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On blowup for semilinear wave equations with a focusing nonlinearity

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Abstract

In this paper we report on numerical studies of the formation of singularities for the semilinear wave equations with a focusing power nonlinearity $u_{tt} - \Delta u = u^p$ in three space dimensions. We show that for generic large initial data that lead to singularities, the spatial pattern of blowup can be described in terms of linearized perturbations about the fundamental self-similar (homogeneous in space) solution. We consider also non-generic initial data which are fine-tuned to the threshold for blowup and identify critical solutions that separate blowup from dispersal for some values of the exponent p.

Mathematics Subject Classification: 74H35

1. Introduction

One of the most interesting features of many nonlinear evolution equations is the spontaneous onset of singularities in solutions starting from smooth initial data. Such a phenomenon, usually called a 'blowup', has been the subject of intensive studies in many fields ranging from fluid dynamics to general relativity. Given a nonlinear evolution equation, the key question is whether or not a blowup can occur for some initial data. Once the existence of the blowup is established for a particular equation, many further questions arise. When and where does the blowup occur? What is the character of blowup and is it universal? What happens at the threshold of blowup?

In this paper, we consider these questions for the simplest nonlinear generalization of the free wave equation: the semilinear wave equation with the power nonlinearity

$$u_{tt} - \Delta u = u^p, \qquad u = u(t, x), \quad x \in R^3, \tag{1}$$

where $p > 1$ is an odd integer. There are many mathematical results for equation (1) concerning its well-posedness in suitable Sobolev spaces and global existence for small initial data—we refer an interested reader to the excellent online overview of the subject together with the complete bibliography by Colliander *et al* [1].

Note that the sign of the nonlinear term corresponds to focusing, that is, it tends to magnify the amplitude of the wave. If u is small this term is negligible and the evolution is essentially linear (actually, for the values of p we consider, one has scattering for $t \to \infty$). However, if u is large the dispersive effect of the linear wave operator may be overcome by the focusing effect of the nonlinearity and a singularity can form. In fact, it is known that if the energy

$$E[u] = \int_{R^3} \left(\frac{1}{2} u_t^2 + \frac{1}{2} (\nabla u)^2 - \frac{1}{p+1} u^{p+1} \right) \mathrm{d}^3 x \tag{2}$$

is negative, then a singularity must form in a finite time [2]. This theorem says only that the solution cannot be continued beyond certain time but it gives no information on what the solution looks like as it approaches the blowup time. Another way to learn about the character of singularities is to look at explicit singular solutions. For equation (1) it is easy to see that

$$u_0 = \frac{a}{(T-t)^\alpha}, \qquad a = \left[\frac{2(p+1)}{(p-1)^2} \right]^{1/(p-1)}, \quad \alpha = \frac{2}{p-1}, \quad T > 0 \tag{3}$$

is the exact solution which blows up as $t \to T$. This solution is obtained by neglecting the Laplacian in (1) and solving the corresponding ordinary differential equation $u_{tt} = u^p$. By the finite speed of propagation, one can truncate this solution in space to get a solution with compactly supported initial data which blows up in finite time. The question is how typical this explicit singular behaviour is. There are several ways to approach this problem. On the analytical side there are Fuchsian methods developed by Kichenassamy [3] which allow us to construct open sets of initial data which blowup on a prescribed spacelike hypersurface, with the leading order asymptotic behaviour being given by the solution u_0 [4]. On the heuristic side there are numerical and perturbative methods which, albeit non-rigorous, allow us to gain more detailed information about the nature of the blowup. In this paper we take the latter approach.

Our main goal was to show that the solution u_0 determines the leading order asymptotics of the blowup for generic large initial data and the spatial pattern of convergence to this solution can be described in terms of the least damped eigenmodes of the linearized perturbations about u_0. We did this in the spherically symmetric case

$$u_{tt} - u_{rr} - \frac{2}{r} u_r = u^p \tag{4}$$

for three representative values of $p = 3$, 5 and 7. These values corresponds to three different classes of criticality of equation (1). To see this, notice that equation (1) has the scaling symmetry: if $u(t, x)$ is the solution, so is

$$u_L(t, x) = L^\alpha u \left(\frac{t}{L}, \frac{x}{L} \right), \qquad \alpha = \frac{2}{1-p}. \tag{5}$$

Under this transformation the energy scales as

$$E[u_L] = L^\beta E[u], \qquad \beta = \frac{p-5}{p-1}, \tag{6}$$

hence equation (1) is subcritical for $p = 3$ ($\beta < 0$), critical for $p = 5$ ($\beta = 0$) and supercritical for $p > 5$ ($\beta > 0$). Since the energy (2) is not positive definite, this distinction is not very important as far as the generic character of blowup is concerned, however, as we shall show below, it is relevant for understanding the behaviour of solutions at the threshold for blowup.

We point out that, besides the scaling power $p_s = 1 + (4/(n-2))$ ($p_s = 5$ for $n = 3$), there is another critical power for equation (1), namely the so-called conformal power

$p_c = 1+(4/(n-1))$ ($p_c = 3$ for $n = 3$). For the conformal power equation (1) is conformally invariant and the homogeneous Sobolev norm $||u||_{\dot{H}^{1/2}}$ is dimensionless. Merle and Zaag [5,6] showed recently that for $p \leqslant p_c$ the blowup rate is given by solution (3). Although the key step in their argument—the existence of a Lyapunov functional—breaks down for $p > p_c$, our results give evidence that in fact there is no qualitative change in the blowup behaviour at $p = p_c$.

The paper is organized as follows. In section 2 we discuss self-similar solutions of equation (4) and analyse their stability. In section 3 we present the results of numerical simulations of the blowup and demonstrate the universality of the blowup profile. Finally, in section 4 we discuss the behaviour of solutions at the threshold for blowup.

2. Self-similar solutions

In order to set the stage for the discussion of singularity formation we first discuss self-similar solutions of equation (4). As we shall see below these solutions play an important role in the process of blowup. By definition, self-similar solutions are invariant under rescaling, see equation (5), hence in the spherically symmetric case they have the form

$$u(t,r) = (T-t)^{-\alpha} U(\rho), \qquad \alpha = \frac{2}{p-1}, \quad \rho = \frac{r}{T-t}, \tag{7}$$

where T is a positive constant, clearly allowed by time translation invariance. Note that each self-similar solution, if it is regular for $t < T$, provides an explicit example of a singularity developing at $r = 0$ in finite time T from nonsingular initial data—for this reason we shall refer to T as the blowup time. Substituting the ansatz (7) into equation (4) one gets the ordinary differential equation for the similarity profile $U(\rho)$

$$(1-\rho^2)U'' + \left(\frac{2}{\rho} - (2+2\alpha)\rho\right) U' - \alpha(\alpha+1)U + U^p = 0. \tag{8}$$

We consider this equation inside the past light cone of the blowup point ($t = T, r = 0$), that is in the interval $0 \leqslant \rho \leqslant 1$. It is easy to see that for any p equation (8) has the constant solution

$$U_0(\rho) = \left[\frac{2(p+1)}{(p-1)^2}\right]^{1/(p-1)}. \tag{9}$$

Of course, this solution corresponds exactly to the solution u_0 of the original equation (4). It turns out that besides this trivial solution, for some values of p there exist also nontrivial profiles. The existence of such solutions can be proved by the shooting technique which goes as follows. Using Fuchsian methods one first shows [7] that at both endpoints of the interval $0 \leqslant \rho \leqslant 1$ there exist one-parameter families of local analytic solutions which behave, respectively, as

$$U(\rho) \sim c + \frac{1}{3}\left(\frac{p+1}{(p-1)^2}c - \frac{1}{2}c^p\right)\rho^2 \qquad \text{for } \rho \to 0 \tag{10}$$

and

$$U(\rho) \sim b + \frac{1}{2}\left(\frac{1}{2}(p-1)b^p - \frac{p+1}{p-1}b\right)(\rho - 1) \qquad \text{for } \rho \to 1, \tag{11}$$

where b and c are free parameters. Having that, the strategy for finding solutions which are regular in the interval $0 \leqslant \rho \leqslant 1$ is simple: one shoots the solution satisfying the initial condition (11) at $\rho = 1$ towards the centre and adjusts the shooting parameter b so that $U'(0) = 0$. Applying this technique one can show [7] existence for a countable set of

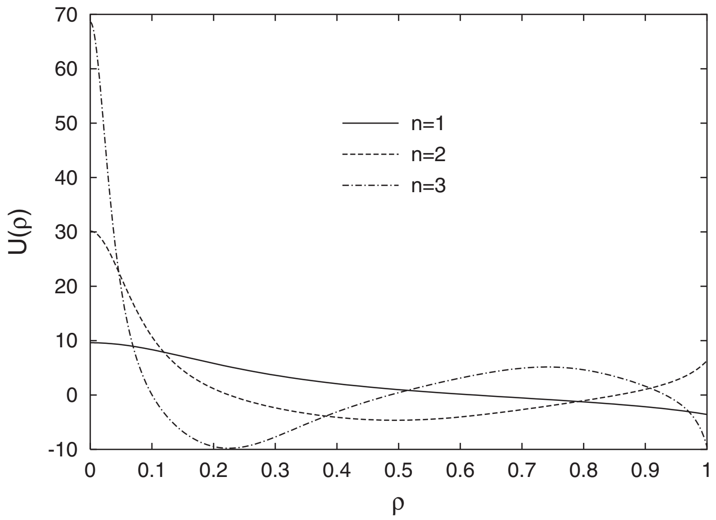

Figure 1. The first three similarity profiles $U_n(\rho)$ for $p = 3$. The index n counts the number of zeros of $U_n(\rho)$ in the interval $0 \leqslant \rho \leqslant 1$. When continued beyond the past light cone, these solutions become singular at some $\rho > 1$.

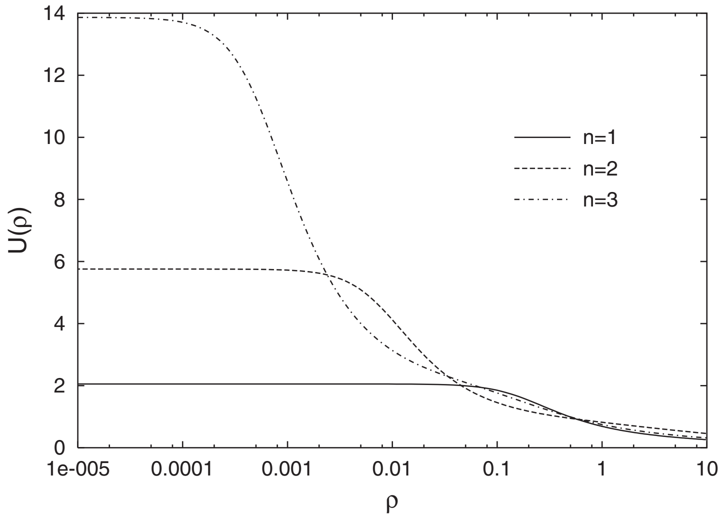

Figure 2. The first three similarity profiles $U_n(\rho)$ for $p = 7$. In contrast to $p = 3$, here all profiles are monotone and have no zeros. For $\rho \to \infty$ they decay as $U(\rho) \sim \rho^{-1/3}$.

parameters b_n ($n = 0, 1, \ldots$) which determine analytic similarity profiles U_n for $p = 3$ and all odd $p \geqslant 7$. The first few similarity profiles U_n for $p = 3$ and 7 are shown in figures 1 and 2.

The behaviour of solutions $U_n(\rho)$ outside the past light cone, that is for $\rho > 1$, depends on p. One can show (see [7]) that for $p = 3$ all $n > 0$ solutions become singular outside the past light cone, namely

$$U(\rho) \sim \frac{d}{\rho_0 - \rho} \qquad \text{for some } \rho_0 > 1. \tag{12}$$

In contrast, for $p = 7, 9, \ldots$ all solutions U_n remain regular outside the past light cone.

For $p = 5$ there are no nontrivial self-similar solutions—this can be shown as follows. Consider the function

$$Q(\rho) = \frac{1}{2}(1-\rho^2)\rho^3 U'^2 + \frac{1}{2}\rho^2(1-\rho^2)UU' + \left[\frac{3(5-p)}{4(p-1)} - \frac{2}{(p-1)^2}\right]\rho^3 U^2 + \frac{1}{p+1}\rho^3 U^{p+1}. \tag{13}$$

This function was introduced by Kavian and Weissler [8] in their study of equation (4). They showed that $Q'(\rho) = 0$ for $p = 5$, hence, in this case Q is the first integral of equation (8). Since $Q(0) = 0$, it follows that $Q(1) = 0$, from which one gets $b = U(1) = (3/4)^{1/4}$. This coincides with U_0 so by uniqueness we conclude that U_0 is the only similarity profile.

In order to determine the role of self-similar solutions in dynamics it is essential to analyse their stability. To this end we define the slow time $\tau = -\ln(T-t)$ and rewrite equation (4) in terms of the new variable $U(\tau, \rho)$ defined by

$$u(t,r) = \mathrm{e}^{\alpha\tau} U(\tau,\rho), \qquad \alpha = \frac{2}{p-1}. \tag{14}$$

We get

$$U_{\tau\tau} + (1+2\alpha)U_\tau + 2\rho U_{\tau\rho} = (1-\rho^2)U_{\rho\rho} + \left(\frac{2}{\rho} - (2+2\alpha)\rho\right) U_\rho - \alpha(\alpha+1)U + U^p. \tag{15}$$

The advantage of this formulation is that self-similar solutions of equation (4) now correspond to τ-independent solutions of equation (15) while the asymptotics of blowup correspond to the behaviour at $\tau \to \infty$. In order to determine the linear stability of solutions $U_n(\rho)$ we seek solutions of (15) in the form $U(\tau,\rho) = U_n(\rho) + \mathrm{e}^{\lambda\tau}\xi(\rho)$. After linearization we get the quadratic eigenvalue problem

$$(1-\rho^2)\xi'' + \left(\frac{2}{\rho} - 2(1+\alpha)\rho\right)\xi' - 2\rho\lambda\xi' + [pU_n^{p-1} - \alpha(\alpha+1) - (1+2\alpha)\lambda - \lambda^2]\xi = 0. \tag{16}$$

Let us consider first the stability of the constant solution U_0. In this case equation (16) becomes

$$(1-\rho^2)\xi'' + \left[\frac{2}{\rho} - 2\left(\frac{p+1}{p-1} + \lambda\right)\rho\right]\xi' + \left[\frac{2(p+1)}{p-1} - \frac{p+3}{p-1}\lambda - \lambda^2\right]\xi = 0. \tag{17}$$

Near $\rho = 0$ the admissible solution has the formal power series expansion

$$\xi(\rho) = \sum_{k=0} a_k \rho^{2k} \tag{18}$$

with the coefficients satisfying the recurrence relation

$$a_{k+1} = \frac{\lambda^2 + (4k + (p+3)/(p-1))\lambda + 2k(2k + (p+3)/(p-1)) - 2((p+1)/(p-1))}{2(k+1)(2k+3)} a_k. \tag{19}$$

Since $a_{k+1}/a_k \to 1$ as $k \to \infty$, the series (18) diverges for $\rho > 1$. In order to be able to continue the solution $\xi(\rho)$ defined by the series (18) analytically through the point $\rho = 1$ we must impose the condition that the series truncates at the kth term

$$\lambda^2 + \left(4k + \frac{p+3}{p-1}\right)\lambda + 2k\left(2k + \frac{p+3}{p-1}\right) - 2\,\frac{p+1}{p-1} = 0. \tag{20}$$

This yields two infinite sequences of pairs of real eigenvalues

$$\lambda_k = 1 - 2k, \quad \bar{\lambda}_k = -\frac{2(p+1)}{p-1} - 2k, \qquad k = 0, 1, \ldots. \tag{21}$$

There is exactly one positive eigenvalue $\lambda_0 = 1$. It corresponds to the gauge mode which is due to the freedom of choosing the blowup time T. All the remaining eigenvalues are negative, hence, for any p the solution U_0 is linearly stable. This suggests that it can appear as an attractor in generic evolution.

Since we do not know the solutions U_n with $n > 0$ in closed form, their spectrum of linear perturbations can be computed only numerically. Our numerical calculations indicate that the solution U_n has n unstable modes (apart from the spurious unstable mode corresponding to the change of blowup time). For this reason the solutions with $n > 0$ are not expected to appear in generic evolution. However, as we shall see later, the solution U_1 with one unstable mode appears as the codimension-one attractor in the evolution of specially prepared initial data.

3. Blowup profile

Having learned about the stability of the solution u_0 we are now prepared to interpret the results of numerical simulations. The main goal of these simulations was to determine the asymptotics of blowup. Before discussing the results let us briefly describe the numerical method we apply in our simulations. To solve equation (4), we rewrite it as the first-order system in time

$$u_t = v, \qquad v_t = u_{rr} + \frac{2}{r}u_r + u^p \tag{22}$$

and use the standard method of lines. To this end we discretize the space variable and replace the system (22) of partial differential equations by a corresponding system of ordinary differential equations. For the discretization we use the uniform fixed grid and approximate the continuous spatial differential operators by standard five-point, fourth-order accurate, finite difference operators. At the inner boundary $r = 0$, the requirement of regularity enforces the boundary condition $u_r(0, t) = 0$. We note that the formal Taylor series representing the solution at the origin contains only even powers of r. In order to get the right-hand sides of the first (i.e. corresponding to the grid point $r = 0$) equations we fit the right-hand sides of several neighbouring grid points. Keeping in mind that the function u (and as a consequence the whole right-hand side) contains only even powers of r we obtain the following, fourth-order accurate, formula

$$r(1) = \frac{3}{2}r(2) - \frac{3}{5}r(3) + \frac{1}{10}r(4),$$

where $r(i)$ denotes the right-hand side of the ordinary differential equation at $r = (i-1)h$ (h is grid spacing). At the outer boundary we approximate the differential operators by appropriate five-point forward difference operators and implement the usual outgoing wave condition. To integrate the resulting system of ordinary differential equations we use the standard fourth-order Runge–Kutta method. As a result we obtain the stable numerical scheme which is fourth-order accurate both in space and time.

In most cases the calculations performed on a single grid were sufficient for our purposes. However, in some cases, especially in the computation of the critical behaviour for $p = 7$, the single grid calculation had insufficient resolution. Since high resolution was needed only in the vicinity of the origin, we used in these cases a system of overlapping fixed grids of progressively finer resolutions. The basic grid with coarse resolution covers the whole computational domain, whereas each additional grid starts from the origin and has a resolution of the previous grid times a constant refinement factor. All additional grids have fixed number of gridpoints and cover only the vicinity of the origin. The value of grid points at the outer boundary of additional finer grids are obtained by interpolation from coarser grids. Using only two additional grids and the refinement factor equal to 10, we improve the resolution near the centre by a factor of hundred.

We found that, for sufficiently large initial data, the amplitude $u(t, r)$ becomes unbounded in a finite time T for some $r = r_S$. More precisely, we have

$$\lim_{t \to T} (T - t)^{\alpha} u(t, r_S) = a = \left[\frac{2(p+1)}{(p-1)^2} \right]^{1/(p-1)}, \tag{23}$$

which confirms the expectation that the solution u_0 determines the leading order asymptotics of the blowup. In this section we wish to show that if the blowup point is at the centre, that is $r_S = 0$, then the spatial pattern of the developing singularity can be described in terms of the least damped eigenmodes about u_0.

Using the results of linear stability analysis we can represent the asymptotic approach to U_0 for $\tau \to \infty$ (i.e. $t \to T$) by the formula

$$U(\tau, \rho) = U_0 + \sum_{k=1} c_k \, \mathrm{e}^{\lambda_k \tau} \xi_k(\rho) + \sum_{k=0} \bar{c}_k \, \mathrm{e}^{\bar{\lambda}_k \tau} \bar{\xi}_k(\rho), \tag{24}$$

where $\xi_k(\rho)$, $\bar{\xi}_k(\rho)$ are the eigenmodes corresponding to the eigenvalues λ_k, $\bar{\lambda}_k$, respectively, and c_k, $\bar{c}_k$ are the expansion coefficients. Keeping the first two least damped eigenmodes we obtain the following expansions in terms of original variables (using the abbreviation $\delta = T - t$)

$p = 3$

$$\delta u(r, t) = \sqrt{2} + c_1 \delta \left(1 - \frac{r^2}{\delta^2}\right) + c_2 \delta^3 \left(1 - \frac{2r^2}{3\delta^2} + \frac{r^4}{5\delta^4}\right) + \mathrm{O}(\delta^4), \tag{25}$$

$p = 5$

$$\sqrt{\delta} u(t, r) = \left(\frac{3}{4}\right)^{1/4} + c_1 \delta \left(1 - \frac{2r^2}{3\delta^2}\right) + \delta^3 \left(\bar{c}_0 + c_2 \frac{r^2}{\delta^2} \left(1 - \frac{r^2}{5\delta^2}\right)\right) + \mathrm{O}(\delta^5), \tag{26}$$

$p = 7$

$$\delta^{1/3} u(t, r) = \left(\frac{2}{3}\right)^{1/3} + c_1 \delta \left(1 - \frac{5r^2}{9\delta^2}\right) + \bar{c}_0 \delta^{8/3} + \mathrm{O}(\delta^3). \tag{27}$$

We claim that these formulae describe accurately the convergence to the blowup profile inside the past light cone of the blowup point ($t = T, r = 0$). The numerical evidence for this assertion is summarized in figures 3 and 4 in the case $p = 3$ (throughout this section we use the case $p = 3$ for illustration—analogous results hold for $p = 5$ and 7).

As shown in figure 4 the formula (25) accurately describes the blowup profile for large τ (i.e. t close to T) not only inside the light cone but even slightly beyond. However, the expansions (25–27) are expected to break down when the linearization is no longer valid; that is, if $r^2/\delta^2 \sim 1/\delta$. In this transition region the leading order approximation for any p reads

$$u(t, r) \simeq \frac{1}{\delta^{\alpha}} \left(a + c_1 d_{12} \frac{r^2}{\delta}\right) \tag{28}$$

with α and a defined as in (3) and d_{12} equal to the coefficient of the quadratic term of the ξ_1 eigenfunction. This indicates the parabolic scaling

$$u(t, r) = \frac{1}{\delta^{\alpha}} F(z), \qquad z = \frac{r}{\sqrt{\delta}}. \tag{29}$$

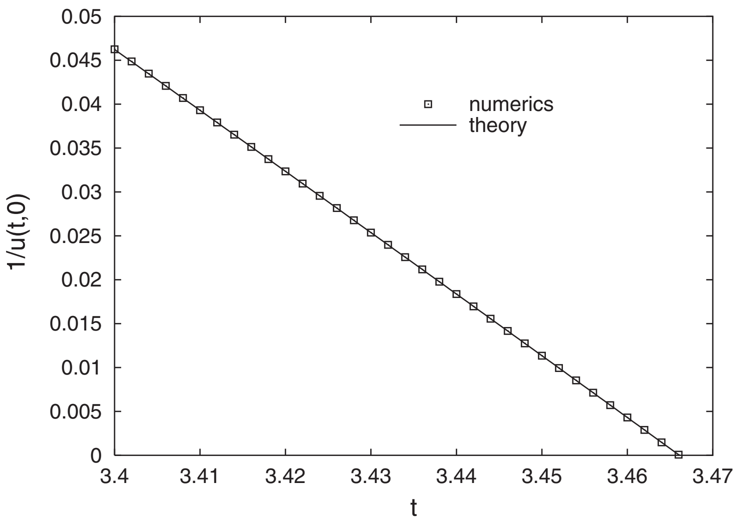

Figure 3. For $p = 3$ we plot the function $1/u(t, 0)$ for the solution that blows up at $r = 0$ as $t \to T \approx 3.466$. The solid line shows the fit to the first order analytic approximation $\delta/(\sqrt{2}+c_1\delta)$.

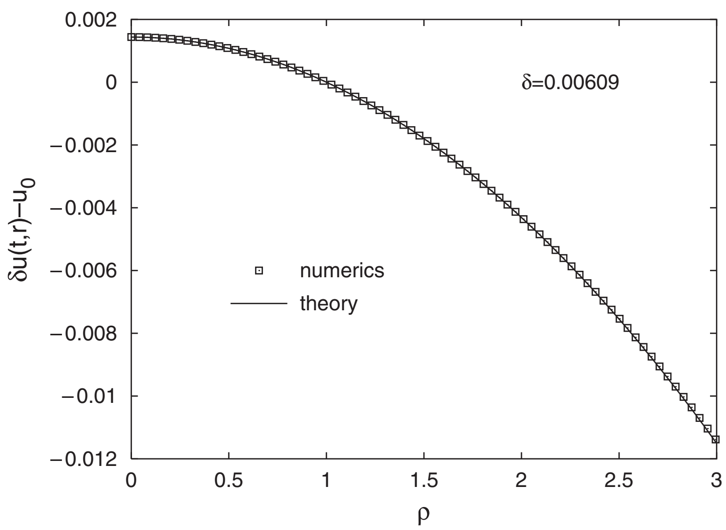

Figure 4. For the same numerical data as in figure 3 we plot the deviation of the rescaled solution $\delta u(t, r/\delta)$ from the constant solution $U_0 = \sqrt{2}$ at time $\delta = 6.09 \times 10^{-3}$. The solid line shows the least damped eigenmode $c_1(1 - \rho^2)\delta$ with the same coefficient c_1 as in figure 3.

Substituting this ansatz into equation (4) and dropping the Laplacian (which becomes negligible as $\delta \to 0$) we get the ordinary differential equation

$$z^2 F'' + (4\alpha + 3)zF' + 4\alpha(\alpha + 1)F - 4F^p = 0, \tag{30}$$

which has a one-parameter family of regular solutions

$$F(z) = \frac{a}{\left(1 + bz^2\right)^{\alpha}}. \tag{31}$$

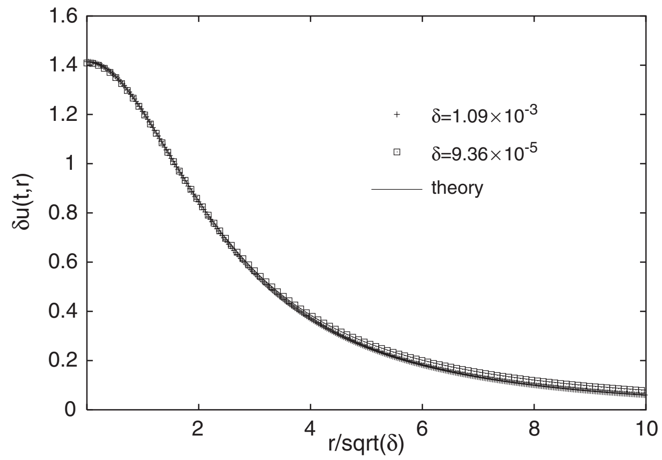

Figure 5. For $p = 3$ the rescaled solution at two moments of time close to the blowup time is shown to collapse to the analytic curve $F(z) = \sqrt{2}/(1 + bz^2)$ with $b = c_1/\sqrt{2}$.

Comparing (28) with (29) and (31) we get the matching condition

$$b = -\frac{d_{12}}{\alpha a} c_1, \tag{32}$$

which, for example, for $p = 3$ gives $b = c_1/\sqrt{2}$. The numerical confirmation of this prediction is shown in figure 5.

We remark that the above result follows immediately from the Fuchsian analysis which predicts the leading order asymptotics on a spacelike blowup curve $T(r)$ in the form

$$u(t, r) = \frac{a}{(T(r) - t)^{\alpha}}. \tag{33}$$

The blowup time is defined as $T = \inf T(r)$. Assuming that this infimum is attained at $r = 0$, we have $T(r) \simeq T + br^2$ for some $b > 0$. Inserting this into (33) we get

$$u(t, r) = \frac{1}{\delta^{\alpha}} \frac{A}{(1 + b(r^2/\delta))^{\alpha}}, \tag{34}$$

which reproduces (29) and (31).

As the coefficient b becomes negative, the first blowup occurs at $r_S > 0$. By fine-tuning initial data to the transition between the blowup at $r_S = 0$ and the blowup at $r_S > 0$ we set $b = 0$ which means that the first eigenmode in the expansion (24) is tuned away. For such codimension-one initial data the formula (28) should be replaced by

$$u(t, r) \simeq \frac{1}{\delta^{\alpha}} \left(a + c_2 d_{23} \frac{r^4}{\delta} \right), \tag{35}$$

where the coefficient d_{23} is equal to the quartic term of the ξ_2 eigenfunction. This gives another scaling (see figure 6)

$$u(t, r) = \frac{1}{\delta^{\alpha}} G(z), \qquad z = \frac{r}{\delta^{1/4}}, \tag{36}$$

where

$$G(z) = \frac{a}{\left(1 + dz^4\right)^{\alpha}}, \qquad d = -\frac{d_{23}}{\alpha a} c_2. \tag{37}$$

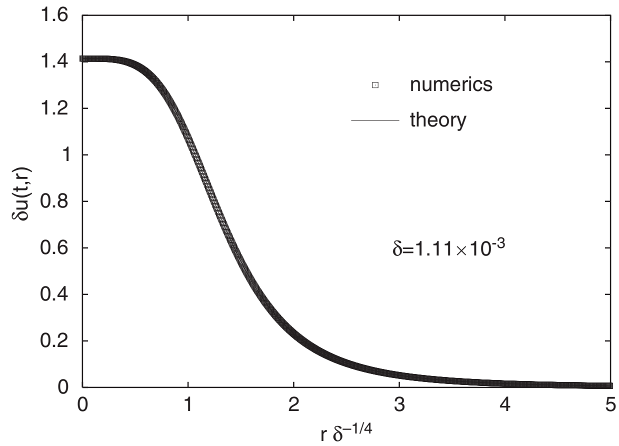

Figure 6. For $p = 3$ we plot the rescaled solution approaching the blowup time for initial data fine-tuned to the borderline of blowup at $r_S = 0$ and $r_S > 0$. The solid line shows the fit to the analytic prediction $G(z) = \sqrt{2}/(1 + dz^4)$.

4. Threshold for blowup

Since solutions of equation (4) disperse for small initial data and blowup for large initial data, there arises a natural question what happens in between. In the following the boundary between initial data that lead to dispersion and initial data that lead to singularity formation will be referred to as the threshold for blowup. The determination of the threshold for blowup and the corresponding dynamics is of great interest in physical models which predict formation of singularities, for example in general relativity. This issue can be studied numerically as follows. Consider a one-parameter family of initial data $\phi(p)$ such that the corresponding solutions exist globally if the parameter p is small or blowup if the parameter p is large. Then, along the curve $\phi(p)$ there must be a critical value p^* (or an interval $[p^*_{\min}, p^*_{\max}]$) which separates these two scenarios. Given two values p small and p large, it is straightforward (in principle but not always in practice) to find p^* by bisection. Repeating this procedure for different interpolating families of initial data one obtains a set of critical data which by construction belongs to the threshold for blowup. Having that, one can look in more detail at the evolution of critical data. The precisely critical data cannot be prepared numerically but in practice it is sufficient to follow the evolution of marginally critical data. Typically, one finds that the evolution of such data has a universal (that is family independent) transient phase during which the solution approaches a kind of an intermediate attractor.

The heuristic explanation of this behaviour is sketched in figure 7. According to this picture the threshold for blowup is given by the codimension-one stable manifold $W_S(u^*)$ of an intermediate attractor u^*, called the critical solution. The critical initial data, corresponding to intersections of $W_S(u^*)$ with different interpolating one-parameter families of initial data, converge[4] along $W_S(u^*)$ towards the critical solution. The marginally critical data, by continuity, initially remain close to $W_S(u^*)$ and approach u^* for intermediate times but eventually are repelled from its vicinity along the one-dimensional unstable manifold. Within this picture the universality of marginally critical dynamics in the intermediate asymptotics follows immediately from the fact that the same unstable mode dominates the evolution of all

[4] It should be stressed that for conservative wave equations, such as (1), the convergence (which is due to radiation of energy to infinity) is always meant in the local sense.

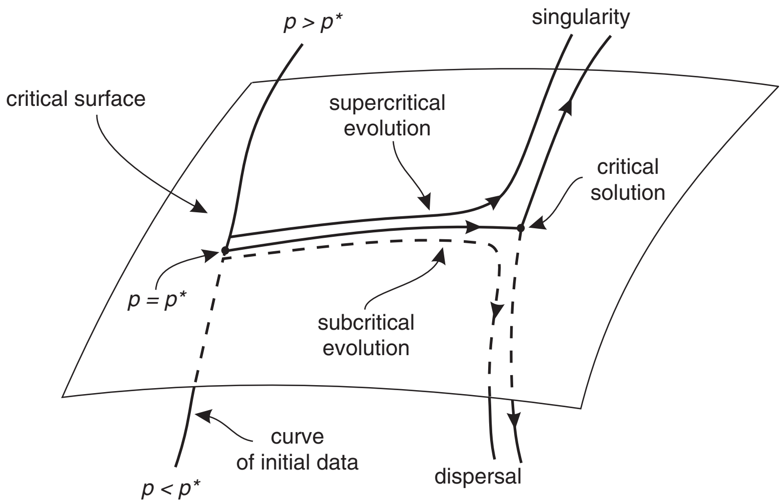

Figure 7. A schematic phase space picture of dynamics at the threshold for blowup.

solutions. The nature of the critical solution itself depends on a model—typically u^* is a static or a self-similar solution with exactly one unstable mode.

To apply the numerical strategy outlined above we solved equation (4) for various one-parameter families of initial conditions which interpolate between small and large initial data. The results described below do not depend on the particular choice of the family—for concreteness we present them for the initial data of the form

$$u(0, r) = Ar^2 \exp\left[-\left(\frac{r-R}{\sigma}\right)^4\right], \qquad u_t(0, r) = 0 \tag{38}$$

with adjustable amplitude A and fixed parameters σ and R. Since the initial data are time symmetric, the initial profile splits into ingoing and outgoing waves travelling with approximately unit speed. Except for very large initial amplitudes for which the singularity forms very fast, before the separation into ingoing and outgoing wave occurs, the evolution of the outgoing wave does not affect the singularity formation so we shall ignore it. The behaviour of the ingoing wave depends on the amplitude A. For large amplitudes we observe the formation of singularity at some $r_S > 0$ in finite time T. As A decreases, the blowup point r_S decreases also and reaches[5] $r_S = 0$ for some value A_0. As we keep decreasing the amplitude below A_0 we eventually reach a critical value A^* below which solutions do not blowup. The asymptotic pattern of blowup described in section 3 applies to solutions with amplitudes $A^* < A < A_0$. The character of the threshold for blowup at A^* depends on p so we discuss three values of p separately.

$p = 7$:
In this case we identify the critical solution as the $n = 1$ self-similar solution

$$u_1(t, r) = (T - t)^{-(1/3)} U_1(\rho). \tag{39}$$

The numerical evidence for the criticality of solution u_1 is presented in figure 8.

[5] The behaviour of the function $r_S(A)$ depends on p. As $A \to A_0$ from above, the function $r_S(A)$ decreases continuously to zero for $p = 3$ but for $p > 3$ it jumps from some $r_S > 0$ to $r_S = 0$.

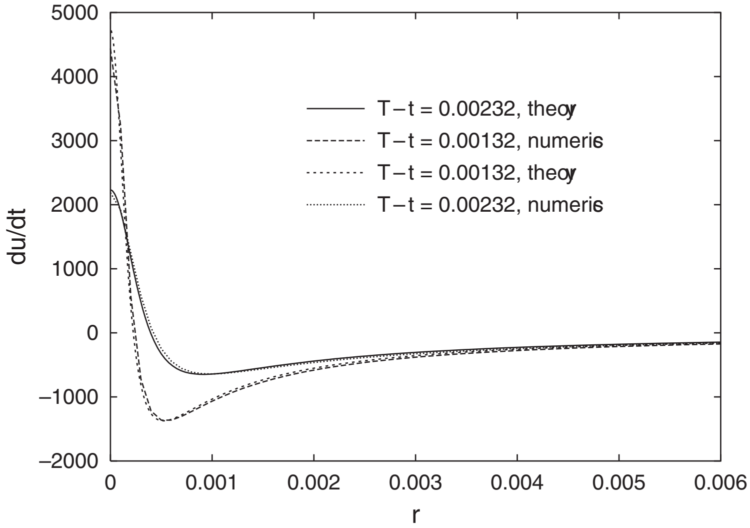

Figure 8. For $p = 7$ the time derivative $u_t(t, r)$ of marginally critical solutions is plotted for two moments of time during the transient phase of evolution and compared to the theoretical prediction $\partial u_1/\partial t = (T - t)^{-4/3}(U_1/3 + \rho U_1')$. The parameter T is the same for both curves.

According to the picture of critical behaviour described earlier, the marginally critical solutions have the following form in the intermediate asymptotics

$$u(t, r) = (T - t)^{-1/3} U_1(\rho) + C(A)(T - t)^{-\lambda_1 - (1/3)} \xi_1(\rho) + \text{damped modes}, \tag{40}$$

where ξ_1 is the single unstable mode about u_1 with the eigenvalue $\lambda_1 = 11.6442$. A small constant $C(A)$, which is the only vestige of initial data, quantifies an admixture of the unstable mode—for precisely critical data $C(A^*) = 0$. We show in figure 9 that the departure from the critical solution proceeds in agreement with equation (40).

$p = 5$:
We know from section 2 that in this case there are no nontrivial self-similar solutions. However, since for $p = 5$ the energy is scale invariant, static solutions with finite energy are possible. Indeed, it is well known that equation (4) has the finite energy solution

$$u_S(r) = \left(1 + \tfrac{1}{3} r^2\right)^{-1/2}. \tag{41}$$

Rescalings of this solution generate the orbit of static solutions $u_S^L = L^{-1/2} u_S(r/L)$. To determine the linear stability of this solution we plug $u(t, r) = u_S(r) + \mathrm{e}^{\mathrm{i}kt} v(r)$ into (4) and linearize. We get the eigenvalue problem in the form of the radial Schrödinger equation

$$-v'' - \frac{2}{r} v' + V v = k^2 v, \qquad V = -\frac{5}{(1 + (1/3) r^2)^2}. \tag{42}$$

Notice that the perturbation induced by rescaling

$$v_0(r) = -\frac{\mathrm{d}}{\mathrm{d}L} u_S^L(r)\bigg|_{L=1} = \frac{(1/2) - (r^2/6)}{(1 + (1/3) r^2)^{3/2}}, \tag{43}$$

satisfies equation (42) for k^2. This is the so-called zero mode. Since the zero mode has one node, it follows by the standard result from Sturm–Liouville theory that the potential V has exactly one bound state, $k_1^2 < 0$, which means that there is exactly one growing mode $\mathrm{e}^{\lambda_1 t} v_1(r)$, where $\lambda_1 = \sqrt{-k_1^2}$. Numerical calculation gives $\lambda_1 \approx 1.1$.

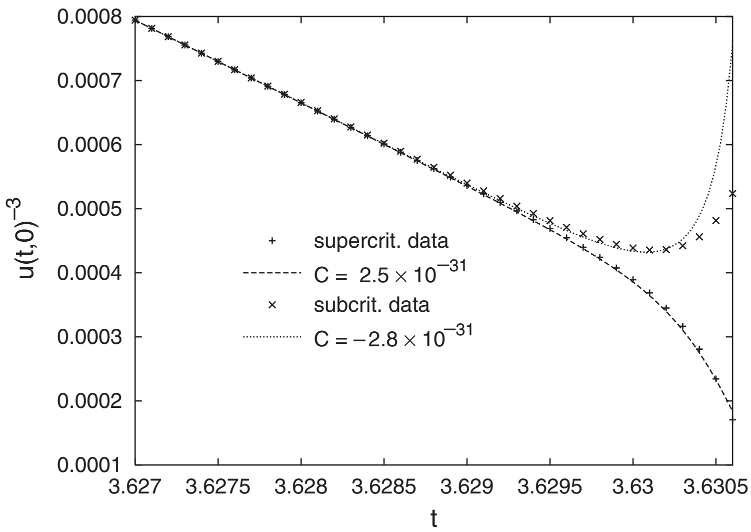

Figure 9. We plot $u^{-3}(t, 0)$ for the pair of marginally critical solutions corresponding to initial data (38) with $A = A^* \pm 10^{-31}$. Initially these solutions are indistinguishable but eventually they split and depart from the critical solution towards blowup and dispersal, respectively. The theoretical curves, corresponding to equation (40) for $r = 0$, with two different fitted coefficients C are superimposed.

Thus, according to our preceding discussion, the solution u_S is a candidate for a critical solution. To verify this, Szpak [9] has investigated the nonlinear evolution of the growing mode. For initial data of the form $u(0, r) = u_S(r) + \epsilon v_1(r)$, $u_t(0, r) = \epsilon\lambda_1 v_1(r)$, he found that depending on the sign of the amplitude ϵ, the solution either disperses or blows up in finite time. This confirmed the expectation that in fact u_S is the critical solution sitting on the saddle separating the blowup from dispersal. Applying bisection to the family of initial data (38) we have obtained the solution u_S as the intermediate attractor with pretty long lifetime. We refer the reader to [9] for more details, in particular the analysis of convergence to u_S.

$p = 3$:

In this case we were not able to identify a critical solution because of two reasons. First, in contrast to the cases described above, in $p = 3$ we do not have a good candidate for the critical solution. The only potential candidate is the self-similar solution u_1 with one unstable mode, however, as mentioned in section 2, this solution is singular outside the past light cone of the blowup point and, therefore, cannot be a *bona fide* critical solution. Second, we face the following difficulty when trying to determine A^*. As we approach the expected value of A^* from above, the wave initially shrinks but at some later time t_1 it bounces back and expands outside with decreasing amplitude. During this period of evolution the amplitude of outer wave front decreases faster than the amplitude of the solution at the centre, so a flat central region that slowly decreases with time develops. After some time t_2 this central part of the solution returns and starts growing again to form a singularity at $r = 0$. If we decrease A further, the time of bounce t_1 almost does not change but the return time t_2 increases significantly. Therefore, approaching A^* we have to evolve the solution longer and longer on larger and larger grids. Since the numerical grid is always finite, we cannot tell if an expanding wave which leaves the grid represents a genuine dispersion or a singular solution with large return time t_2. Figures 10 and 11 illustrate this difficulty.

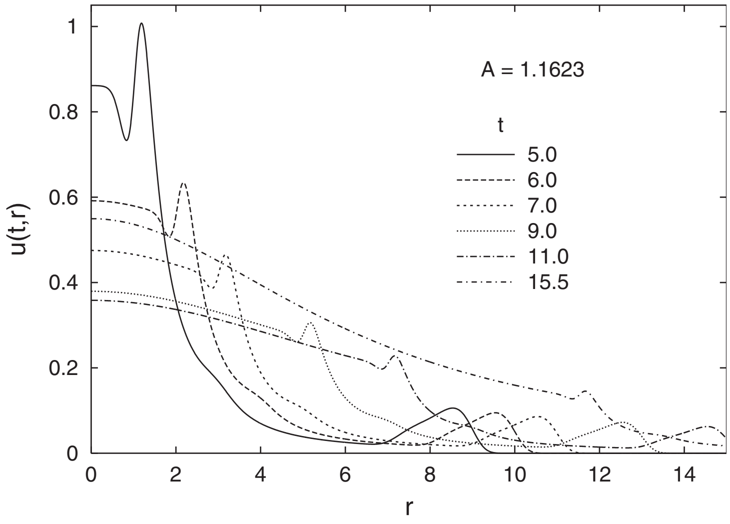

Figure 10. For $p = 3$ we plot the snapshots from the evolution of the wave that has bounced back from the centre. After the bounce the amplitude at the centre initially decreases but later the wave returns and the amplitude starts growing again.

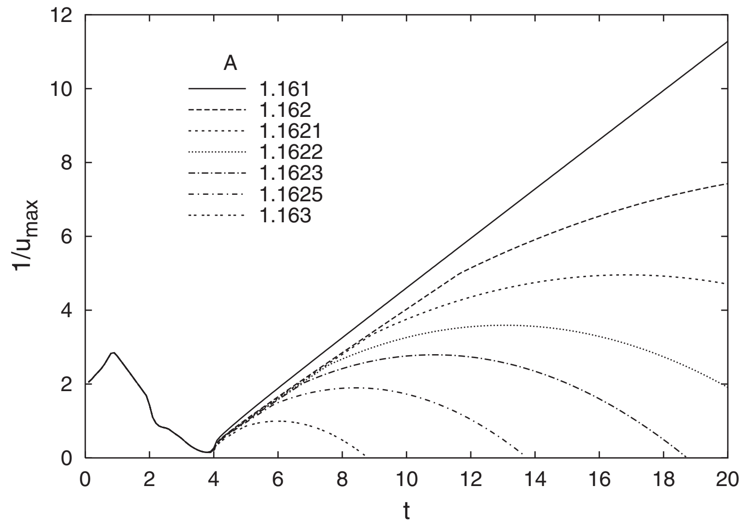

Figure 11. The same data as in figure 10. The first minimum of $1/u_{\rm max}$ corresponds to the bounce. The second local maximum corresponds to the return.

5. Conclusions

We have studied the formation of singularities for the spherically symmetric semilinear wave equation with the focusing power nonlinearity u^p for three representative values of the exponent p: $p = 3$ (subcritical case) $p = 5$ (critical case), and $p = 7$ (supercritical case). We showed that in all these cases the asymptotic behaviour of the blowup can be understood in terms of decaying perturbations about the fundamental (homogeneous in space) self-similar solution. We showed also that the nature of the critical solution, whose codimension-one stable manifold separates the blowup from dispersal, depends on p: for $p = 7$ the critical solution is self-similar while for $p = 5$ it is static. For $p = 3$ we were not able to identify a critical solution—determining the character of the threshold for blowup in this case remains an open problem.

Acknowledgment

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