On bisectors in Minkowski normed space. *

Á.G.Horváth
Department of Geometry,
Technical University of Budapest,
H-1521 Budapest,
Hungary

November 6, 1997

Abstract

In this paper we discuss the concept of the bisector of a segment in a Minkowski normed n-space. We prove that if the unit ball K of the space is strictly convex then all bisectors are topological images of a hyperplane of the embedding Euclidean n-space. The converse statement is not true. We give an example in the three-space showing that all bisectors are topological planes however K contains segments on its boundary. Strictly convexity ensures the normality of Dirichlet-Voronoi-type K-subdivison of any point lattice.

1 Introduction, historical remarks

If K is a 0-symmetric, bounded, convex body in the Euclidean n-space E^n (with a fixed origin O) then it defines a norm whose unit ball is K itself (see [8] or [13]). Such a space is called **Minkowski normed space**. In fact, the norm is a continuous function which is considered (in the geometric terminology as in [8]) gauge function. The metric (the so-called Minkowski metric), the distance of two points, induced by this norm, is invariant with respect to the translations of the space.

The unit ball is said to be **strictly convex** if its boundary contains no line segment. A body is said to be **smooth** if each point on its boundary has a unique supporting hyperplane. There are dual notions with respect to the scalar product of the embedding Euclidean space. The dual body K^* of K is

$$K^* = \{\mathbf{y} | < \mathbf{x}, \mathbf{y} > \leq 1 \text{ for all } \mathbf{x} \in K\}$$

where $< \cdot, \cdot >$ means the inner product of the embedding Euclidean space. It can be shown (see [1]) that the (convex) unit ball K is strictly convex if and only if its dual body K^* is smooth.

In this paper we shall examine the boundary of the unit ball of the norm and give two theorems (Theorem 2 and Theorem 3) similar to the characterization of the Euclidean norm investigated by H.Mann, A.C.Woods and P.M.Gruber in [10], [14], [5], [6] and [7], respectively. H.Mann proved that a Minkowskian normed space is Euclidean one (so its unit ball is an ellipsoid) if and only if all **Leibnizian halfspaces** (containing those points of the space which are closer to the origin than to another point $\mathbf{x}$) are convex. A.C.Woods proved the analogous statement for such a distance function whose unit ball is bounded but is not necessarily centrally symmetric or convex. P.M Gruber extended the theorem for distance functions whose unit ball is a ray set. (This unit ball is star-shaped with respect to the origin but in general is not necessarily bounded,

*Supported by Hung.Nat.Found for Sci.Reseach (OTKA) grant no.T020498 (1996)

convex or centrally symmetric.) P.M. Gruber generalized the Woods' theorem in another way, too. He showed (see Satz.5 in [5]) that a bounded distance function gives a Euclidean norm if and only if there is a subset T of the $(n-1)$-dimensional unit sphere whose relative interior (with respect to the sphere) is not empty, having the property: each of the pairs of points $\{0,\mathbf{x}\}$ where $\mathbf{x} \in T$ the corresponding Leibnizian halfspace is convex.

From the convexity of the Leibnizian halfspaces follows that the collection of all points of the space whose distances from two distinct points are equal are hyperplanes. We call such a set the **bisector** of the considered points. Thus from Mann's theorem follows a theorem stated first explicitly by M.M.Day in [2]:

Theorem 1 ([2]) *All of the bisectors, with respect to the Minkowski norm defined by the body K, are hyperplanes if and only if K is an ellipsoid.*

Day pointed out that this result is an immediate Corollary of a result of James [9].

We note that Day's theorem is also a consequence of a third (ellipsoid characterization) theorem proved by P.M.Gruber ([6] Satz.3) which says that if K_1 is a convex body in E^d $(d \geq 3)$, and the intersection of the boundaries of the bodies K_2' and K_1 is contained in a hyperplane for all translates K_2' of K_1 with $K_2' \neq K_1$ then K_1 is an ellipsoid. P.R.Goodey gave a little bit more general form of this theorem in ([3] and [4]), showing that if K_2 is another convex body of the space as K_1, and the intersection of the boundaries of the bodies K_2' and K_1 is contained in a hyperplane for all translates K_2' of K_2 with $K_2' \neq K_1$, then K_1 and K_2 are homothetic ellipsoids.

The second question concerning Theorem 1 also was posed by H.Mann in [10]. He proved that if for all lattices of the embedding space the closed Dirichlet-Voronoi cell of a lattice point (determined by the Minkowski norm) is convex (in the usual Euclidean sense) then the norm is Euclidean one, too. This theorem was also extended by P.M.Gruber for a distance function with bounded star-shaped unit ball.

It is possible that the interior (with respect to the Minkowski metric) of a Dirichlet-Voronoi cell is convex while the closed one is not, thus we have to distinguish the open and the closed Dirichlet-Voronoi cells from each other. The "walls" such a closed cell may be an n-dimensional set in the Euclidean n-space. (See the simple example of Fig.4 in the last pharagraph before the Definition 2.) It is also possible that the bisector of $\{0, \mathbf{x}\}$ is an n-dimensional part of the space. This is the case, e.g., if the unit ball is a square of the plane and the vector $\mathbf{x}$ is parallel to one of the edges of this square.

Our first purpose to prove that if a Minkowski normed space is strictly convex then every bisector is a topological hyperplane (Theorem 2). Conversely we show that if a bisector $H_{\mathbf{x}}$ is a topological hyperplane with a maximal cylinder C with generators parallel to $\mathbf{x}$ on its boundary then the dimension of C is $n-2$ (Theorem 3). Example 3 shows that strict convexity does not follows from the fact that all bisectors are topological hyperplanes.

We recall the concept of normal subdivision: the Dirichlet-Voronoi cell system of a lattice L yields a **normal subdivision** of the embedding Euclidean space if the boundary of any cell does not contain Euclidean n-ball and we show that the Dirichlet-Voronoi cell system of an arbitrary lattice L gives a normal subdivision of the embedding Euclidean space if and only if the bisectors are topological hyperplanes. Especially the strict convexity of the unit ball ensures the normality of Dirichlet-Voronoi type K-subdivision of any point lattice (Theorem 4).

2 Basic properties of the bisector

Let K be an 0-symmetric bounded convex body and let N_K be the corresponding Minkowski norm function. First of all we define the bisector of a segment with respective ends 0 and $\mathbf{x}$. It is clear that the geometric properties (in the embedding Euclidean space) of the bisector depend on the direction and length of the segment but do not depend on its position.

Definition 1 *The* **bisector** *of the segment, corresponding to the position vector* $\mathbf{x}$*, is*

$$H_{\mathbf{x}} := \{\mathbf{y} \in E^n | N_K(\mathbf{y}) = N_K(\mathbf{y} - \mathbf{x})\}.$$

We denote by $H_{\mathbf{x},\mathbf{0}}$ *and* $H_{\mathbf{x},\mathbf{x}}$ *the* **Leibnizian halfspaces** *to the segments* $[0, \mathbf{x}]$ *and* $[\mathbf{x}, 0]$*, respectively, as the set of those points which are closer (with respect to the norm* N_K*) to the first end than to the second one.*

It is clear that if $\mathrm{cl}_K S$ denotes the closure of the set S with respect to the norm N_K we have

$$H_{\mathbf{x}} = \mathrm{cl}_K H_{\mathbf{x},\mathbf{0}} \cap \mathrm{cl}_K H_{\mathbf{x},\mathbf{x}}.$$

Now, we prove some properties of the Leibnizian halfspaces and the bisectors which we shall use in this paper.

Lemma 1 *With respect to the Euclidean metric topology of the embedding n-space the following properties hold:*

1. $H_{\mathbf{x}}$ *is a closed, connected set which is convex in the direction of the vector* $\mathbf{x}$*, i.e. if a line parallel to* $\mathbf{x}$ *intersects* $H_{\mathbf{x}}$ *in two distinct points, then the whole segment with these endpoints also belongs to* $H_{\mathbf{x}}$*.*
2. $H_{\mathbf{x},\mathbf{0}}$ *and* $H_{\mathbf{x},\mathbf{x}}$ *are open, connected sets separated by the bisector* $H_{\mathbf{x}}$*.*

Proof: From the continuity of the norm function it is easy to prove that the sets

$$H_{\mathbf{x},\mathbf{0}} := \{\mathbf{y} \in E^n | N_K(\mathbf{y}) < N_K(\mathbf{x} - \mathbf{y})\}$$

$$H_{\mathbf{x},\mathbf{x}} := \{\mathbf{y} \in E^n | N_K(\mathbf{y}) > N_K(\mathbf{x} - \mathbf{y})\}$$

are open with respect to the Euclidean metric topology, too. This means that $H_{\mathbf{x}}$ is closed.

Using the triangle inequality (by the convexity of K) it is easy to see that $H_{\mathbf{x},\mathbf{0}}$ is a star-shaped set. This means that it is connected, too.

Prove now that $H_{\mathbf{x}}$ is convex in the direction of $\mathbf{x}$. Let $\mathbf{y}$ and $\mathbf{z}$ be two points of $H_{\mathbf{x}}$ for which $\mathbf{y} - \mathbf{z}$ parallel to $\mathbf{x}$ and $N_K(\mathbf{y}) \geq N_K(\mathbf{z})$. Consider the points $\mathbf{u} = \mathbf{y} - \mathbf{z}$, $\mathbf{v} = \mathbf{y} - \mathbf{z} + \mathbf{x}$, 0 and $\mathbf{x}$. If $N_K(\mathbf{y}) < N_K(\mathbf{z})$ (see Figure 1) then we have

$$N_K(\mathbf{u} - \mathbf{y}) = N_K(\mathbf{v} - \mathbf{y}) = N_K(\mathbf{z} - \mathbf{x}) = N_K(\mathbf{z}) > N_K(\mathbf{y}) = N_K(0 - \mathbf{y}) = N_K(\mathbf{x} - \mathbf{y}).$$

Thus $\mathbf{u}$,$\mathbf{v}$ are on the boundary of the N_K-ball with center $\mathbf{y}$ and radius $N_K(\mathbf{z})$, while the points 0 and $\mathbf{x}$ are in the interior of this ball. This means that the points $\mathbf{u}, \mathbf{v}, 0, \mathbf{x}$ in their line must have the order $[\mathbf{u}, 0, \mathbf{x}, \mathbf{v}]$. It is impossible because $\mathbf{v} - \mathbf{u} = \mathbf{x}$. From this we get that $N_K(\mathbf{y}) = N_K(\mathbf{z})$. Let now E, F, E', F' be the ends of the position vectors $\mathbf{y}$, $\mathbf{z}$, $\mathbf{y} - \mathbf{x}$ and $\mathbf{z} - \mathbf{x}$, respectively. These points are on the boundary of the K-ball with center 0 and radius $N_K(\mathbf{y})$ which means that the segment $\mathrm{conv}\{E, F, E', F'\}$ belongs to the boundary of this ball. (At least three of these points are distinct.) So the intersection of the considered line with the bisector $H_{\mathbf{x}}$ contains the segment $[EF]$ as we stated.

Since the intersection of a line l parallel to $\mathbf{x}$ with a closed K-ball is a compact segment, if we consider another K-ball K_1 intersecting the line l, the following non-empty set

$$(K_1 \cup (K_1 + \mathbf{x})) \cap l$$

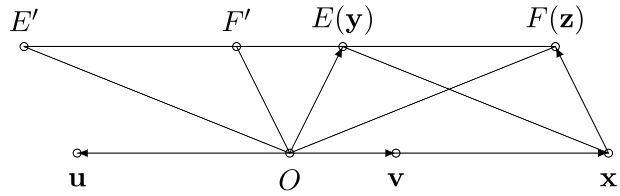

Figure 1:

is also compact. The complement of this set on the line l contains two open half lines l_- and l_+ satisfying the properties that the points of K_1 separate the points of l_- from the right endpoint of $(K_1 + \mathbf{x}) \cap l$ and the points of $K_1 + \mathbf{x}$ separate the points of l_+ from the left endpoint of $K_1 \cap l$, respectively. It is easy to see that the points of l_- belong to $H_{\mathbf{x},\mathbf{0}}$ and the points of l_+ belong to $H_{\mathbf{x},\mathbf{x}}$, respectively. So by the continuity of the Minkowski norm, every line parallel to $\mathbf{x}$ can be divided into three non-empty parts: a compact segment (may be degenerated to a point) belongs to $H_{\mathbf{x}}$ and two open halflines belong to $H_{\mathbf{x},\mathbf{0}}$ and $H_{\mathbf{x},\mathbf{x}}$, respectively.

Consider now a hyperplane orthogonal to the vector $\mathbf{x}$ and take the orthogonal projection of $H_{\mathbf{x}}$ into this $(n-1)$-dimensional Euclidean space. If we assume that $H_{\mathbf{x}}$ can be decomposed into the union of two disjoint closed subsets of it, then the images of these components (by the convexity in the direction of $\mathbf{x}$ and the above trisection of any projection line) are disjoint closed subsets whose union is this hyperplane. Using now the connectivity of the hyperplane we get that this decomposition is trivial and in fact $H_{\mathbf{x}}$ is connected, too.

The last statement of this lemma is the separating property of the bisector. Consider an elementary curve γ which connects a point $\mathbf{y}$ of $H_{\mathbf{x},\mathbf{0}}$ with a point $\mathbf{z}$ of $H_{\mathbf{x},\mathbf{x}}$. Since $H_{\mathbf{x},\mathbf{0}}$ and $H_{\mathbf{x},\mathbf{x}}$ are open with respect to the Euclidean topology of the space, the sets $H_{\mathbf{x},\mathbf{0}} \cap \gamma$ and $H_{\mathbf{x},\mathbf{x}} \cap \gamma$ are open in the induced topology of the connected curve γ. However, these sets are non-empty and disjoint hence there is (at least one) point of γ which lies in the complement of $H_{\mathbf{x},\mathbf{0}} \cup H_{\mathbf{x},\mathbf{x}}$, i.e. in $H_{\mathbf{x}}$. So for every pairs of such points $\mathbf{y}$, $\mathbf{z}$ and their connecting curve γ there is a point of $\gamma \cap H_{\mathbf{x}}$ which separates the endpoints of γ. □

The results of the following two lemmas seem to be new. The first one is an important consequence of the statements of Lemma 1.

Lemma 2 *The boundary of K does not contain any line segment parallel to $\mathbf{x}$ if and only if for each line l parallel to $\mathbf{x}$ the set*

$$H_{\mathbf{x}} \cap l$$

contains exactly one point.

Proof: Assume indirectly that the boundary of K, denoted by $\mathrm{bd}K$, contains a non-degenerate segment s parallel to $\mathbf{x}$ (see Figure 2).

For the line l containing s we have

$$\mathrm{bd}K \cap l = s \qquad \text{and} \qquad (\mathrm{bd}K + \mathbf{x}) \cap l = s + \mathbf{x}.$$

This means that for a sufficiently large real number r the set

$$\mathrm{bd}(rK) \cap \mathrm{bd}(rK) + \mathbf{x}$$

contains the non-degenerate segment $r \cdot s \cap r \cdot s + \mathbf{x}$. This proves one direction of the lemma.

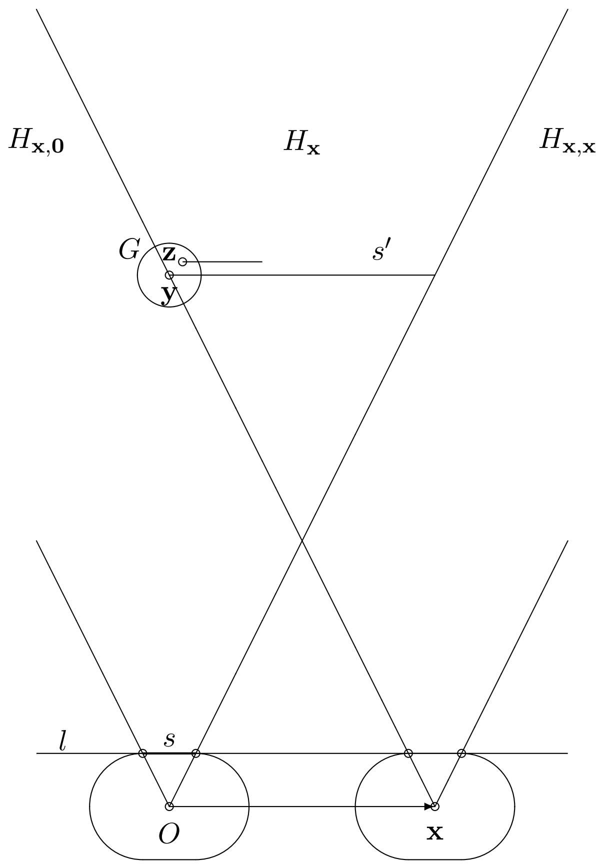

Figure 2: Maximal segment s' in $H_{\mathbf{x}}$.

Conversely, if $H_{\mathbf{x}} \cap l$ contains the points $\mathbf{y}$ and $\mathbf{z}$ then as we saw in the proof of the convexity part of the proof of Lemma 1 (to Figure 1), the following equalities hold

$$N_K(\mathbf{y}) = N_K(\mathbf{z}) = N_K(\mathbf{y}-\mathbf{x}) = N_K(\mathbf{z}-\mathbf{x}),$$

which means again that the set $H_{\mathbf{x}} \cap l$ contains at least three distinct points of the boundary of the K-ball with center 0 and radius $N_K(\mathbf{y})$. This means that the boundary of this ball contains a segment parallel to $\mathbf{x}$ which proves our Lemma 2. □

Our last lemma formulates a topological property of the bisector. We shall use the natural notion of **maximal segment s' belonging to $H_{\mathbf{x}}$ parallel to $\mathbf{x}$** and the **left or right end** of s'. (Left end of s' is from which any other point of s' can be get by adding a positive multiples of $\mathbf{x}$.) It is possible that a left end of a maximal segment belonging to $H_{\mathbf{x}}$ is an inner point of the closed set $\mathrm{cl}_{E^n} H_{\mathbf{x},\mathbf{0}}$ meaning that there exists an open Euclidean n-ball G around this left end which does not intersect the other Leibnitzian halfspace $H_{\mathbf{x},\mathbf{x}}$. We prove that in this case the bisector does not a topological hyperplane.

Lemma 3 *Let $\mathbf{y} \in H_{\mathbf{x}}$ be a left end of a maximal segment s' belonging to $H_{\mathbf{x}}$ parallel to $\mathbf{x}$ and having non-zero length. If there is an n-dimensional open Euclidean ball G with center $\mathbf{y}$ for which $H_{\mathbf{x},\mathbf{x}} \cap G$ is empty then $H_{\mathbf{x}}$ does not homeomorphic to a hyperplane.*

Before the proof of this lemma we recall the definition of topological manifold with relative boundary points. An $n-1$-dimensional **topological manifold** is a separable topological space having a countable base and holds the property that each of its points has a neighbourhood homeomorphic either to an open subset of E^{n-1} or to a halfspace E^{n-1}_{+}. We note that this definition of topological manifold (see e.g. [11]) in our paper may be applied well. A relative boundary point of an $n-1$-manifold, lies on a bounding $n-2$-manifold of the original one. We note that the concept of boundary point of such a manifold is a topological invariant and a set homeomorphic to an $n-1$-dimensional hyperplane is a topological manifold without boundary points.

Proof: Consider the boundary $\mathrm{bd}_{E^n}\mathrm{cl}_{E^n}H_{\mathbf{x},\mathbf{x}}$ of closed set $\mathrm{cl}_{E^n}H_{\mathbf{x},\mathbf{x}}$ (relative to the topology of E^n). (In general this set is a proper subset of $\mathrm{bd}_{E^n}H_{\mathbf{x},\mathbf{x}}$.) By the assumption for $\mathbf{y}$ we see that this set does not contain $\mathbf{y}$ meaning that $H_{\mathbf{x}}$ contains an $n-1$-dimensional (separation) set $\mathrm{bd}_{E^n}\mathrm{cl}_{E^n}H_{\mathbf{x},\mathbf{x}}$ and at least one maximal segment s' does not belong to this set. Since the set $\mathrm{bd}_{E^n}\mathrm{cl}_{E^n}H_{\mathbf{x},\mathbf{x}}$ is closed there is a maximal nondegenerated subsegment s'' of s' (without right endpoint) which disjoint from $\mathrm{bd}_{E^n}\mathrm{cl}_{E^n}H_{\mathbf{x},\mathbf{x}}$. If the point $\mathbf{z}$ is in $H_{\mathbf{x}}\cap G'$ where G' is a smaller as G closed ball with center $\mathbf{y}$ then it has the same property as $\mathbf{y}$, namely it has also a non-trivial segment in $H_{\mathbf{x}}\setminus \mathrm{cl}_{E^n}H_{\mathbf{x},\mathbf{x}}$. All of the segments parallel to $\mathbf{x}$ connecting the points of $H_{\mathbf{x}}\cap G'$ with a corresponding point of $\mathrm{bd}_{E^n}\mathrm{cl}_{E^n}H_{\mathbf{x},\mathbf{x}}$ determine a cylinder C with generator segments parallel to $\mathbf{x}$. Of course the point $\mathbf{y}$ is an endpoint of a generator of this cylinder. Assuming now that $H_{\mathbf{x}}$ is a topological hyperplane C is a topological manifold, too. Thus C is a topological cylinder of dimension $n-1$. If now G'' is a smaller open ball as G' with center $\mathbf{y}$ then $G''\cap H_{\mathbf{x}} = G''\cap C$ proving that $\mathbf{y}$ is a relative boundary point of $H_{\mathbf{x}}$. This is a contradiction because the relative boundary of a topological hyperplane is empty. □

3 The theorems and normality of K-subdivisions.

In this section we can give conditions for the situation when all of the bisectors are $(n-1)$-dimensional topological hyperplanes. The first one a sufficint one.

Theorem 2 *If the unit ball K of a Minkowski normed space is strictly convex then all bisectors are homeomorphic to a hyperplane.*

Proof: Since the Minkowski metric is invariant under translations we have to prove that if K is strictly convex then all sets $H_{\mathbf{x}}$ are homeomorphic image of a hyperplane.

Assume that the unit ball K is strictly convex. Let $\mathbf{x}$ be an arbitrary point of the space. Since K does not contain a segment on its boundary, from Lemma 2 we obtain that the intersections of $H_{\mathbf{x}}$ with every lines parallel to $\mathbf{x}$ contain exactly one point. Let now H be the $n-1$-dimensional subspace of E^n orthogonal to $\mathbf{x}$ and incident to the origin O and F be a map from this hyperplane H to $H_{\mathbf{x}}$ by $\mathbf{x}$-projection with the definition:

$$F : H \longrightarrow H_{\mathbf{x}}, \mathbf{y} \longrightarrow F(\mathbf{y}) = H_{\mathbf{x}} \cap \{\mathbf{y} + t\mathbf{x} | \mathbf{t} \in \mathbf{R}\}.$$

From Lemma 1 it follows that F is a bijective mapping from H to $H_{\mathbf{x}}$ we have to prove only that it is continuous one, with respect to the Euclidean metric topology. (The continuity of the inverse map will be a consequence of the fact that H is locally compact set.) Let now $\mathbf{y}$ be any point of H and $\epsilon > 0$ be arbitrary real number. Let $\mathbf{z}$ be a point of H for which the line $\mathbf{z}+t\mathbf{x}$ intersects the boundary of the K-ball K_1 with center 0 and radius $N_K(F(\mathbf{y}))$. We have two parameters say t_1 and t_2 for which

$$N_K(\mathbf{z} + t_1\mathbf{x}) = N_K(F(\mathbf{y}))$$

and

$$N_K((\mathbf{z} + t_2\mathbf{x} - \mathbf{x})) = N_K(F(\mathbf{y}) - \mathbf{x})) = N_K(F(\mathbf{y})).$$

Since K convex compact body, the function from $H = \mathbf{R}^{(n-1)}$ to $\mathbf{R}$ giving those half of the boundary of K which contains the point $F(\mathbf{y})$ (with respect to an orthonormal base containing a unit vector parallel to $\mathbf{x}$) is continuous. This means that we can choose a number $\delta > 0$ that if the

Euclidean distance of $\mathbf{z}$ and $\mathbf{y}$ is less than δ then the distances of the points $\mathbf{z}+t_1\mathbf{x}, \mathbf{z}+t_2\mathbf{x}, F(\mathbf{y})$ are less than ϵ, respectively. Since the points $\mathbf{z}+t_1\mathbf{x}, \mathbf{z}+t_2\mathbf{x}$ belong to $H_{\mathbf{x},0}$ and $H_{\mathbf{x},\mathbf{x}}$ or $H_{\mathbf{x},\mathbf{x}}$ and $H_{\mathbf{x},0}$, respectively, we get that the corresponding segment $[\mathbf{z}+t_1\mathbf{x}, \mathbf{z}+t_2\mathbf{x}]$ contains the point $F(\mathbf{z})$. So the Euclidean distance of the image points $F(\mathbf{z})$ and $F(\mathbf{y})$ is also less than ϵ, meaning that F is continuous, so it is a homeomorphism. This proves the theorem. □

Illustrating the difficulties of the reversal problem now we consider three important examples.

Example 1. Let the unit ball K be the cylinder defined by

$$K = \{(x,y,z) \in E^3 | -1 \le x \le 1, \quad y^2+z^2 \le 1\}.$$

The Leibnizian halfspaces of the vector $(2,0,0)$ are truncated open convex cones

$$\{(x,y,z) \in E^3 | x < 1, \quad 2-x > \sqrt{y^2+z^2}\} \text{ and } \{(x,y,z) \in E^3 | x > 1, \quad x > \sqrt{y^2+z^2}\},$$

respectively. The topological dimension of $H_{\mathbf{x}}$ is three showing that it is not homeomorphic to a 2-plane.

Example 2. A more interesting fact that the unit sphere defined by the compact surface

$$\mathbf{r}(t,s) := (2-s^2)\cos(t)\mathbf{e}_1 + (1-s^2)\sin(t)\mathbf{e}_2 + s\mathbf{e}_3, \text{ where } -1 \le s \le 1, \text{ and } 0 \le t < 2\pi,$$

contains exactly two (opposite) segments with parameter values $s = \pm 1$. The bisector $H_{\mathbf{x}}$ of the vector $\mathbf{x} = 4\mathbf{e}_1$ is the union of the plane $x = 2$ and the angular domains defined by the inequalities $\{y = 0, \quad x-4 \ge z \ge x\}$ and $\{y = 0, \quad -x+4 \le z \le -x\}$, respectively. This means that $H_{\mathbf{x}}$ belongs to two orthogonal planes of the space. For the proof that this set is not homeomorphic to a plane we have to see only that a set which is the union of two open circular disk with a common diameter can not be embedded topologically into a plane. In this topological space the separation theorem of Jordan does not hold because a closed Jordan curve in the plane of the first disk intersecting in two points of the common diameter, does not separate the all space. Hence this space is not homeomorphic an Euclidean plane as we stated.

From this two examples it can be thought that if all bisectors are topological hyperplanes then K is strictly convex. The following example shows that it is not true in general.

Example 3. K is an O-symmetric convex body of the three dimensional space bounded by the compact surface $\mathbf{r}(u,v)$ defined by the following manner. Let $\gamma_u(v)$ be a closed parabolic Bezier spline containing the parabola segments determined by the points $P_i(u)$ $P_{i+2}(u)$ and the corresponding tangent lines $P_i(u)P_{i+1}(u)$ and $P_{i+1}(u)P_{i+2}(u)$, respectively, where $i = 0,2,4,6,8,10$; $P_0(u) = P_{12}(u)$; $P_{6+i}(u) = -P_i(u) + [0,0,2\sin u]^T$ and the coordinates of the first six $P_i(u)$'s are

$$P_0(u) = \begin{bmatrix} 1+\varepsilon\cos u \\ 0 \\ \sin u \end{bmatrix} P_1(u) = \begin{bmatrix} 1+\varepsilon\cos u \\ \cos u \\ \sin u \end{bmatrix} P_2(u) = \begin{bmatrix} 1 \\ \cos u \\ \sin u \end{bmatrix} P_3(u) = \begin{bmatrix} 1-\cos u \\ \cos u \\ \sin u \end{bmatrix}$$

$$P_4(u) = \begin{bmatrix} -1 \\ \varepsilon\cos u \\ \sin u \end{bmatrix} P_5(u) = \begin{bmatrix} -1-\varepsilon\cos u \\ \varepsilon\cos u \cdot \frac{2-(2-\varepsilon)\cos u}{2-\cos u} \\ \sin u \end{bmatrix} P_6(u) = \begin{bmatrix} -1-\varepsilon\cos u \\ 0 \\ \sin u \end{bmatrix},$$

respectively. In Fig. 3 we can see the basic points $P_i(u)$ $(i = 0, \ldots 6)$ for the parameter values $u = 0, \frac{\pi}{3}$ and $\frac{\pi}{2}$, respectively. Here ε is a non-negative constant (less or equal to $\frac{1}{2}$) $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$ is fixed and the parameter of $\gamma_u(v)$ is v, mapping the interval $[0,6)$ onto the points of $\gamma_u(v)$. (The interval $[0,1]$ mapped on the first parabola segment the interval $[1,2]$ on the second one, etc.) Obviously $-\gamma_u(v) = \gamma_{-u}(3+v)$. The boundary of K is defined by the surface

$$\mathbf{r}(u,v) := \{\gamma_u(v) | -\frac{\pi}{2} \le u \le \frac{\pi}{2}, 0 \le v < 6\}.$$

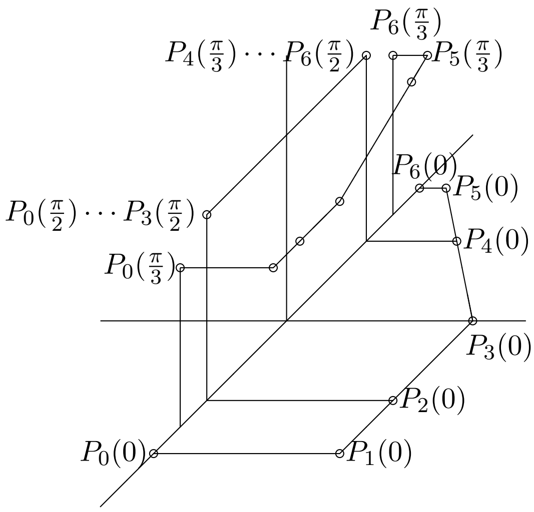

Figure 3: The controll points of three section splines of the unit ball K.

K is centrally symmetric convex body with origin O for every $0 \leq \varepsilon \leq \frac{1}{2}$. If ε is positive that it is smooth and contains precisely two opposite segments at the parameter valus $u = \pm\frac{\pi}{2}$. From the proof of the previous theorem we see that if the direction of $\mathbf{x}$ is not $[1,0,0]^T$ then $H_{\mathbf{x}}$ homeomorphic to a hyperplane. If now $\mathbf{x} = [2+2\varepsilon, 0, 0]^T$ then $H_{\mathbf{x}}$ also homeomorphic to a hyperplane, though it contains two 2-dimensional angular domains of the plane $y = 0$. To prove this fact we note that the intersection of the two enlarged copies λK and $\lambda K + [2+2\varepsilon, 0, 0]^T$ in the case when $\lambda \geq 1+\varepsilon$ is a closed Jordan curve, containing the parallel segments $s_1 = [[\lambda, 0, \lambda]^T, [[2+2\varepsilon-\lambda, 0, \lambda]^T]$, $s_2 = [[\lambda, 0, -\lambda]^T, [[2+2\varepsilon-\lambda, 0, -\lambda]^T]$ and two opposite (with respect to the center $P_0(0)$) curves connecting the point pairs $\{[\lambda, 0, \lambda]^T, [\lambda, 0, -\lambda]^T\}$, $\{[2+2\varepsilon-\lambda, 0, \lambda]^T, [2+2\varepsilon-\lambda, 0, -\lambda]^T\}$ where these curves are in the opposite space quarters $\{x \geq 1+\varepsilon, y \geq 0\}$, $\{x \leq 1+\varepsilon, y \leq 0\}$, respectively and if $1 \leq \lambda \leq 1+\varepsilon$ holds then this opposite parallel segments degenerate a point pair of the vertical segment $[[1, 0, 1+\varepsilon]^T, [1, 0, -1-\varepsilon]^T]$. Illustrating this situation we can figure of the most simple case when the parabola segments defined by the point pairs P_2P_4 and P_8P_{10} substituted by the line segments P_2P_4 and P_8P_{10}, respectively and $\varepsilon = 0$ and so the boundary of K is a ruled surface defined by two opposite closed half-circle). (See Fig.4.)

This third example shows that a bisector $H_{\mathbf{x}}$ is homeomorphic to a hyperplane can contain $n-1$-dimensional cylinder with generators parallel to $\mathbf{x}$ implying the existance of a precisely $n-2$-dimensional cylinder on the boundary of K. We now formulate this observation in the following theorem.

Theorem 3 *Let n be greater then two. If each of the bisectors is a topological hyperplane, then there is no $n-1$-dimensional cylinder on the boundary of K. Furthermore if $H_{\mathbf{x}}$ is a topological hyperplane and C is a maximal cylinder with generators parallel to $\mathbf{x}$ lying on bd$K$ then it has dimension $n-2$.*

Proof: The first statement of the theorem can be proved easily from the fact that every segment on the boundary induce an angular domain in the bisector $H_{\mathbf{x}}$ as we saw in the proof of Lemma 2. Hence If the boundary of K contains an $n-1$-dimensional cylinder then $H_{\mathbf{x}}$ contains an n-dimensional one.

We now prove the second statement of the theorem. Let C be any maximal cylinder of bd$K$ with generators parallel to $\mathbf{x}$. This means that the boundary of K in the direction of $\mathbf{x}$ contains C but there is no cylinder C' with the same direction of generators containing C and belonging

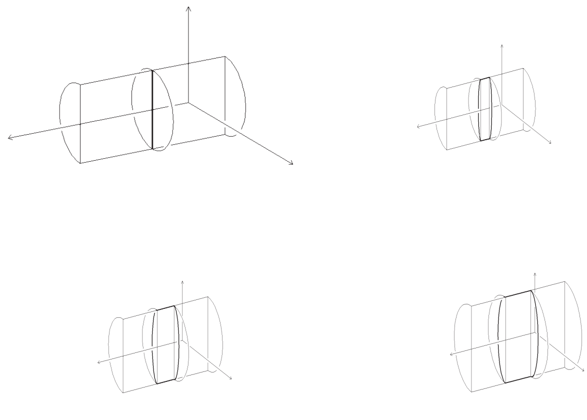

Figure 4: Four intersection curves in the case when $\varepsilon = 0$.

also to bd$K$ having greater dimension as of C. Let this dimension be k. C now induces a $k+1$-dimensional cylinder C^* with generators parallel to $\mathbf{x}$ in $H_{\mathbf{x}}$ containing maximal segments of $H_{\mathbf{x}}$ with the same direction. In Lemma 3 we showed that if $H_{\mathbf{x}}$ is topological hyperplane then all left end of every maximal segments of $H_{\mathbf{x}}$ containing the closed set $\text{bd}_{E^n}\text{cl}_{E^n}H_{\mathbf{x},\mathbf{x}}$. Obviously, the analogous statement is true for a right end of a maximal segment in $H_{\mathbf{x}}$, meaning that it is in $\text{bd}_{E^n}\text{cl}_{E^n}H_{\mathbf{x},\mathbf{0}}$. Thus we have that in this case

$$H_{\mathbf{x}} = \text{bd}_{E^n}\text{cl}_{E^n}H_{\mathbf{x},\mathbf{0}} = \text{bd}_{E^n}\text{cl}_{E^n}H_{\mathbf{x},\mathbf{x}}.$$

(The left ends and right ends of maximal segments evidently belong to $\text{bd}_{E^n}\text{cl}_{E^n}H_{\mathbf{x},\mathbf{0}}$ and $\text{bd}_{E^n}\text{cl}_{E^n}H_{\mathbf{x},\mathbf{x}}$, respectively, and these two sets are also convex in the direction of $\mathbf{x}$ as $H_{\mathbf{x}}$.) Let G be an n-dimensional ball with the radius ε. The points of $G+C^*$ can be divided into three sets S_0, S and $S_{\mathbf{x}}$ of $H_{\mathbf{x},\mathbf{0}}$, $H_{\mathbf{x}}$ and $H_{\mathbf{x},\mathbf{x}}$, respectively. Since the n-dimensional cylinder $G+C^*$ separated by S the dimension of S is at least $n-1$. Since $C^* \subset S$ we have two possibilities. In the first one S is a cylinder in $H_{\mathbf{x}}$ containing C^* and having greater dimension as of C^* while in the second case the two dimension is equal. The first possibility implies a cylinder in the boundary of K containing C and having dimension greater then of C. This contradicts to the assumption gave for C so the dimension of C^* is greater or equal to $n-1$. Thus the dimension of C is greater or equal to $n-2$. From the first note of this proof we can preclude the possibility of that this dimension is $n-1$ proving the second statement of the theorem. □

We now turn out the problem of Dirichlet-Voronoi cells on the base of a K-ball above. First of all consider the following interesting example:

Example 4: Let the unit ball is the square $[-1,1]^2$ of the plane and consider the lattice generated by the orthogonal vectors $(2,0)$ and $(0,16)$. (See Fig.4.) The interior (open) Dirichlet-Voronoi cell of the point $(0,0)$ is the open convex hexagon bounded by the lines $x = \pm 1, y = \pm x \pm 2$, respectively. The exterior (closed) Dirichlet-Voronoi cell of the origin is the closure of the union of the interior Dirichlet-Voronoi cell and two concave pentagon with vertices $\{(0,2),(1,1),(8,8),(-8,8),(-1,1)\}$ and $\{(0,-2),(1,-1),(8,-8),(-8,-8),(-1,-1)\}$, respectively.

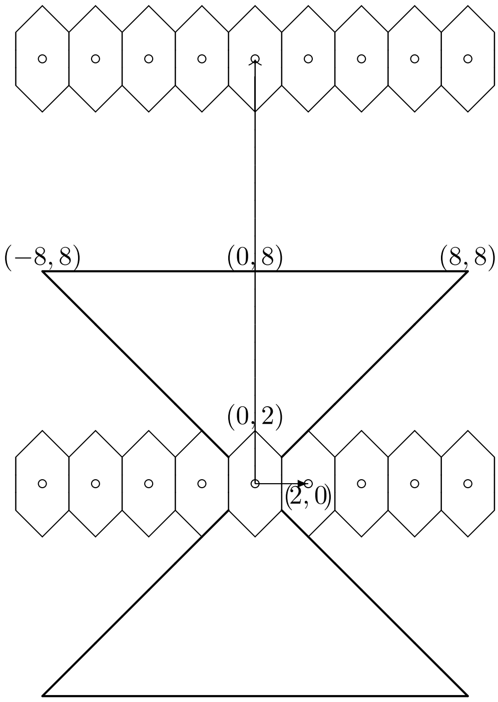

Figure 5: Convex open Dirichlet-Voronoi cell which closure is concave

The "wall" of this cell is a 2-dimensional subset of the plane. In this terminology the result of H.Mann says that if for all lattices of the space the exterior Dirichlet-Voronoi cells with respect to the considered Minkowski norm are convex then the unit ball of the norm is an ellipsoid. Because it is possible (using Theorem 3) that the exterior Dirichlet-Voronoi cell is the Euclidean closure of the interior cell and they are not convex, we now introduce the normality of the subdivision of the space generated by a lattice with respect to the examined Minkowski norm.

Definition 2 *The Dirichlet-Voronoi cell system of a lattice L gives a* **normal subdivision** *of the embedding Euclidean space if the boundary of the cells does not contain n-balls.*

The following theorem gives a necessary and sufficient condition that all of the subdivisions being normal in the space.

Theorem 4 *The Dirichlet-Voronoi cell system of an arbitrary lattice L gives a normal subdivision of the embedding Euclidean space if and only if all bisectors are topological hyperplanes. Especially if the unit ball of the Minkowski norm is strictly convex then a lattice-like Dirichlet-Voronoi K-subdivision of any point lattice is normal.*

Proof: If in the space there is a lattice which Dirichlet-Voronoi cell does not give a normal subdivision then there is n-dimensional ball belonging to the boundary of a cell. This means that there is a bisector which contains an n-dimensional ball.

Conversely, if all lattice-like Dirichlet-Voronoi cell subdivision are normal then all bisector is a topological hyperplane.

In fact, if $H_{\mathbf{x}}$ is bisector G is an arbitrary open Euclidean ball with radius r and center $\frac{1}{2}\mathbf{x}$, there is a lattice L for wich the common wall of the Dirichlet-Voronoi cells of the origin and

$\mathbf{x}$ (which are lattice points) contains the set $H_{\mathbf{x}} \cap G$. (It is enough to choose a brick lattice generated by $\mathbf{x}$ and certain large vectors from its orthogonal complement.) Using normality and the fact that the exterior Dirichlet-Voronoi cell is a topological ball we get that this part of $H_{\mathbf{x}}$ is an elementary hypersurface. If now the radius r tends to infinity the statement is given.

Now the theorem follows from Theorem 2. $\square$

References

[1] M.M.Day, *Normed linear spaces.* Springer-Verlag, Berlin, 1958.

[2] M.M.Day, Some characterization of inner-product spaces. *Trans. Amer. Math. Soc.* **62** (1947) 320–337.

[3] P.R.GOODEY–M.M.WOODCOCK, Intersections of convex bodies with their translates, *The Geometric Vein* ed. C.Davis, B.Grünbaum and F.A.Sherk (Springer-Verlag, 1982.)

[4] P.R.GOODEY, Homothetic ellipsoids. *Math. Proc. Comb. Phil. Soc.* **93** (1983) 25–34.

[5] P.M.GRUBER, Kennzeichnende Eigenschaften von euklidischen Räumen und Ellipsoiden. I. *J. reine angew. Math.* **256** (1974) 61–83.

[6] P.M.GRUBER, Kennzeichnende Eigenschaften von euklidischen Räumen und Ellipsoiden. II. *J. reine angew. Math.* **270** (1974) 123–142.

[7] P.M.GRUBER, Kennzeichnende Eigenschaften von euklidischen Räumen und Ellipsoiden. III. *Monatsh. Math.* **78** (1974) 311–340.

[8] P.M.GRUBER–C.G.LEKKERKERKER, *Geometry of numbers.* North-Holland Amsterdam-New York-Oxford-Tokyo 1987.

[9] R.L.JAMES, Orthogonality in normed linear spaces. *Duke Math J.* **12** (1945) 291–302.

[10] H.MANN, Untersuchungen über Wabenzellen bei allgemeiner Minkowskischer Metrik. *Mh. Math. Phys.* **42**, 417–424, (1935).

[11] L.S.PONTRIAGIN, *Smooth manifolds and its applications in homotopy theory.* Trudy. Mat. Inst. AN CCCP. **45** (1955).

[12] M.POSTNIKOV, *Lectures in Geometry, semester III, Smooth Manifolds*, MIR PUBLISHERS, Moscow 1989.

[13] W.RUDIN, *Functional analysis.* McGraw-Hill Book Company 1973.

[14] A.C.WOODS, A characteristic property of ellipsoids. *Duke Math. J.* **36** (1969), 1–6.