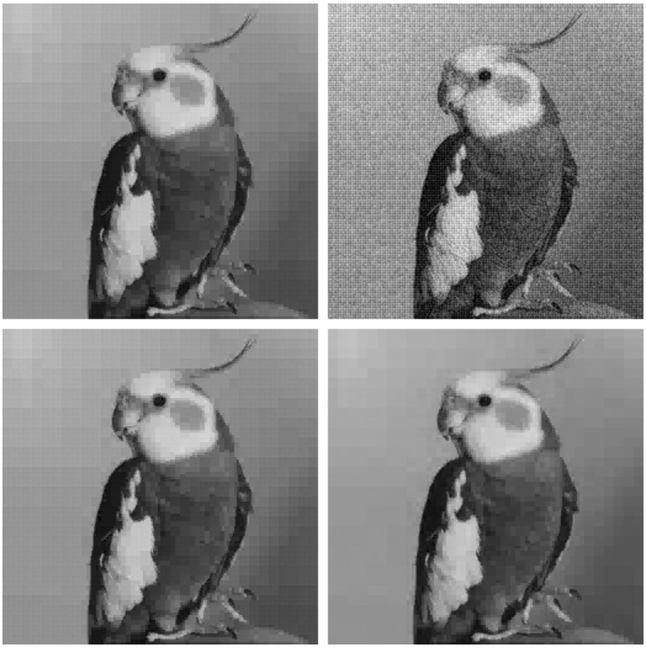

Fig. 3. Reconstructed "bird" image for Haar (top left), D8 (top right) and DBW8 (bottom) wavelets.

at the subsegment replaces by its center). In Step 2 of **Method C** we used the corresponding algorithm from the JPEG2000 standard. So, each wavelet coefficient $a_{i,j}$ was quantized by the formula

$$a_{i,j} = \operatorname{sign}(a_{i,j}) \left\lfloor \frac{a_{i,j}}{\Delta} \right\rfloor,$$

where Δ is the length of the quantization interval. The most significant results were obtained using Method B with $\Delta = 50$; see Table 1 and the reconstructed images in Figs. 1–3. We see that for considered images the introduced dyadic wavelets have an advantage over the standard Haar, Daubechies', and biorthogonal 9/7 wavelets.

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