On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains

Martin Costabel Alan McIntosh

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Abstract

We study integral operators related to a regularized version of the classical Poincaré path integral and the adjoint class generalizing Bogovskiĭ's integral operator, acting on differential forms in $\mathbb{R}^n$. We prove that these operators are pseudodifferential operators of order -1. The Poincaré-type operators map polynomials to polynomials and can have applications in finite element analysis.

For a domain starlike with respect to a ball, the special support properties of the operators imply regularity for the de Rham complex without boundary conditions (using Poincaré-type operators) and with full Dirichlet boundary conditions (using Bogovskiĭ-type operators). For bounded Lipschitz domains, the same regularity results hold, and in addition we show that the cohomology spaces can always be represented by $\mathscr{C}^\infty$ functions.

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1 Introduction

In [3], Bogovskiĭ introduced an integral operator T with two remarkable properties:
- If f is a function satisfying $\int f(x)dx = 0$, then $u = Tf$ solves the partial differential equation $\operatorname{div} u = f$, and
- If the bounded domain $\Omega \subset \mathbb{R}^n$ is starlike with respect to an open ball B, then T maps the Sobolev space $W_0^{m-1,p}(\Omega)$ boundedly to $W_0^{m,p}(\Omega)^n$ for all $m \geq 0$ and $1 < p < \infty$.

This implies for a large class of domains Ω, including all bounded Lipschitz domains, the solvability in $W_0^{m,p}(\Omega)^n$ of the equation $\operatorname{div} u = f$ for $f \in W_0^{m-1,p}(\Omega)$ satisfying the integrability condition $\int f dx = 0$. This means that there is no loss of regularity, and the support is preserved.

This operator is now a classical tool in the theory of the equations of hydrodynamics [5]. It was recently noticed that its range of continuity can be extended to Sobolev spaces of negative order of regularity [6], and the study of more refined mapping properties has been instrumental in obtaining sharp regularity estimates for powers of the Stokes operator [12].

Bogovskiĭ's integral operator T makes use of a smoothing function

$$\theta \in \mathscr{C}_0^\infty(\mathbb{R}^n)\,, \quad \operatorname{supp}\theta \subset B\,, \quad \int \theta(x)\,dx = 1 \tag{1.1}$$

when Ω is starlike with respect to an open ball B, and is defined by

$$Tf(x) = \int_\Omega f(y) \frac{x-y}{|x-y|^n} \int_{|x-y|}^\infty \theta\Big(y + r\frac{x-y}{|x-y|}\Big)\, r^{n-1}\, dr\, dy\,. \tag{1.2}$$

Applying the change of variables $(y,r) \mapsto (a,t) = (x + r\frac{y-x}{|x-y|}, 1 - \frac{|x-y|}{r})$, one sees that the formally adjoint integral operator T' is given by a smoothed-out path integral which defines the potential $v = T'u$ of a conservative vector field u, thus giving a solution of the equation $\mathbf{grad}\, v = u$:

$$T'u(x) = -\int \theta(a) J_a u(x)\, da\ , \quad J_a u(x) = (x-a) \cdot \int_0^1 u\big(a + t(x-a)\big)\, dt\ . \tag{1.3}$$

The standard proof of Poincaré's lemma in differential geometry via "Cartan's magic formula" [15, Theorem 13.2] uses a generalization of the path integral J_a in (1.3) to construct a right inverse of the exterior derivative operator for closed differential forms. A typical example in $\mathbb{R}^3$ is the path integral

$$R_a u(x) = -(x-a) \times \int_0^1 u\big(a + t(x-a)\big)\, t\, dt \tag{1.4}$$

which provides a solution of the equation $\mathbf{curl}\, v = u$ for a divergence-free vector field u. Under the name "Poincaré map", this integral operator has recently been used in the analysis of finite element methods for Maxwell's equations [7, 4]. Three properties of the operator R_a are important for this application:

- R_a maps polynomial vector fields to polynomial vector fields
- If Ω is starlike with respect to a, then the restriction of $R_a u$ to Ω depends only on the restriction of u to Ω
- R_a maps $L^2(\Omega)^3$ boundedly to itself.

One of the results of the present paper is that the regularized version R of R_a, given by

$$Ru(x) = \int \theta(a) R_a u(x)\, da\ ,$$

while still preserving polynomials and the local domain of influence, defines a bounded operator from $W^{s,p}(\Omega)$ to $W^{s+1,p}(\Omega)$ for all $s \in \mathbb{R}$ and $1 < p < \infty$, if Ω is starlike with respect to the ball B. Such an operator was used in Section 4 of [2] to obtain an inverse to the exterior derivative operator in L^2 spaces.

In [11], Mitrea studied the generalization of both the Bogovskiĭ-type and the regularized Poincaré-type integral operators acting on differential forms with coefficients in Besov or Triebel-Lizorkin spaces. In [10], Mitrea, Mitrea and Monniaux extended this analysis to obtain sharp regularity estimates for the "natural" boundary value problems of the exterior derivative operator on Lipschitz domains. There the non-smoothness of the boundary of the domain implies that the solutions of these boundary value problems are singular, and therefore the solution operator is bounded for certain intervals of the regularity index s depending on the exponent p, whereas for certain critical indices the boundary value problem does not define an operator with closed range.

In this paper, we prove that the Bogovskiĭ-type and the regularized Poincaré-type integral operators are classical pseudodifferential operators of order -1 with symbols in the Hörmander class $S^{-1}_{1,0}(\mathbb{R}^n)$. As is well known [17, Chapter 6], this implies immediately that the operators act as bounded operators in a wide range of function spaces including Hölder, Hardy or Sobolev spaces, or more generally the Besov spaces B^s_{pq} for $0 < p, q \le \infty$, and the Triebel-Lizorkin spaces F^s_{pq} for $0 < p < \infty$, $0 < q \le \infty$. In each case, the operators map differential forms with coefficients of regularity s boundedly to differential forms of regularity $s+1$ and, if Ω is bounded and starlike with respect to a ball, the Bogovskiĭ-type operators act between spaces of distributions with compact support in $\overline{\Omega}$, and the Poincaré-type operators act between spaces of restrictions to Ω.

As a consequence, we obtain regularity results for the exterior derivative operator on bounded Lipschitz domains, either in spaces with compact support, or in spaces without boundary conditions, and these regularity results hold without restriction on the regularity index s. In particular, we show that the cohomology spaces of the de Rham complex on a bounded Lipschitz domain, either with compact support, or without boundary conditions, can be represented independently of the regularity index s by finite dimensional spaces of differential forms with $\mathscr{C}^\infty$ coefficients.

Thus, by the end of the paper, we will have employed the Bogovskiĭ-type and the regularized Poincaré-type integral operators to construct finite dimensional spaces $\mathscr{H}_\ell(\overline{\Omega}) \subset \mathscr{C}^\infty(\overline{\Omega}, \Lambda^\ell)$ and $\mathscr{H}_{\overline{\Omega},\ell}(\mathbb{R}^n) \subset \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)$, each independent of the degree of regularity s, such that all of the following direct sum decompositions hold true. To do this we use finitely many coverings of $\overline{\Omega}$, each by finitely many starlike domains. (A similar procedure would work for a Lipschitz domain in a compact Riemannian manifold.) See the next section for definitions.

Theorem 1.1 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, and let $0 \le \ell \le n$. Then for the spaces without boundary conditons,*

$$\ker\Big(d : \mathscr{C}^\infty(\overline{\Omega}, \Lambda^\ell) \to \mathscr{C}^\infty(\overline{\Omega}, \Lambda^{\ell+1})\Big) = d\,\mathscr{C}^\infty(\overline{\Omega}, \Lambda^{\ell-1}) \oplus \mathscr{H}_\ell(\overline{\Omega}) ,$$

$$\ker\Big(d : H^s(\Omega, \Lambda^\ell) \to H^{s-1}(\Omega, \Lambda^{\ell+1})\Big) = d\,H^{s+1}(\Omega, \Lambda^{\ell-1}) \oplus \mathscr{H}_\ell(\overline{\Omega})$$

where the H^s ($-\infty < s < \infty$) denote Sobolev spaces, and, more generally,

$$\ker\Big(d : B^s_{pq}(\Omega, \Lambda^\ell) \to B^{s-1}_{pq}(\Omega, \Lambda^{\ell+1})\Big) = d\,B^{s+1}_{pq}(\Omega, \Lambda^{\ell-1}) \oplus \mathscr{H}_\ell(\overline{\Omega}) ,$$

$$\ker\Big(d : F^s_{pq}(\Omega, \Lambda^\ell) \to F^{s-1}_{pq}(\Omega, \Lambda^{\ell+1})\Big) = d\,F^{s+1}_{pq}(\Omega, \Lambda^{\ell-1}) \oplus \mathscr{H}_\ell(\overline{\Omega})$$

where the B^s_{pq} ($-\infty < s < \infty$, $0 < p, q \le \infty$) denote Besov spaces, and the F^s_{pq} ($-\infty < s < \infty$, $0 < p < \infty$, $0 < q \le \infty$) denote Triebel-Lizorkin spaces.

For the spaces with compact support, and the same values of s, p and q, we have

$$\ker\Big(d : \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell) \to \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell+1})\Big) = d\,\mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1}) \oplus \mathscr{H}_{\overline{\Omega},\ell}(\mathbb{R}^n) ,$$

$$\ker\Big(d : H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell) \to H^{s-1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell+1})\Big) = d\,H^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1}) \oplus \mathscr{H}_{\overline{\Omega},\ell}(\mathbb{R}^n) ,$$

$$\ker\Big(d : B^s_{pq\,\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell) \to B^{s-1}_{pq\,\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell+1})\Big) = d\,B^{s+1}_{pq\,\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1}) \oplus \mathscr{H}_{\overline{\Omega},\ell}(\mathbb{R}^n) ,$$

$$\ker\Big(d : F^s_{pq\,\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell) \to F^{s-1}_{pq\,\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell+1})\Big) = d\,F^{s+1}_{pq\,\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1}) \oplus \mathscr{H}_{\overline{\Omega},\ell}(\mathbb{R}^n) .$$

We remark without further discussion that this result has applications for the local Hardy spaces $h^1_r(\Omega, \Lambda^\ell) = F^0_{12}(\Omega, \Lambda^\ell)$ and $h^1_z(\Omega, \Lambda^\ell) = F^0_{12\,\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)$.

2 Notation and definitions

For a bounded domain Ω in $\mathbb{R}^n$, we consider four spaces of infinitely differentiable functions. Besides $\mathscr{C}^\infty(\Omega)$, the space of all infinitely differentiable functions in Ω, and $\mathscr{C}^\infty_0(\Omega)$, the functions with compact support in Ω, we also use the space of restrictions to Ω

$$\mathscr{C}^\infty(\overline{\Omega}) = \{u \in \mathscr{C}^\infty(\Omega) \mid \exists \tilde{u} \in \mathscr{C}^\infty(\mathbb{R}^n) :\ u = \tilde{u} \text{ on } \Omega\}$$

and the space of functions with support in $\overline{\Omega}$

$$\mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n) = \{u \in \mathscr{C}^\infty(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\} \ .$$

Thus $\mathscr{C}^\infty(\overline{\Omega})$ is a quotient space of $\mathscr{C}^\infty(\mathbb{R}^n)$ (or $\mathscr{C}^\infty_0(\mathbb{R}^n)$) modulo functions vanishing on Ω, and $\mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n)$ is a subspace of $\mathscr{C}^\infty(\mathbb{R}^n)$ (or $\mathscr{C}^\infty_0(\mathbb{R}^n)$). Likewise, for functions or distributions of regularity $s \in \mathbb{R}$, we consider spaces of restrictions to Ω and spaces with compact support in $\overline{\Omega}$.

By the term *bounded Lipschitz domain* Ω in $\mathbb{R}^n$ we mean a connected bounded open set which is strongly Lipschitz in the sense that in the neighborhood of each point of $\overline{\Omega} = \Omega \cup \partial\Omega$ it is congruent to the domain below the graph of a scalar Lipschitz continuous function of $n-1$ variables.

A domain Ω is *starlike* with respect to a set B if for every $x \in \Omega$ the convex hull of $\{x\} \cup B$ is contained in Ω. It is not hard to see that a bounded domain which is starlike with respect to an open ball is Lipschitz, and that conversely, every bounded Lipschitz domain is the union of a finite number of domains, each of which is starlike with respect to an open ball.

To keep the notation simple, we use the Sobolev space $H^s = W^{s,2}$ as representative for a space of regularity s. But, as already mentioned, many of the following arguments remain valid if the L^2-based Sobolev space H^s is replaced by the Sobolev-Slobodeckii space $W^{s,p}$ or the Bessel potential space H^s_p $(1 < p < \infty)$ or, more generally, by any of B^s_{pq} $(0 < p, q \le \infty)$ or F^s_{pq} $(0 < p < \infty, 0 < q \le \infty)$.

We let $H^s(\Omega)$ denote the quotient space of $H^s(\mathbb{R}^n)$ by the subspace of distributions vanishing in Ω, while we let $H^s_{\overline{\Omega}}(\mathbb{R}^n)$ denote the subspace of $H^s(\mathbb{R}^n)$ consisting of all distributions with support in $\overline{\Omega}$. Thus $H^s(\Omega)$, for which also equivalent intrinsic definitions exist, can be considered as a space of distributions on Ω, whereas $H^s_{\overline{\Omega}}(\mathbb{R}^n)$ is a space of distributions on $\mathbb{R}^n$.

Let us mention some well-known properties of these spaces that hold if Ω is a bounded Lipschitz domain. Proofs (for the spaces $W^{s,p}$, $s \in \mathbb{R}$, $1 < p < \infty$) can be found in [8, Chapter 1]: The intersection of all $H^s(\Omega)$, $s \in \mathbb{R}$, is $\mathscr{C}^\infty(\overline{\Omega})$ and the union of all $H^s(\Omega)$ is the space of all distributions on Ω that allow an extension to a neighborhood of $\overline{\Omega}$. Likewise, the intersection of all $H^s_{\overline{\Omega}}(\mathbb{R}^n)$ is $\mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n)$ and the union of all $H^s_{\overline{\Omega}}(\mathbb{R}^n)$ is the space of all distributions on $\mathbb{R}^n$ with support in $\overline{\Omega}$. It is also well known that $H^s_{\overline{\Omega}}(\mathbb{R}^n)$, for which also Triebel's notation $\widetilde{H}^s(\Omega)$ is commonly used, can be identified with the space $H^s_0(\Omega)$, the closure of $\mathscr{C}^\infty_0(\Omega)$ in $H^s(\Omega)$, if s is positive and $s - \frac{1}{2}$ is not an integer. For any $s \in \mathbb{R}$, $H^s_{\overline{\Omega}}(\mathbb{R}^n)$ is the closure of $\mathscr{C}^\infty_0(\Omega)$ in $H^s(\mathbb{R}^n)$. In our Hilbert space setting, for all $s \in \mathbb{R}$ the space $H^s_{\overline{\Omega}}(\mathbb{R}^n)$ is in a natural way isomorphic to the dual space of $H^{-s}(\Omega)$.

For differential forms we use standard notation which is, for example, defined in [13, 15]. The exterior algebra of $\mathbb{R}^n$ is Λ^ℓ, $0 \le \ell \le n$, where Λ^0 and Λ^1 are identified with $\mathbb{R}$ and $\mathbb{R}^n$, respectively, and we set $\Lambda^\ell = \{0\}$ if $\ell < 0$ or $\ell > n$.

Differential forms of order ℓ with coefficients in H^s are denoted by $H^s(\Omega, \Lambda^\ell)$ and $H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)$. With the exterior derivative d satisfying $d \circ d = 0$ we then have the *de Rham complex without boundary conditions*

$$0 \to H^s(\Omega, \Lambda^0) \xrightarrow{d} H^{s-1}(\Omega, \Lambda^1) \xrightarrow{d} \cdots \xrightarrow{d} H^{s-n}(\Omega, \Lambda^n) \to 0 \tag{2.1}$$

and the *de Rham complex with compact support*

$$0 \to H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^0) \xrightarrow{d} H^{s-1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^1) \xrightarrow{d} \cdots \xrightarrow{d} H^{s-n}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^n) \to 0 \tag{2.2}$$

Besides these complexes we also consider the *extended* de Rham complexes without boundary conditions

$$0 \to \mathbb{R} \xrightarrow{\iota} H^s(\Omega, \Lambda^0) \xrightarrow{d} H^{s-1}(\Omega, \Lambda^1) \xrightarrow{d} \cdots \xrightarrow{d} H^{s-n}(\Omega, \Lambda^n) \to 0 \tag{2.3}$$

and with compact support

$$0 \to H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^0) \xrightarrow{d} H^{s-1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^1) \xrightarrow{d} \cdots \xrightarrow{d} H^{s-n}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^n) \xrightarrow{\iota^*} \mathbb{R} \to 0 \tag{2.4}$$

Here the mapping denoted by ι in (2.3) is the natural inclusion of constant functions, and ι^* in (2.4) is the generalization to distributional coefficients with compact support of the integral $u \mapsto \iota^* u = \int_{\mathbb{R}^n} u$ for an n-form u with integrable coefficients.

The extended de Rham complexes (2.3) and (2.4) are exact at the left end because Ω is connected, and their exactness at the right end is the subject of Bogovskiĭ's theorem mentioned in the introduction. We will show in Section 4 below that for bounded domains starlike with respect to a ball, both complexes (2.3) and (2.4) are exact for any $s \in \mathbb{R}$, and that for bounded Lipschitz domains both complexes (2.1) and (2.2) have finite dimensional cohomology spaces whose dimension does not depend on s.

We will make use of the following standard algebraic operations in the exterior algebra which then also extend as pointwise operations to differential forms on domains of $\mathbb{R}^n$:

the exterior product:	$\wedge$	:	$\Lambda^\ell \times \Lambda^m \to \Lambda^{\ell+m}$
the interior product or contraction:	$\lrcorner$	:	$\Lambda^\ell \times \Lambda^m \to \Lambda^{m-\ell}$
the euclidean inner product:	$\langle a, b \rangle$	:	$\Lambda^\ell \times \Lambda^\ell \to \mathbb{R}$
the Hodge star operator:	$\star$	:	$\Lambda^\ell \to \Lambda^{n-\ell}$

We now give a list of well-known properties of these operations which will be sufficient for verifying the arguments used in our proofs below.

In particular we need the exterior product and the contraction with a vector $a \in \mathbb{R}^n$, identified with a 1-form. For $a = (a_1, \dots, a_n)$ and $u = dx_{j_1} \wedge \dots \wedge dx_{j_\ell}$ with $j_1 < \cdots < j_\ell$, the contraction is given by

$$a \lrcorner u = \sum_{k=1}^{\ell} (-1)^{k-1} a_{j_k} dx_{j_1} \wedge \dots \wedge \widehat{dx}_{j_k} \wedge \dots \wedge dx_{j_\ell}$$

where the notation $\widehat{dx}_{j_k}$ means that the corresponding factor is to be omitted. In the special case of $\mathbb{R}^3$, this corresponds to the following classical operations of vector algebra:

u scalar, interpreted as 0-form:	$a \wedge u = ua$	$a \lrcorner u = 0$
u scalar, interpreted as 3-form:	$a \wedge u = 0$	$a \lrcorner u = ua$
u vector, interpreted as 1-form:	$a \wedge u = a \times u$	$a \lrcorner u = a \cdot u$
u vector, interpreted as 2-form:	$a \wedge u = a \cdot u$	$a \lrcorner u = -a \times u$

Some useful formulas for $u, v \in \Lambda^\ell$, $w \in \Lambda^{\ell+1}$, $a \in \Lambda^1$ are:

$$\star \star u = (-1)^{\ell(n-\ell)} u \tag{2.5}$$

$$\star(a \wedge u) = (-1)^\ell \, a \lrcorner (\star u) \tag{2.6}$$

$$\langle u, v \rangle = \star(u \wedge \star v) = \langle \star u, \star v \rangle \tag{2.7}$$

$$\langle w, a \wedge u \rangle = \langle u, a \lrcorner w \rangle \tag{2.8}$$

We note the product rule of the exterior derivative for an ℓ-form u and an m-form v

$$d(u \wedge v) = (du) \wedge v + (-1)^{\ell} u \wedge (dv) \,. \tag{2.9}$$

Finally, with the L^2 scalar product for ℓ-forms u and v,

$$(u, v) = \int_\Omega \langle u(x), v(x) \rangle \, dx$$

and the co-derivative δ, there holds

$$(\delta u, v) = (u, dv) \;, \tag{2.10}$$

$$\star\,\delta \;=\; (-1)^{\ell}\, d\,\star \quad \text{ and } \quad \star\, d \;=\; (-1)^{\ell-1}\,\delta\,\star \quad \text{ on } \ell\text{-forms} \,. \tag{2.11}$$

3 The Bogovskiĭ and Poincaré integral operators

In this section, we fix a function $\theta \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with support in a ball B satisfying $\int \theta(x)\, dx = 1$.

3.1 Definition, support properties

For $\ell \in \{0, \dots, n\}$, define the kernel G_ℓ by

$$G_\ell(x, y) = \int_1^\infty (t-1)^{n-\ell} t^{\ell-1} \theta\big(y + t(x-y)\big)\, dt \,. \tag{3.1}$$

Definition 3.1 *For a differential form $u \in \mathscr{C}_0^\infty(\mathbb{R}^n, \Lambda^\ell)$, define four integral operators:*

$$Q_\ell u(x) = \int G_{\ell+1}(y, x)\, (y-x) \wedge u(y)\, dy \quad (0 \le \ell \le n-1) \tag{3.2}$$

$$R_\ell u(x) = \int G_{n-\ell+1}(y, x)\, (x-y) \lrcorner\, u(y)\, dy \quad (1 \le \ell \le n) \tag{3.3}$$

$$S_\ell u(x) = \int G_{n-\ell}(x, y)\, (y-x) \wedge u(y)\, dy \quad (0 \le \ell \le n-1) \tag{3.4}$$

$$T_\ell u(x) = \int G_\ell(x, y)\, (x-y) \lrcorner\, u(y)\, dy \quad (1 \le \ell \le n) \tag{3.5}$$

We refer to Q_ℓ and R_ℓ as Poincaré-type operators, and to S_ℓ and T_ℓ as Bogovskiĭ-type operators.

In order to see that the integrals in Definition 3.1 exist, we rewrite the kernel G_ℓ:

$$\begin{aligned} G_\ell(x, y) &= \int_0^\infty \tau^{n-\ell} (\tau+1)^{\ell-1} \theta\big(x + \tau(x-y)\big)\, d\tau \\ &= \sum_{k=0}^{\ell-1} \tbinom{\ell-1}{k} \int_0^\infty \tau^{n-k-1} \theta\big(x + \tau(x-y)\big)\, d\tau \\ &= \sum_{k=0}^{\ell-1} \tbinom{\ell-1}{k} |x-y|^{k-n} \int_0^\infty r^{n-k-1} \theta\big(x + r\frac{x-y}{|x-y|}\big)\, dr \,. \end{aligned} \tag{3.6}$$

This representation as a finite sum of homogeneous functions gives a bound

$$|G_\ell(x,y)\,(x-y)| \le C(x)\,|x-y|^{-n+1}\,, \tag{3.7}$$

where $C(x)$ depends on $\|\theta\|_{L^\infty}$ and the size of the ball B, and is uniformly bounded for x in a bounded set. Hence the integrals in Definition 3.1 are weakly singular and therefore convergent.

As one can readily see from the definitions, the four integral operators are related by duality, Q_ℓ with R_ℓ and S_ℓ with T_ℓ by Hodge star duality, and Q_ℓ with T_ℓ and R_ℓ with S_ℓ by L^2 duality. More precisely, we have for $0 \le \ell \le n-1$ and $u \in \mathscr{C}_0^\infty(\mathbb{R}^n, \Lambda^\ell)$

$$\star Q_\ell\, u = (-1)^{\ell-1}\, R_{n-\ell}\,(\star u) \quad \text{ and } \quad \star\, S_\ell\, u = (-1)^{\ell-1}\, T_{n-\ell}\,(\star u) \tag{3.8}$$

and for $v \in \mathscr{C}_0^\infty(\mathbb{R}^n, \Lambda^{\ell+1})$

$$\big(v, Q_\ell u\big) = \big(T_{\ell+1} v, u\big) \quad \text{ and } \quad \big(v, S_\ell u\big) = \big(R_{\ell+1} v, u\big)\,. \tag{3.9}$$

Since the Hodge star operator has no influence on the regularity nor on the support of the coefficients of a differential form, we can restrict most of the following discussions to the two operators R_ℓ and T_ℓ; the results for the other two operators then follow from the star duality relations (3.8). From (3.8) and (3.9) we get the following relation between the Poincaré-type operators R_ℓ and the Bogovskiĭ-type operators T_ℓ; here the prime means the formal adjoint operator with respect to the L^2 duality.

$$\star R_\ell = (-1)^\ell\, T'_{n-\ell+1}\,\star \tag{3.10}$$

In order to see other properties of the operators, we apply a different change of variables. Let us write this in detail for the operator R_ℓ. We use the change of variables $a = x + t(y-x)$ and then replace $(t-1)/t$ by t.

$$\begin{aligned} R_\ell u(x) &= \int\!\int_1^\infty (t-1)^{\ell-1} t^{n-\ell}\theta\big(x+t(y-x)\big)\,(x-y) \lrcorner\, u(y)\,dt\,dy \\ &= \int\!\int_1^\infty (t-1)^{\ell-1} t^{-\ell-1}\theta(a)\,(x-a) \lrcorner\, u\big(x+(a-x)/t\big)\,dt\,da \\ &= \int \theta(a)\,(x-a) \lrcorner \int_0^1 t^{\ell-1}\, u\big(a+t(x-a)\big)\,dt\,da\,. \end{aligned} \tag{3.11}$$

From this form of R_ℓ, one sees immediately that it maps differential forms with polynomial coefficients to differential forms with polynomial coefficients and $\mathscr{C}^\infty(\mathbb{R}^n, \Lambda^\ell)$ to $\mathscr{C}^\infty(\mathbb{R}^n, \Lambda^{\ell-1})$, and that $R_\ell u(x)$ depends only on the values of u in the convex hull of $B \cup \{x\}$, that is, the starlike hull of $\{x\}$ with respect to the ball B. This implies in particular that if Ω is open and starlike with respect to B, then R_ℓ maps $\mathscr{C}^\infty(\Omega, \Lambda^\ell)$ to $\mathscr{C}^\infty(\Omega, \Lambda^{\ell-1})$ and also $\mathscr{C}^\infty(\overline{\Omega}, \Lambda^\ell)$ to $\mathscr{C}^\infty(\overline{\Omega}, \Lambda^{\ell-1})$.

Rewriting T_ℓ in the same way, we get

$$T_\ell u(x) = -\int \theta(a)\,(x-a) \lrcorner \int_1^\infty t^{\ell-1}\, u\big(a+t(x-a)\big)\,dt\,da\,. \tag{3.12}$$

From this form of T_ℓ, because of the unbounded interval of integration in t, one cannot immediately conclude that T_ℓ maps $\mathscr{C}^\infty$ functions to $\mathscr{C}^\infty$ functions. But if $u \in \mathscr{C}_0^\infty(\mathbb{R}^n, \Lambda^\ell)$, one sees that $T_\ell u$ is $\mathscr{C}^\infty$ on $\mathbb{R}^n \setminus \operatorname{supp}\theta$, and that $T_\ell u(x) = 0$ unless x lies in the starlike hull of $\operatorname{supp} u$ with respect to B. Thus if Ω is open and starlike with respect to B, then $u \in \mathscr{C}_0^\infty(\Omega, \Lambda^\ell)$ implies $\operatorname{supp} T_\ell u \subset \Omega$, and, if Ω is bounded, then $u \in \mathscr{C}_{\overline{\Omega}}^\infty(\mathbb{R}^n, \Lambda^\ell)$ implies $\operatorname{supp} T_\ell u \subset \overline{\Omega}$. The fact that T_ℓ indeed maps $\mathscr{C}_0^\infty(\mathbb{R}^n, \Lambda^\ell)$ to $\mathscr{C}_0^\infty(\mathbb{R}^n, \Lambda^{\ell-1})$ will be a consequence of Theorem 3.2 below.

3.2 Homotopy relations

Cartan's formula for the Lie derivative of a differential form with respect to a vector field can be written as

$$\frac{d}{dt}F_t^* u = F_t^*\big(d(X_t \lrcorner u) + X_t \lrcorner du\big) \ ,$$

where F_t^* denotes the pull-back by the flow F_t associated with the vector field X_t. Here we consider the special case of the dilation flow with center a

$$F_t(x) = a + t(x-a) \quad \text{with vector field } X_t = x - a \ ,$$

which gives a pull-back of

$$F_t^* u(x) = t^\ell \, u\big(a + t(x-a)\big) \quad \text{for an } \ell\text{-form } u \ .$$

This leads to the formula

$$\frac{d}{dt}(t^\ell u\big(a+t(x-a)\big) = d\Big(t^{\ell-1}(x-a) \lrcorner u\big(a+t(x-a)\big)\Big) + t^\ell (x-a) \lrcorner du\big(a+t(x-a)\big) \quad (3.13)$$

which can also be verified elementarily from the formulas we gave in Section 2.

Integrating (3.13) from 0 to 1 and comparing with (3.11), we find the homotopy relations, valid for all $u \in \mathscr{C}_0^\infty(\mathbb{R}^n, \Lambda^\ell)$

$$\begin{aligned} dR_\ell u + R_{\ell+1} du &= u && (1 \le \ell \le n-1) \ ; \\ R_1 du &= u - \big(\theta, u\big) && (\ell = 0) \ ; \\ dR_n u &= u && (\ell = n) \ . \end{aligned} \quad (3.14)$$

One could be tempted to integrate Cartan's formula from 1 to ∞ and compare with (3.12), thus formally obtaining a similar homotopy relation for T_ℓ directly. The result is indeed true except for $\ell = n$, but we prefer a different proof: Use the duality relations (3.8) and (3.9) to deduce corresponding anticommutation relations for the other three families of integral operators from the relations (3.14) which are already proved. Here is what one obtains for $u \in \mathscr{C}_0^\infty(\mathbb{R}^n, \Lambda^\ell)$:

$$\begin{aligned} \delta Q_\ell u + Q_{\ell-1} \delta u &= u && (1 \le \ell \le n-1) \ ; \\ \delta Q_0 u &= u && (\ell = 0) \ ; \\ Q_{n-1} \delta u &= u - \star\big(\theta, \star u\big) && (\ell = n) \ . \end{aligned} \quad (3.15)$$

$$\begin{aligned} \delta S_\ell u + S_{\ell-1} \delta u &= u && (1 \le \ell \le n-1) \ ; \\ \delta S_0 u &= u - (\textstyle\int u)\, \theta && (\ell = 0) \ ; \\ S_{n-1} \delta u &= u && (\ell = n) \ . \end{aligned} \quad (3.16)$$

$$\begin{aligned} dT_\ell u + T_{\ell+1} du &= u && (1 \le \ell \le n-1) \ ; \\ T_1 du &= u && (\ell = 0) \ ; \\ dT_n u &= u - (\textstyle\int u) \star \theta && (\ell = n) \ . \end{aligned} \quad (3.17)$$

Here we consider θ as an element of $\mathscr{C}_0^\infty(\mathbb{R}^n, \Lambda^0)$, so that for another 0-form u we have the L^2 scalar product $\big(\theta, u\big) = \int \theta(a) u(a) da$, and $\star\theta$ is the n-form $\theta(x) dx_1 \wedge \ldots \wedge dx_n$. For an n-form u, the expression $\star\big(\theta, \star u\big)$ denotes the n-form $(\int \theta u)\, dx_1 \wedge \ldots \wedge dx_n$.

For the operators R_ℓ and T_ℓ, the formulas for the endpoints $\ell = 0$ and $\ell = n$ correspond to the two extended de Rham complexes without boundary conditions and with compact support, see (2.3) and (2.4). To see this, let us extend the definition of the exterior derivative by writing $\overline{d}$ for all the mappings of the complex

$$0 \to \mathbb{R} \xrightarrow{\iota} \mathscr{C}^\infty(\overline{\Omega}, \Lambda^0) \xrightarrow{d} \mathscr{C}^\infty(\overline{\Omega}, \Lambda^1) \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{C}^\infty(\overline{\Omega}, \Lambda^n) \to 0$$

and $\underline{d}$ for all the mappings of the complex

$$0 \to \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^0) \xrightarrow{d} \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^1) \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^n) \xrightarrow{\iota^*} \mathbb{R} \to 0$$

where ι is the inclusion mapping for constant functions and $\iota^* = (\star\iota)'$ denotes the integral $u \mapsto \int u$ for n-forms.

If we correspondingly extend the definitions of R_ℓ and T_ℓ by

$$\begin{aligned} R_0 u &:= \big(\theta, u\big) \text{ for 0-forms } u\,, & R_{n+1} &:= 0\,,\\ T_{n+1} u &:= \star(u\theta) \text{ for } u \in \mathbb{R}\,, & T_0 &:= 0\,, \end{aligned}$$

then we can write the relations (3.14) and (3.17) simply as

$$\overline{d}\, R_\ell u + R_{\ell+1}\, \overline{d} u = u \quad \text{ and } \quad \underline{d}\, T_\ell u + T_{\ell+1}\, \underline{d} u = u \quad \text{ for all } 0 \le \ell \le n. \tag{3.18}$$

3.3 Continuity

The most important result about analytic properties of our integral operators is the following.

Theorem 3.2 *The operators Q_ℓ, R_ℓ, S_ℓ, T_ℓ defined in Definition 3.1 are pseudodifferential operators on $\mathbb{R}^n$ of order -1 with symbols in the Hörmander symbol class $S^{-1}_{1,0}(\mathbb{R}^n)$.*

Proof: For basic facts about pseudodifferential operators, see for example [14, 16, 18]. We are using here the local symbol class $S^{-1}_{1,0}(\mathbb{R}^n)$ that consists of functions $a \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisifying for any compact set $M \subset \mathbb{R}^n$ and any multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, estimates of the form

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}(M)\,(1+|\xi|)^{-1-|\beta|} \quad \forall (x,\xi) \in M \times \mathbb{R}^n\,. \tag{3.19}$$

The proof will show that the constants $C_{\alpha\beta}$ are polynomially bounded in $x \in \mathbb{R}^n$, but this is not important here, since we are only interested in the local behavior.

We give the proof for the operator T_ℓ. For the other three operators the result then follows by either applying the Hodge star operator which is a purely algebraic operation on basis vectors in the exterior algebra and does not change coefficients of differential forms, or by taking L^2 adjoints which according to the calculus of pseudodifferential operators does not lead out of this class.

Thus we consider the integral operator defined by

$$T_\ell u(x) = \int G_\ell(x,y)\,(x-y) \lrcorner\, u(y)\, dy$$

with the kernel G_ℓ given in (3.1). Writing the differential forms in components, we see that for $j, \ell \in \{1, \dots, n\}$ we need to study the following operator K acting on scalar functions u:

$$Ku(x) = \int_{\mathbb{R}^n} k(x, x-y)\, u(y)\, dy$$
$$\text{with } k(x,z) = z_j \int_0^\infty s^{n-\ell}(s+1)^{\ell-1}\theta(x+sz)\, ds \ \text{ for } x, z \in \mathbb{R}^n\,. \quad (3.20)$$

We write $k(x,z) = k_0(x,z) + k_1(x,z)$ with

$$k_0(x,z) = z_j \int_0^1 s^{n-\ell}(s+1)^{\ell-1}\theta(x+sz)\, ds\,,$$
$$k_1(x,z) = z_j \int_1^\infty s^{n-\ell}(s+1)^{\ell-1}\theta(x+sz)\, ds\,.$$

It is clear that $k_0 \in \mathscr{C}^\infty(\mathbb{R}^{2n})$, and therefore only k_1 needs to be analyzed. If $\operatorname{supp}\theta \subset B_\epsilon(0)$, then

$$k_1(x,z) = 0 \quad \text{for } |z| \geq |x| + \epsilon\,,$$

and we have already seen in (3.7) that $z \mapsto k_1(x,z)$ is weakly singular. It is therefore integrable over $\mathbb{R}^n$, so we can write its Fourier transform as the convergent integral

$$\begin{aligned}\hat{k}_1(x,\xi) &= \int_{\mathbb{R}^n} e^{-i\langle\xi,z\rangle} k_1(x,z)\, dz \\ &= \int_1^\infty s^{n-\ell}(s+1)^{\ell-1} \int e^{-i\langle\xi,z\rangle} z_j \theta(x+sz)\, dz\, ds\,,\end{aligned}$$

and we can represent the operator K as

$$Ku(x) \;=\; \int_{\mathbb{R}^n} k_0(x, x-y)\, u(y)\, dy + (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle\xi,x\rangle} \hat{k}_1(x,\xi)\, \hat{u}(\xi)\, d\xi\,.$$

The proof will be complete once we show that the symbol $\hat{k}_1$ of the operator K satisfies the estimates (3.19), namely for any multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ and $x, \xi \in \mathbb{R}^n$:

$$|\partial_x^\alpha \partial_\xi^\beta \hat{k}_1(x,\xi)| \leq C_{\alpha\beta}(x)\, (1+|\xi|)^{-1-|\beta|} \quad (3.21)$$

where $C_{\alpha\beta}(x)$ is bounded for x in any compact set.
With the change of variables $(t,y) = (1/s, x+sz)$ we can write

$$\begin{aligned}\hat{k}_1(x,\xi) &= \int_0^1 (t+1)^{\ell-1} e^{it\langle\xi,x\rangle} \int e^{-it\langle\xi,y\rangle}\, (y_j - x_j)\theta(y)\, dy\, dt \\ &= \int_0^1 (t+1)^{\ell-1} e^{it\langle\xi,x\rangle} \Big(i(\partial_j\hat{\theta})(t\xi) - x_j \hat{\theta}(t\xi)\Big)\, dt\,. \quad (3.22)\end{aligned}$$

Here $\hat{\theta}$ is the Fourier transform of $\theta \in \mathscr{C}_0^\infty(\mathbb{R}^n)$, thus a rapidly decreasing $\mathscr{C}^\infty$ function. The representation (3.22) shows that

$$\hat{k}_1 \in \mathscr{C}^\infty(\mathbb{R}^{2n}) \quad \text{and} \ |\hat{k}_1(x,\xi)| \leq C_\theta(1+|x|) \quad (3.23)$$

where C_θ depends only on θ. Writing $\tau = t|\xi|$ and $\omega = \xi/|\xi|$, we find

$$\hat{k}_1(x,\xi) \;=\; |\xi|^{-1}\int_0^{|\xi|}\big(1+\frac{\tau}{|\xi|}\big)^{\ell-1}e^{i\tau\langle\omega,x\rangle}\Big(i(\partial_j\hat\theta)(\tau\omega)-x_j\hat\theta(\tau\omega)\Big)\,d\tau \tag{3.24}$$

and hence

$$|\hat{k}_1(x,\xi)| \le |\xi|^{-1}2^{\ell-1}\int_0^{\infty}\big(|\partial_j\hat\theta)(\tau\omega)|+|x_j\hat\theta(\tau\omega)|\big)\,d\tau \le (1+|x|)\,C_\theta\,|\xi|^{-1}\;.$$

Thus we have shown (3.21) for $|\alpha|=|\beta|=0$.
Similarly, by taking derivatives in (3.22), we can write for any multi-indices α,β:

$$\partial_x^\alpha\partial_\xi^\beta\hat{k}_1(x,\xi) \;=\; \int_0^1 (t+1)^{\ell-1}e^{it\langle\xi,x\rangle}t^{|\beta|}\Big(p_{\alpha\beta}(x,t\xi,\partial)\hat\theta\Big)(t\xi)\,dt\;, \tag{3.25}$$

where $p_{\alpha\beta}(x,\xi,\partial)$ is a partial differential operator of order $|\beta|+1$ with polynomial coefficients of degree $\le|\beta|+1$ in x and $\le|\alpha|$ in ξ. We obtain an immediate estimate

$$|\partial_x^\alpha\partial_\xi^\beta\hat{k}_1(x,\xi)| \le C_{\alpha\beta}(x)\,(1+|\xi|)^{|\alpha|}\;, \tag{3.26}$$

and after the change of variables $\tau = t|\xi|$ with $\omega=\xi/|\xi|$:

$$\partial_x^\alpha\partial_\xi^\beta\hat{k}_1(x,\xi) \;=\; \int_0^{|\xi|}\big(1+\frac{\tau}{|\xi|}\big)^{\ell-1}e^{i\tau\langle\omega,x\rangle}\tau^{|\beta|}\Big(p_{\alpha\beta}(x,\tau\omega,\partial)\hat\theta\Big)(\tau\omega)\,d\tau\,|\xi|^{-1-|\beta|}\;. \tag{3.27}$$

This gives a second estimate

$$|\partial_x^\alpha\partial_\xi^\beta\hat{k}_1(x,\xi)| \le C_{\alpha\beta}(x)\,|\xi|^{-1-|\beta|}\;. \tag{3.28}$$

In (3.26) and (3.28), $C_{\alpha\beta}(x)$ is bounded for x in any compact set. (One can see that $C_{\alpha\beta}(x)\le C_{\alpha\beta,\theta}\cdot(1+|x|)^{1+|\beta|}$ where $C_{\alpha\beta,\theta}$ depends only on α, β and θ.)
This shows (3.21) and completes the proof. □

An immediate consequence of the theorem is that the four integral operators map differential forms with $\mathscr{C}_0^\infty$ coefficients to differential forms with $\mathscr{C}^\infty$ coefficients. Taking into account the support properties deduced above from the representations (3.11) and (3.12), we get the following statements, where we use the standard topologies for the function spaces.

Corollary 3.3 *The integral operators defined in Definition 3.1 define continuous mappings*

$$Q_\ell : \mathscr{C}^\infty(\mathbb{R}^n,\Lambda^\ell)\to\mathscr{C}^\infty(\mathbb{R}^n,\Lambda^{\ell+1})\,,\qquad R_\ell : \mathscr{C}^\infty(\mathbb{R}^n,\Lambda^\ell)\to\mathscr{C}^\infty(\mathbb{R}^n,\Lambda^{\ell-1})\,,$$
$$S_\ell : \mathscr{C}_0^\infty(\mathbb{R}^n,\Lambda^\ell)\to\mathscr{C}_0^\infty(\mathbb{R}^n,\Lambda^{\ell+1})\,,\qquad T_\ell : \mathscr{C}_0^\infty(\mathbb{R}^n,\Lambda^\ell)\to\mathscr{C}_0^\infty(\mathbb{R}^n,\Lambda^{\ell-1})\,.$$

If $\Omega\subset\mathbb{R}^n$ is a bounded domain starlike with respect to a ball B containing $\operatorname{supp}\theta$*, then the operators define continuous mappings*

$$Q_\ell : \mathscr{C}^\infty(\Omega,\Lambda^\ell)\to\mathscr{C}^\infty(\Omega,\Lambda^{\ell+1})\,,\qquad R_\ell : \mathscr{C}^\infty(\Omega,\Lambda^\ell)\to\mathscr{C}^\infty(\Omega,\Lambda^{\ell-1})\,,$$
$$Q_\ell : \mathscr{C}^\infty(\overline\Omega,\Lambda^\ell)\to\mathscr{C}^\infty(\overline\Omega,\Lambda^{\ell+1})\,,\qquad R_\ell : \mathscr{C}^\infty(\overline\Omega,\Lambda^\ell)\to\mathscr{C}^\infty(\overline\Omega,\Lambda^{\ell-1})\,,$$
$$S_\ell : \mathscr{C}_0^\infty(\Omega,\Lambda^\ell)\to\mathscr{C}_0^\infty(\Omega,\Lambda^{\ell+1})\,,\qquad T_\ell : \mathscr{C}_0^\infty(\Omega,\Lambda^\ell)\to\mathscr{C}_0^\infty(\Omega,\Lambda^{\ell-1})\,,$$
$$S_\ell : \mathscr{C}_{\overline\Omega}^\infty(\mathbb{R}^n,\Lambda^\ell)\to\mathscr{C}_{\overline\Omega}^\infty(\mathbb{R}^n,\Lambda^{\ell+1})\,,\qquad T_\ell : \mathscr{C}_{\overline\Omega}^\infty(\mathbb{R}^n,\Lambda^\ell)\to\mathscr{C}_{\overline\Omega}^\infty(\mathbb{R}^n,\Lambda^{\ell-1})\,.$$

Either by duality or by extension using standard continuity properties of pseudodifferential operators, the four operators can be defined on differential forms with distributional coefficients, in the case of the Poincaré-type operators Q_ℓ and R_ℓ for arbitrary distributions from $\mathscr{D}'(\mathbb{R}^n, \Lambda^\ell)$ and in the case of the Bogovskiĭ-type operators S_ℓ and T_ℓ for distributions with compact support in $\mathbb{R}^n$.

For finite regularity, the standard continuity properties of pseudodifferential operators together with the support properties immediately imply results of the following type.

Corollary 3.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain starlike with respect to a ball B containing* $\operatorname{supp}\theta$*. Then the four integral operators define bounded operators for any $s \in \mathbb{R}$:*

$$Q_\ell : H^s(\Omega, \Lambda^\ell) \to H^{s+1}(\Omega, \Lambda^{\ell+1}), \qquad R_\ell : H^s(\Omega, \Lambda^\ell) \to H^{s+1}(\Omega, \Lambda^{\ell-1}),$$
$$S_\ell : H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell) \to H^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell+1}), \qquad T_\ell : H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell) \to H^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1}).$$

Remark 3.5 Corollary 3.4 remains valid when H^s is replaced by B^s_{pq} $(0 < p \le \infty, 0 < q \le \infty)$, or by F^s_{pq} $(0 < p < \infty, 0 < q \le \infty)$. The spaces $B^s_{pq}(\Omega, \Lambda^\ell)$ and $F^s_{pq}(\Omega, \Lambda^\ell)$ are defined as quotient spaces, and the spaces $B^s_{pq\,\overline{\Omega}}(R^n, \Lambda^\ell)$ and $F^s_{pq\,\overline{\Omega}}(R^n, \Lambda^\ell)$ are defined as subspaces, in an analogous way to the spaces $H^s(\Omega, \Lambda^\ell)$ and $H^s_{\overline{\Omega}}(R^n, \Lambda^\ell)$. They include the special cases of Sobolev spaces $W^{s,p} = F^s_{p,2}$, and local Hardy spaces $h^1_r(\Omega, \Lambda^\ell) = F^0_{1,2}(\Omega, \Lambda^\ell)$ and $h^1_z(\Omega, \Lambda^l) = F^0_{1,2\,\overline{\Omega}}(R^n, \Lambda^\ell)$. See Chapter 6 of [16]. △

In all these cases, the commutation relations (3.14)–(3.18) remain valid. What this implies for the regularity of the de Rham complex and its cohomology is the subject of the next section.

4 Regularity of the de Rham complex

4.1 Starlike domains

The homotopy relations (3.18) together with the mapping properties from Corollary 3.4 imply the existence of regular solutions of the equation $du = 0$, as we now state. There are similar results in the $\mathscr{C}^\infty$ spaces which follow from Corollary 3.3.

Proposition 4.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, starlike with respect to a ball B.*
(i) *For any $s \in \mathbb{R}$ and $\ell \in \{1, \dots, n\}$, let $u \in H^s(\Omega, \Lambda^\ell)$ satisfy $du = 0$ in Ω. Then there exists $v \in H^{s+1}(\Omega, \Lambda^{\ell-1})$ such that $dv = u$, and there is a constant C independent of u such that*

$$\|v\|_{H^{s+1}(\Omega)} \le C\,\|u\|_{H^s(\Omega)}\,.$$

For $\ell = n$ the condition $du = 0$ is always satisfied.
(ii) *For any $s \in \mathbb{R}$ and $\ell \in \{1, \dots, n\}$, let $u \in H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)$ satisfy $du = 0$ in $\mathbb{R}^n$, and $\int u = 0$ if $\ell = n$. Then there exists $v \in H^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1})$ such that $dv = u$, and there is a constant C independent of u such that*

$$\|v\|_{H^{s+1}(\mathbb{R}^n)} \le C\,\|u\|_{H^s(\mathbb{R}^n)}\,.$$

Proof: With $du = 0$ ($\underline{d}u = 0$ in case (ii)), the relations (3.18) reduce to

$$u = d\,R_\ell u \quad \text{and} \quad u = d\,T_\ell u\,.$$

Therefore in case (i) we take $v = R_\ell u$ and in case (ii) $v = T_\ell u$. The estimates are a consequence of the boundedness of the operators R_ℓ and T_ℓ as given in Corollary 3.4. □

In the case $s = 0$, there is a natural isomorphism (extension by zero outside Ω) between the spaces $L^2(\Omega, \Lambda^\ell)$ and $L^2_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)$. Thus for a differential form $u \in L^2(\Omega, \Lambda^\ell)$, both (i) and (ii) of the Proposition can be applied, giving a solution v of $dv = u$ with coefficients in $H^1(\Omega)$ for case (i) and – apparently stronger – in $H^1_0(\Omega)$ for case (ii). It is important to notice, however, that the condition $du = 0$ does not mean the same thing in both cases:

In case (i), it simply means $du = 0$ in the sense of distributions in the open set Ω. In case (ii), the condition is $du = 0$ in the sense of distributions on $\mathbb{R}^n$, and this is stronger: It includes not only $du = 0$ inside Ω, but also a boundary condition $\nu \wedge u = 0$ on $\partial\Omega$ in a weak sense.

4.2 Differential forms with polynomial coefficients

As we have seen, the Poincaré-type operator R_ℓ (and its Hodge star dual twin Q_ℓ) preserves the class of differential forms with polynomial coefficients. This class has recently attracted some attention in the field of finite element methods. For quite a while already in relation with numerical methods for electromagnetism [9], but more recently also in other applications including elasticity theory [1], finite dimensional subcomplexes of the de Rham complex generated by polynomials have been studied.

For the following, we assume we have a piece of such a complex, namely for some $\ell \in \{1, \dots, n\}$ two spaces $P(\Lambda^{\ell-1})$ and $P(\Lambda^\ell)$ of differential forms of order $\ell - 1$ and ℓ with coefficients which are polynomials in $x_1, \dots, x_n$, which we require to satisfy the following two conditions:

1. The space $P(\Lambda^\ell)$ is invariant with respect to dilations and translations, that is

$$\text{For any } t \in \mathbb{R}, a \in \mathbb{R}^n: \text{ If } u \in P(\Lambda^\ell), \text{ then } \big(x \mapsto u(tx + a)\big) \in P(\Lambda^\ell)\ .$$

2. The interior product ("Koszul" multiplication) $x \lrcorner : u \mapsto x \lrcorner u$ maps $P(\Lambda^\ell)$ to $P(\Lambda^{\ell-1})$.

Then, as in Section 3, we fix a function $\theta \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with support in a ball B satisfying $\int \theta(x)\, dx = 1$, and we define the Poincaré-type operator R_ℓ as in Definition 3.1.

Proposition 4.2 *The operator R_ℓ maps $P(\Lambda^\ell)$ into $P(\Lambda^{\ell-1})$, and for any bounded domain $\Omega \subset \mathbb{R}^n$ that is starlike with respect to the ball B and for any $s \in \mathbb{R}$ there is a constant C such that for all $u \in P(\Lambda^\ell)$*

$$\|R_\ell u\|_{H^{s+1}(\Omega)} \le C\, \|u\|_{H^s(\Omega)}\ .$$

In addition, we have for all $u \in P(\Lambda^\ell)$

$$u = d\, R_\ell u + R_{\ell+1}\, du\ .$$

Proof: That R_ℓ maps $P(\Lambda^\ell)$ into $P(\Lambda^{\ell-1})$ is a consequence of the representation (3.11) and conditions 1. and 2. The estimate follows from the continuity stated in Corollary 3.4. □

In [1], complexes of polynomial differential forms are studied that satisfy conditions 1. and 2. above, and in fact a more restrictive condition than 1., namely invariance with respect to all affine transformations. The latter condition is suitable for finite elements on simplicial meshes, but our more general condition 1. covers also some cases of polynomials used in finite elements on tensor product meshes. A well-known example in 3 dimensions is the complex studied for example in

[4], which uses spaces Q^{p_1,p_2,p_3} of polynomials of partial degree p_j in the variable x_j, $j = 1, 2, 3$. The complex is then for a given $p \in \mathbb{N}$

$$P(\Lambda^0) \xrightarrow{d} P(\Lambda^1) \xrightarrow{d} P(\Lambda^2) \xrightarrow{d} P(\Lambda^3)$$

with

$$\begin{aligned}
&P(\Lambda^0) = Q^{p,p,p}(\Lambda^0)\ ,\\
&P(\Lambda^1) = \big\{u_1 dx_1 + u_2 dx_2 + u_3 dx_3 \mid u_1 \in Q^{p-1,p,p},\ u_2 \in Q^{p,p-1,p},\ u_3 \in Q^{p,p,p-1}\big\}\ ,\\
&P(\Lambda^2) = \big\{u_1 dx_2 \wedge dx_3 + u_2 dx_3 \wedge dx_1 + u_3 dx_1 \wedge dx_2 \mid\\
&\qquad\qquad u_1 \in Q^{p,p-1,p-1},\ u_2 \in Q^{p-1,p,p-1},\ u_3 \in Q^{p-1,p-1,p}\big\}\ ,\\
&P(\Lambda^3) = Q^{p-1,p-1,p-1}(\Lambda^3)\ .
\end{aligned}$$

It is clear that these spaces form a subcomplex of the de Rham complex, and that they satisfy conditions 1. and 2. above.

4.3 Bounded Lipschitz domains

In this subsection we draw some conclusions from Theorem 3.2 that are valid for bounded Lipschitz domains. The main property of a bounded Lipschitz domain Ω that is relevant here is the existence of a finite covering of $\overline{\Omega}$ by open sets U_i, $i = 1, \dots, m$ such that each $U_i \cap \Omega$ is starlike with respect to a ball B_i, and a subordinate partition of unity $(\chi_i)_{i=1,\dots,m}$. This means that $\chi_i \in \mathscr{C}_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\chi_i \subset U_i$, and $\sum_{i=1}^m \chi_i(x) = 1$ for all x in a neighborhood of $\overline{\Omega}$.

For each $i = 1, \dots, m$ we can choose a smoothing function θ_i supported in B_i and satisfying $\int \theta_i(x)dx = 1$ and define the integral operators $R_{\ell,i}$ and $T_{\ell,i}$ accordingly. By Theorem 3.2, these are all pseudodifferential operators of order -1 on $\mathbb{R}^n$. They all satisfy the homotopy relations (3.18), but they do not have good support properties with respect to Ω, only with respect to their respective $U_i \cap \Omega$. We then define operators R_ℓ and T_ℓ according to

$$R_\ell u = \sum_{i=1}^m \chi_i R_{\ell,i} u \quad \text{and } T_\ell u = \sum_{i=1}^m T_{\ell,i}(\chi_i u) \quad \text{for } u \in \mathscr{C}_0^\infty(\mathbb{R}^n, \Lambda^\ell),\ 1 \le \ell \le n\,. \tag{4.1}$$

These operators are still pseudodifferential operators of order -1 on $\mathbb{R}^n$, but they have better support properties with respect to Ω:

If $u \in \mathscr{C}_0^\infty(\mathbb{R}^n, \Lambda^\ell)$ vanishes in Ω, then it vanishes in $U_i \cap \Omega$, and since $U_i \cap \Omega$ is starlike with respect to B_i, $R_{\ell,i}u$ vanishes in $U_i \cap \Omega$ and therefore $\chi_i R_{\ell,i}u$ vanishes in all of Ω. Hence $R_\ell u$ vanishes in Ω. In other words, the restriction of $R_\ell u$ to Ω depends only on the restriction of u to Ω.

For T_ℓ the argument is similar: If $\operatorname{supp} u \subset \overline{\Omega}$, then $\operatorname{supp}\chi_i u \subset \overline{U_i \cap \Omega}$, and therefore $\operatorname{supp} T_{\ell,i}(\chi_i u) \subset \overline{U_i \cap \Omega} \subset \overline{\Omega}$. Hence $\operatorname{supp} T_\ell u \subset \overline{\Omega}$.

As a result, we immediately get the same mapping properties as in Corollaries 3.3 and 3.4.

Lemma 4.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let the operators R_ℓ and T_ℓ for $1 \le \ell \le n$ be defined from a finite starlike open cover of $\overline{\Omega}$ as in* (4.1)*. Then R_ℓ defines continuous mappings from $\mathscr{C}^\infty(\Omega, \Lambda^\ell)$ to $\mathscr{C}^\infty(\Omega, \Lambda^{\ell-1})$, from $\mathscr{C}^\infty(\overline{\Omega}, \Lambda^\ell)$ to $\mathscr{C}^\infty(\overline{\Omega}, \Lambda^{\ell-1})$, and for any $s \in \mathbb{R}$ from $H^s(\Omega, \Lambda^\ell)$ to $H^{s+1}(\Omega, \Lambda^{\ell-1})$. The operator T_ℓ defines continuous mappings from $\mathscr{C}_0^\infty(\Omega, \Lambda^\ell)$ to $\mathscr{C}_0^\infty(\Omega, \Lambda^{\ell-1})$, from $\mathscr{C}_{\overline{\Omega}}^\infty(\mathbb{R}^n, \Lambda^\ell)$ to $\mathscr{C}_{\overline{\Omega}}^\infty(\mathbb{R}^n, \Lambda^{\ell-1})$, and for any $s \in \mathbb{R}$ from $H_{\overline{\Omega}}^s(\mathbb{R}^n, \Lambda^\ell)$ to $H_{\overline{\Omega}}^{s+1}(\mathbb{R}^n, \Lambda^{\ell-1})$.*

On the other hand, the simple anticommutation relations (3.18) are, of course, no longer valid for these composite operators R_ℓ and T_ℓ. Instead we have for $1 \le \ell \le n-1$

$$\begin{aligned}\big(dR_\ell + R_{\ell+1}d\big)u &= d\sum_{i=1}^m \chi_i R_{\ell,i}u + \sum_{i=1}^m \chi_i R_{\ell+1,i}du \\ &= \sum_{i=1}^m \chi_i\big(dR_{\ell,i} + R_{\ell+1,i}d\big)u + \sum_{i=1}^m [d,\chi_i]R_{\ell,i}u \\ &= \sum_{i=1}^m \chi_i u - K_\ell u \quad \text{ with } K_\ell u = -\sum_{i=1}^m [d,\chi_i]R_{\ell,i}u\ .\end{aligned}$$

On a neighborhood of $\overline{\Omega}$, this reduces to $\big(dR_\ell + R_{\ell+1}d\big)u = u - K_\ell u$.

From the product rule $d(\chi_i u) = (d\chi_i)\wedge u + \chi_i du$ we obtain the commutator $[d,\chi_i]u = (d\chi_i)\wedge u$, and hence the expression for K_ℓ:

$$K_\ell u = -\sum_{i=1}^m (d\chi_i)\wedge R_{\ell,i}u, \quad 1 \le \ell \le n\ . \tag{4.2}$$

This shows immediately that K_ℓ is a pseudodifferential operator of order -1 on $\mathbb{R}^n$, and that it has the same support properties as the operator R_ℓ.

To complete the family for the endpoints $\ell = 0$ and $\ell = n$, we notice that for a 0-form u

$$R_1 du = \sum_{i=1}^m \chi_i R_{1,i}du = \sum_{i=1}^m \chi_i\big(u - (\theta_i, u)\big)$$

and for an n-form u

$$dR_n u = d\sum_{i=1}^m \chi_i R_{n,i}u = \sum_{i=1}^m \chi_i dR_{n,i}u + \sum_{i=1}^m [d,\chi_i]R_{n,i}u = \sum_{i=1}^m \chi_i u + \sum_{i=1}^m d\chi_i \wedge R_{n,i}u$$

Therefore if we set $H^s(\Omega,\Lambda^{-1}) = H^s(\Omega,\Lambda^{n+1}) = \{0\}$, $R_0 u = 0$, $K_0 = \sum_{i=1}^m (\theta_i, u)\chi_i$, $R_{n+1} = 0$, we obtain the homotopy relation for the de Rham complex without boundary conditions (2.1)

$$d\,R_\ell u + R_{\ell+1}\,du = u - K_\ell u \quad \text{ for all } 0 \le \ell \le n\,. \tag{4.3}$$

Note that this relation is now valid only in a neighborhood of $\overline{\Omega}$, not in all of $\mathbb{R}^n$. As a consequence of (4.3) we get

$$dK_\ell u = du - dR_{\ell+1}du = K_{\ell+1}du \quad \text{ for all } 0 \le \ell \le n\ .$$

For the operator T_ℓ we obtain similarly, when $1 \le \ell \le n-1$,

$$\big(dT_\ell + T_{\ell+1}d\big)u = \Big(\sum_{i=1}^m \chi_i\Big)u - L_\ell u \quad \text{ with } L_\ell u = \sum_{i=1}^m T_{\ell+1,i}[d,\chi_i]u\ .$$

On a neighborhood of $\overline{\Omega}$, this reduces to $\big(dT_\ell + T_{\ell+1}d\big)u = u - L_\ell u$ with the pseudodifferential operator L_ℓ of order -1 given by

$$L_\ell u = \sum_{i=1}^m T_{\ell+1,i}((d\chi_i)\wedge u), \quad 0 \le \ell \le n-1\ . \tag{4.4}$$

We complete this with $H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{-1}) = H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{n+1}) = \{0\}$, $T_0 = 0$, $T_{n+1} = 0$, and $L_n u = \sum(\int \chi_i u) \star \theta_i$ and obtain the homotopy relation for the de Rham complex with compact support (2.2)

$$d\,T_\ell u + T_{\ell+1}\,du = u - L_\ell u \quad \text{for all } \ 0 \le \ell \le n. \tag{4.5}$$

This relation is valid in a neighborhood of $\overline{\Omega}$, but now if we apply it to a u with support in $\overline{\Omega}$, it will be valid in all of $\mathbb{R}^n$. Again as before we obtain

$$dL_\ell u = L_{\ell+1} du \quad \text{for all } 0 \le \ell \le n\ .$$

Remark 4.4 In this subsection on Lipschitz domains, we are using the extended de Rham complexes (2.1) and (2.2), rather than the sequences (2.3) and (2.4) as we did for starlike domains. For this reason, we now have $R_0 = 0$, $T_0 = 0$, $R_{n+1} = 0$ and $T_{n+1} = 0$. $\triangle$

Before drawing conclusions, we prove a stronger version of the relations (4.3) and (4.5), where the perturbations of the identity K_ℓ and L_ℓ are not just of order -1, but in fact infinitely smoothing in a neighborhood of $\overline{\Omega}$.

Let $x_0 \in \mathbb{R}^n$. We shall say that the family of functions $(\chi_i)_{i=1,\dots,m}$ is *flat at* x_0 if each χ_i is constant in a neighborhood of x_0. We will also call an open covering $(U_i)_{i=1,\dots,m}$ of $\overline{\Omega}$ by a slight abuse of language *starlike* if each $U_i \cap \Omega$ is starlike with respect to some open ball B_i.

Lemma 4.5 *Let Ω be a bounded Lipschitz domain. Then there exists a finite number of starlike finite open coverings $(U_i^{(j)})_{i=1,\dots,m^{(j)}}$, $j = 1, \dots, k$, of $\overline{\Omega}$ and subordinate partitions of unity, such that for any $x_0 \in \mathbb{R}^n$ at least one of the partitions of unity is flat at x_0.*

Proof: In a first step we show that for a given $x_0 \in \mathbb{R}^n$ there exists a starlike finite open covering $(U_i)_{i=0,\dots,m}$ of $\overline{\Omega}$ and a partition of unity subordinate to this covering which is flat at x_0.
Let first $x_0 \in \overline{\Omega}$. Let U_0 be a neighborhood of x_0 such that $U_0 \cap \Omega$ is starlike with respect to a ball, V_0 another neighborhood of x_0 such that $\overline{V}_0 \subset U_0$ and $\chi_0 \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\chi_0 \subset U_0$ and $\chi_0 \equiv 1$ on a neighborhood of $\overline{V}_0$. We may assume that $\Omega \setminus \overline{V}_0$ is still Lipschitz. Choose a finite open covering $(U_i)_{i=1,\dots,m}$ of $\overline{\Omega \setminus V_0}$ such that each $U_i \cap \Omega$ is starlike with respect to a ball. Let $\{\tilde\chi_i \mid i = 1, \cdots, m\}$ be a subordinate partition of unity which therefore satisfies

$$\sum_{i=1}^m \tilde\chi_i(x) = 1 \quad \text{for all } x \text{ in a neighborhood of } \overline{\Omega \setminus V_0}\,.$$

Then defining for $i = 1, \cdots, m$:

$$\chi_i = (1 - \chi_0)\tilde\chi_i$$

we have a starlike covering $(U_i)_{i=0,\dots,m}$ of $\overline{\Omega}$ and a subordinate partition of unity $(\chi_i)_{i=0,\dots,m}$ which is flat at x_0.
If now $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$, then from any partition of unity subordinate to an open covering of $\overline{\Omega}$ we get another one which is flat at x_0 by multiplying with a cut-off function which is 1 on a neighborhood of $\overline{\Omega}$ and vanishes on a neighborhood of x_0.
In a second step we choose $R > 0$ such that $\overline{\Omega} \subset B_R(0)$. To any $x_0 \in \overline{B}_R(0)$ there exists, as we have proved in the first step, a neighborhood $V(x_0)$ and a starlike open covering $(U_i^{(x_0)})_i$ of $\overline{\Omega}$ with a subordinate partition of unity $(\chi_i^{(x_0)})_i$ which is flat at any point of $V(x_0)$. The open covering

$\big(V(x_0)\big)_{x_0 \in \overline{B}_R(0)}$ of the compact set $\overline{B}_R(0)$ contains a finite subcovering associated with points $x_0 = x_1, \dots, x_k \in \overline{B}_R(0)$. The corresponding family of open coverings $(U_i^{(x_j)})$ and partitions of unity $(\chi_i^{(x_j)})$ for $j = 1, \dots, k$ will have the required properties for all points $x_0 \in \overline{B}_R(0)$. For the remaining points $x_0 \in \mathbb{R}^n \setminus \overline{B}_R(0)$, one adds one of the previous partitions of unity, after multiplying each of its functions by a $\mathscr{C}^\infty$ cut-off function that is 1 in a neighborhood of $\overline{\Omega}$ and has its support in $B_R(0)$. □

Theorem 4.6 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then for $\ell = 0, 1, \dots, n$, there exist pseudodifferential operators R_ℓ, T_ℓ of order -1 and K_ℓ, L_ℓ of order $-\infty$ on $\mathbb{R}^n$ with the following properties:*
(i) *The operators define continuous mappings*

$$R_\ell : \mathscr{C}^\infty(\overline{\Omega}, \Lambda^\ell) \to \mathscr{C}^\infty(\overline{\Omega}, \Lambda^{\ell-1}) \qquad \text{and} \qquad T_\ell : \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell) \to \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1})\,,$$

and for any $s \in \mathbb{R}$

$$\begin{aligned} R_\ell : H^s(\Omega, \Lambda^\ell) \to H^{s+1}(\Omega, \Lambda^{\ell-1}) \qquad &\text{and} \qquad T_\ell : H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell) \to H^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1})\,,\\ K_\ell : H^s(\Omega, \Lambda^\ell) \to \mathscr{C}^\infty(\overline{\Omega}, \Lambda^\ell) \qquad &\text{and} \qquad L_\ell : H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell) \to \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)\,. \end{aligned}$$

(ii) *On a neighborhood of $\overline{\Omega}$, there holds for $\ell = 0, 1, \dots, n$ and any ℓ-form u on $\mathbb{R}^n$ with compact support*

$$d\,R_\ell u + R_{\ell+1}\,du = u - K_\ell u \quad \text{and} \quad d\,T_\ell u + T_{\ell+1}\,du = u - L_\ell u\,. \tag{4.6}$$

(iii) *In particular, K_0 is a finite-dimensional operator mapping $H^s(\Omega, \Lambda^0)$ continuously to $\mathscr{C}^\infty(\overline{\Omega}, \Lambda^0)$ for any $s \in \mathbb{R}$, L_n is a finite-dimensional operator mapping $H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^n)$ continuously to $\mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^n)$ for any $s \in \mathbb{R}$, and one has in a neighborhood of $\overline{\Omega}$:*

$$\begin{aligned} R_1\,du = u - K_0 u\,, \quad T_1\,du = u - L_0 u\,, \quad &\text{when } \ell = 0\,,\\ d\,R_n u = u - K_n u\,, \quad d\,T_n u = u - L_n u\,, \quad &\text{when } \ell = n\,. \end{aligned}$$

Proof: We give the details of the proof for the Poincaré-type operators R_ℓ. For the Bogovskiĭ-type operators T_ℓ, the proof is the same.

The crucial observation is that in the definitions (4.2) of the perturbation operator K_ℓ and (4.4) of L_ℓ, the factors $d\chi_i$ are all zero in a neighborhood of any point x_0 in which the partition of unity $(\chi_i)_{i=1,\dots,m}$ is flat. The images $K_\ell u$ and $L_\ell u$ are therefore $\mathscr{C}^\infty$ in the neighborhood of such a point (in fact, $K_\ell u$ is even zero there).

We choose now a finite number of starlike finite open coverings $(U_i^{(j)})_{i=1,\dots,m^{(j)}}, j = 1, \dots, k$, of $\overline{\Omega}$ and subordinate partitions of unity $(\chi_i^{(j)})_{i=1,\dots,m^{(j)}}$, $j = 1, \dots, k$ which exist according to Lemma 4.5 in such a way that for any $x_0 \in \mathbb{R}^n$ at least one of the partitions of unity is flat at x_0. For each $j = 1, \dots, k$, we construct the operators $R_\ell^{(j)}$ and $K_\ell^{(j)}$ associated with the corresponding partition of unity. They satisfy the equivalent of (4.3) on a neighborhood of $\overline{\Omega}$, namely

$$\begin{aligned} \big(d\,R_\ell^{(j)} + R_{\ell+1}^{(j)}\,d\big)u = u - K_\ell^{(j)} u\,,\\ dK_\ell^{(j)} u = K_{\ell+1}^{(j)}\,du\,. \end{aligned} \tag{4.7}$$

We can then define

$$
\begin{aligned}
R_\ell &= R_\ell^{(1)} + K_{\ell-1}^{(1)} R_\ell^{(2)} + K_{\ell-1}^{(1)} K_{\ell-1}^{(2)} R_\ell^{(3)} + \cdots + K_{\ell-1}^{(1)} \cdots K_{\ell-1}^{(k-1)} R_\ell^{(k)} \\
K_\ell &= K_\ell^{(1)} \cdots K_\ell^{(k)} \, .
\end{aligned}
$$

Using the relations (4.7), one can easily verify that on a neighborhood of $\overline{\Omega}$ we have

$$
\big(d\, R_\ell + R_{\ell+1}\, d\big)u = u - K_\ell u \quad \text{and} \quad dK_\ell u = K_{\ell+1} du \, . \tag{4.8}
$$

In addition, we find that the operator K_ℓ is not only a pseudodifferential operator of order $-k$ as a product of pseudodifferential operators of order -1, but actually of order $-\infty$, that is, an integral operator with $\mathscr{C}^\infty$ kernel, continuously mapping $\mathscr{D}'(\mathbb{R}^n)$ to $\mathscr{C}^\infty(\mathbb{R}^n)$. The reason for this is that for any $x_0 \in \mathbb{R}^n$, at least one of the partitions of unity $(\chi_i^{(j)})_{i=1,\ldots,m^{(j)}}$ is flat at x_0, and that therefore the corresponding factor $K_\ell^{(j)}$ maps to functions which are $\mathscr{C}^\infty$ in a neighborhood of x_0. The other factors in the definition of K_ℓ are pseudodifferential operators, hence pseudo-local, and therefore the product K_ℓ maps to functions that are $\mathscr{C}^\infty$ in a neighborhood of x_0, too. □

The relations (4.6) imply regularity results for the d operator. These can be expressed as existence of solutions of maximal regularity if the solvability conditions are satisfied. We consider this first for the inhomogeneous equation $dv = u$ and then for the homogeneous equation $du = 0$. Finally we obtain a regularity result for the cohomology spaces of the two de Rham complexes (2.1) and (2.2).

Corollary 4.7 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.*
For $1 \le \ell \le n$ and any $s, t \in \mathbb{R}$ we have:
(a) *If $u \in H^s(\Omega, \Lambda^\ell)$ satisfies $u = dv$ for some $v \in H^t(\Omega, \Lambda^{\ell-1})$, then there exists $w \in H^{s+1}(\Omega, \Lambda^{\ell-1})$ such that $u = dw$, and there is a constant C independent of u and v with*

$$
\|w\|_{H^{s+1}(\Omega)} \le C\left(\|u\|_{H^s(\Omega)} + \|v\|_{H^t(\Omega)}\right) .
$$

(b) *If $u \in H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)$ satisfies $u = dv$ for some $v \in H^t_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1})$, then there exists $w \in H^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1})$ such that $u = dw$, and there is a constant C independent of u and v with*

$$
\|w\|_{H^{s+1}(\mathbb{R}^n)} \le C\left(\|u\|_{H^s(\mathbb{R}^n)} + \|v\|_{H^t(\mathbb{R}^n)}\right) .
$$

Proof: (a) If $u = dv$, then with $v = dR_{\ell-1}v + R_\ell dv + K_{\ell-1}v$ we get $u = d\big(R_\ell u + K_{\ell-1}v\big)$, and $w = R_\ell u + K_{\ell-1}v$ belongs to $H^{s+1}(\Omega, \Lambda^{\ell-1})$ if $u \in H^s(\Omega, \Lambda^\ell)$. The estimate follows from the fact that R_ℓ is of order -1 and that $K_{\ell-1}$ maps $H^t(\Omega, \Lambda^{\ell-1})$ continuously to $H^{s+1}(\Omega, \Lambda^{\ell-1})$ for any s and t.
(b) Likewise, $u = dv$ implies $u = dw$ with $w = T_\ell u + L_{\ell-1}v \in H^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1})$ if $u \in H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)$. □

Next we consider the special case of relations (4.6) where $du = 0$.

Corollary 4.8 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. For any $s \in \mathbb{R}$ and $1 \le \ell \le n$ we have:*

(a) $\quad u \in H^s(\Omega, \Lambda^\ell),\ du = 0$ *in* $\Omega \quad \Longrightarrow \quad u = dR_\ell u + K_\ell u \quad$ *in* Ω
Here $R_\ell u \in H^{s+1}(\Omega, \Lambda^{\ell-1})$ and $K_\ell u \in \mathscr{C}^\infty(\overline{\Omega}, \Lambda^\ell)$.

(b) $\quad u \in H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell),\ du = 0$ *in* $\mathbb{R}^n \quad \Longrightarrow \quad u = dT_\ell u + L_\ell u \quad$ *in* $\mathbb{R}^n$

Here $T_\ell u \in H^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1})$ *and* $L_\ell u \in \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)$.

For a bounded Lipschitz domain Ω, we consider now the cohomology spaces of regularity s of the two de Rham complexes, without boundary conditions (2.1), and with compact support (2.2). Thus we introduce the corresponding two variants of the cohomology spaces, *without boundary conditions*

$$\mathscr{H}^s_\ell(\Omega) := \frac{\ker\big(d : H^s(\Omega, \Lambda^\ell) \to H^{s-1}(\Omega, \Lambda^{\ell+1})\big)}{\operatorname{im}\big(d : H^{s+1}(\Omega, \Lambda^{\ell-1}) \to H^s(\Omega, \Lambda^\ell)\big)} \tag{4.9}$$

and *with compact support*:

$$\mathscr{H}^s_{\overline{\Omega},\ell}(\mathbb{R}^n) = \frac{\ker\big(d : H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell) \to H^{s-1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell+1})\big)}{\operatorname{im}\big(d : H^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1}) \to H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)\big)}\ . \tag{4.10}$$

Here we can consider the full range $0 \le \ell \le n$, if we complete the complexes by 0 as we did in (2.1) and (2.2).

Theorem 4.9 *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$, *and let* $0 \le \ell \le n$.
(a) *For any* $s \in \mathbb{R}$, *the exterior derivatives*

$$d : H^{s+1}(\Omega, \Lambda^{\ell-1}) \to H^s(\Omega, \Lambda^\ell) \qquad \textit{and} \quad d : H^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1}) \to H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)$$

define bounded operators with closed range $dH^{s+1}(\Omega, \Lambda^{\ell-1})$ *and* $dH^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1})$.
(b) *The dimension of* $\mathscr{H}^s_\ell(\Omega)$ *is a finite number* b_ℓ *independent of* $s \in \mathbb{R}$. *Moreover there is a* b_ℓ*-dimensional subspace* $\mathscr{H}_\ell(\overline{\Omega})$ *of* $\mathscr{C}^\infty(\overline{\Omega}, \Lambda^\ell)$ *such that, for all* $s \in \mathbb{R}$,

$$\ker\Big(d : H^s(\Omega, \Lambda^\ell) \to H^{s-1}(\Omega, \Lambda^{\ell+1})\Big) \ = \ d\, H^{s+1}(\Omega, \Lambda^{\ell-1}) \,\oplus\, \mathscr{H}_\ell(\overline{\Omega})\ . \tag{4.11}$$

That is, for any $u \in H^s(\Omega, \Lambda^\ell)$ *satisfying* $du = 0$ *in* Ω, *there exists* $v \in H^{s+1}(\Omega, \Lambda^{\ell-1})$ *and a unique* $w \in \mathscr{H}_\ell(\overline{\Omega})$, *such that*

$$u = dv + w \quad \textit{with} \quad \|v\|_{H^{s+1}(\Omega)} + \|w\|_{H^s(\Omega)} \le C_s \|u\|_{H^s(\Omega)}\ .$$

(c) *The dimension of* $\mathscr{H}^s_{\overline{\Omega},\ell}(\mathbb{R}^n)$ *is a finite number* $\tilde{b}_\ell$ *independent of* $s \in \mathbb{R}$. *Moreover there is a* $\tilde{b}_\ell$*-dimensional subspace* $\mathscr{H}_{\overline{\Omega},\ell}(\mathbb{R}^n)$ *of* $\mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)$ *such that, for all* $s \in \mathbb{R}$,

$$\ker\Big(d : H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell) \to H^{s-1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell+1})\Big) \ = \ d\, H^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1}) \,\oplus\, \mathscr{H}_{\overline{\Omega},\ell}(\mathbb{R}^n)\ . \tag{4.12}$$

That is, for any $u \in H^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)$ *satisfying* $du = 0$ *in* $\mathbb{R}^n$, *there exists* $v \in H^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1})$ *and a unique* $w \in \mathscr{H}_{\overline{\Omega},\ell}(\mathbb{R}^n)$, *such that*

$$u = dv + w \quad \textit{with} \quad \|v\|_{H^{s+1}(\mathbb{R}^n)} + \|w\|_{H^s(\mathbb{R}^n)} \le C_s \|u\|_{H^s(\mathbb{R}^n)}\ .$$

(d) *The dimensions* b_ℓ *and* $\tilde{b}_\ell$ *are related by*

$$\tilde{b}_{n-\ell} = b_\ell\ .$$

Proof: We give the proof for the case without boundary conditions. The proof for the case with compact support is similar if one takes into account the mapping properties of the operators T_ℓ and L_ℓ.

Fix $\ell \in \{0, \dots, n\}$. For $s \in \mathbb{R}$, define

$$N_\ell^s = \ker\bigl(d : H^s(\Omega, \Lambda^\ell) \to H^{s-1}(\Omega, \Lambda^{\ell+1})\bigr)$$

with in particular, $N_n^s = H^s(\Omega, \Lambda^n)$. This is a closed subspace of $H^s(\Omega, \Lambda^\ell)$, and for the study of the range of d, we can replace $H^{s+1}(\Omega, \Lambda^{\ell-1})$ by the quotient space

$$X_{\ell-1}^{s+1} := H^{s+1}(\Omega, \Lambda^{\ell-1}) / N_{\ell-1}^{s+1}$$

with its natural quotient norm. We will now study the properties of d as a mapping

$$d : X_{\ell-1}^{s+1} \to N_\ell^s \ . \tag{4.13}$$

We know from (4.8) that the nullspace of d is an invariant subspace of the operator K_ℓ, and K_ℓ is a compact operator in N_ℓ^s. By the same token, $K_{\ell-1}$ is defined in a natural way on the quotient space $X_{\ell-1}^{s+1}$, and it is a compact operator there.

Also from (4.8) follows that for $u \in H^{s+1}(\Omega, \Lambda^{\ell-1})$ we have

$$R_\ell du = u - K_{\ell-1}u - dR_{\ell-1}u \quad \equiv u - K_{\ell-1}u \mod N_{\ell-1}^{s+1} \ ,$$

and for $v \in N_\ell^s$ we have

$$dR_\ell v = v - K_\ell v \ .$$

Together, this means that if we consider R_ℓ as a bounded operator from N_ℓ^s to $X_{\ell-1}^{s+1}$, it defines a two-sided regularizer (inverse modulo compact operators) of the operator d in (4.13). By the well-known theory of Fredholm operators, this implies that d in (4.13) is a Fredholm operator. Its image is therefore closed, which proves point (a), and it has finite codimension, which shows that $\mathscr{H}_\ell^s(\Omega)$ is finite dimensional.

Let us now define the direct summand $\mathscr{H}_\ell(\overline{\Omega})$. Let $b_\ell = \dim\, \mathscr{H}_\ell^0$. It is a consequence of the above results that $dH^1(\Omega, \Lambda^{\ell-1})$ has a b_ℓ-dimensional direct summand, say $\tilde{\mathscr{H}}(\Omega, \Lambda^\ell)$ in N_ℓ^0. That is

$$N_\ell^0 = dH^1(\Omega, \Lambda^{\ell-1}) \oplus \tilde{\mathscr{H}}(\Omega, \Lambda^\ell) \ .$$

Define

$$\mathscr{H}_\ell(\overline{\Omega}) = K_\ell \tilde{\mathscr{H}}(\Omega, \Lambda^\ell) \subset \mathscr{C}^\infty(\overline{\Omega}, \Lambda^\ell) \ .$$

Then, by (4.8), $\mathscr{H}_\ell(\overline{\Omega}) \subset N_\ell^s$ for all s. Moreover $dH^{s+1}(\Omega, \Lambda^{\ell-1}) \cap \mathscr{H}_\ell(\overline{\Omega}) = \{0\}$. To see this, suppose that $dv = K_\ell w$ where $v \in H^{s+1}(\Omega, \Lambda^{\ell-1})$ and $w \in \tilde{\mathscr{H}}(\Omega, \Lambda^\ell)$. Thus, using (4.6), $d(R_\ell dv + K_{\ell-1}v) = w - dR_\ell w$ and hence $du = w$ where $u = R_\ell K_\ell w + K_{\ell-1}v + R_\ell w \in \mathscr{C}^\infty(\overline{\Omega}, \Lambda^{\ell-1}) \subset H^1(\Omega, \Lambda^{\ell-1})$. So, by the definition of $\tilde{\mathscr{H}}(\Omega, \Lambda^\ell)$, $w = 0$ and then again, $dv = K_\ell w = 0$. In a similar way, we can show that K_ℓ is one–one on $\mathscr{H}_\ell(\overline{\Omega})$, so that $\dim \mathscr{H}_\ell(\overline{\Omega}) = b_\ell$.

We next prove (4.11). Given $u \in N_\ell^s$, write $u = dR_\ell u + K_\ell(dR_\ell u + K_\ell u)$. Now $K_\ell u \in \mathscr{C}^\infty(\overline{\Omega}, \Lambda^\ell) \subset H^0(\Omega, \Lambda^\ell)$, so by the definition of $\tilde{\mathscr{H}}(\Omega, \Lambda^\ell)$, we can write

$$K_\ell u = dv' + w' \quad \text{with} \quad v' \in H^1(\Omega, \Lambda^{\ell-1}),\ w' \in \tilde{\mathscr{H}}(\Omega, \Lambda^\ell) \ .$$

Hence $u = dv + w$ with $v = R_\ell u + K_{\ell-1}R_\ell u + K_{\ell-1}v' \in H^{s+1}(\Omega, \Lambda^{\ell-1})$, $w = K_\ell w' \in \mathscr{H}_\ell(\overline{\Omega})$, and $\|v\|_{H^{s+1}} + \|w\|_{H^s} \le C_s \|u\|_{H^s}$.

It is a consequence of (4.11) that $\mathscr{H}_\ell^s(\Omega)$ is isomorphic to $\mathscr{H}_\ell(\overline{\Omega})$, and hence $\dim \mathscr{H}_\ell^s(\Omega) = b_\ell$ for all s.
To prove part (d), observe that

$$\Big\{\ker\Big(d : H^s(\Omega,\Lambda^\ell) \to H^{s-1}(\Omega,\Lambda^{\ell+1})\Big)\Big\}^\perp = \delta H_{\overline{\Omega}}^{-s+1}(\mathbb{R}^n,\Lambda^{\ell+1}) \\ = *dH_{\overline{\Omega}}^{-s+1}(\mathbb{R}^n,\Lambda^{n-\ell-1})$$

and

$$\big\{dH^{s+1}(\Omega,\Lambda^{\ell-1})\big\}^\perp = *\ker\Big(d : H_{\overline{\Omega}}^{-s}(\mathbb{R}^n,\Lambda^{n-\ell}) \to H_{\overline{\Omega}}^{-s-1}(\mathbb{R}^n,\Lambda^{n-\ell+1})\Big)\ .$$

Therefore, by duality,

$$b_\ell = \dim\left\{\frac{\ker\big(d : H^s(\Omega,\Lambda^\ell) \to H^{s-1}(\Omega,\Lambda^{\ell+1})\big)}{\mathrm{im}\big(d : H^{s+1}(\Omega,\Lambda^{\ell-1}) \to H^s(\Omega,\Lambda^\ell)\big)}\right\} \\ = \dim\left\{\frac{\ker\big(d : H_{\overline{\Omega}}^{-s}(\mathbb{R}^n,\Lambda^{n-\ell}) \to H_{\overline{\Omega}}^{-s-1}(\mathbb{R}^n,\Lambda^{n-\ell+1})\big)}{\mathrm{im}\big(d : H_{\overline{\Omega}}^{-s+1}(\mathbb{R}^n,\Lambda^{n-\ell-1}) \to H_{\overline{\Omega}}^{-s}(\mathbb{R}^n,\Lambda^{n-\ell})\big)}\right\} = \tilde{b}_{n-\ell}\ .$$

□

Remark 4.10 When $\ell = 0$, then

$$\mathscr{H}_0(\overline{\Omega}) = \ker\big(d : H^s(\Omega,\Lambda^0) \to H^{s-1}(\Omega,\Lambda^1)\big) = \mathbb{R} \quad \text{(the constant functions) and}$$
$$\mathscr{H}_{\overline{\Omega},0}(\mathbb{R}^n) = \ker\big(d : H_{\overline{\Omega}}^s(\mathbb{R}^n,\Lambda^0) \to H_{\overline{\Omega}}^{s-1}(\mathbb{R}^n,\Lambda^1)\big) = \{0\}$$

so, by duality,

$$dH^{s+1}(\Omega,\Lambda^{n-1}) = H^s(\Omega,\Lambda^n)\ , \quad \mathscr{H}_n(\overline{\Omega}) = \{0\}\ ,$$
$$dH_{\overline{\Omega}}^{s+1}(\mathbb{R}^n,\Lambda^{n-1}) = \{u \in H_{\overline{\Omega}}^s(\mathbb{R}^n,\Lambda^n) : \textstyle\int u = 0\}\ ,$$

and $\mathscr{H}_{\overline{\Omega},n}(\mathbb{R}^n)$ can be taken to be $\{cL_n 1_{\overline{\Omega}} \mid c \in \mathbb{R}\}$ where $1_{\overline{\Omega}}$ is the characteristic function of $\overline{\Omega}$.
Therefore $b_0 = \tilde{b}_n = 1$ and $b_n = \tilde{b}_0 = 0$.
When $1 \le \ell \le n-1$, we can take $\tilde{\mathscr{H}}(\Omega,\Lambda^\ell)$ to be the orthogonal complement of $dH^1(\Omega,\Lambda^{\ell-1})$ in N_ℓ^0, so that

$$\mathscr{H}_\ell(\overline{\Omega}) = K_\ell\{u \in L^2(\Omega,\Lambda^\ell) \mid du = 0, \delta u = 0 \text{ and } \nu \lrcorner u = 0 \text{ on } \partial\Omega\}\ ,$$

Similarly we can take

$$\mathscr{H}_{\overline{\Omega},\ell}(\mathbb{R}^n) = L_\ell \mathscr{E}_\ell\{u \in L^2(\Omega,\Lambda^\ell) \mid du = 0, \delta u = 0 \text{ and } \nu \wedge u = 0 \text{ on } \partial\Omega\}$$

where $\mathscr{E}_\ell : L^2(\Omega,\Lambda^\ell) \to L^2(\mathbb{R},\Lambda^\ell)$ denotes extension by zero.
The integers b_ℓ are the Betti numbers of Ω. △

Note that the sequence of Betti numbers $b_0,\dots,b_n$ will in general be different from the sequence $\tilde{b}_0,\dots,\tilde{b}_n$. For example, for the standard torus embedded in $\mathbb{R}^3$, one finds without difficulties the two sequences $1,1,0,0$ and $0,0,1,1$, and for the ball with a hole $B_2(0) \setminus \overline{B_1(0)}$, one gets the two sequences $1,0,1,0$ and $0,1,0,1$.

Classically, one considers the de Rham complexes for differential forms with smooth coefficients

$$0 \to \mathscr{C}^\infty(\overline{\Omega}, \Lambda^0) \xrightarrow{d} \mathscr{C}^\infty(\overline{\Omega}, \Lambda^1) \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{C}^\infty(\overline{\Omega}, \Lambda^n) \to 0 \tag{4.14}$$

and

$$0 \to \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^0) \xrightarrow{d} \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^1) \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^n) \to 0 \tag{4.15}$$

With the same arguments as in the preceding proof one can see that the associated cohomology spaces are isomorphic to those with finite regularity considered in Theorem 4.9. It suffices to notice that pseudodifferential operators map $\mathscr{C}^\infty$ functions to $\mathscr{C}^\infty$ functions.

Corollary 4.11 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and $0 \le \ell \le n$.*
(a) *The cohomology space without boundary condition*

$$\frac{\ker\big(d : \mathscr{C}^\infty(\overline{\Omega}, \Lambda^\ell) \to \mathscr{C}^\infty(\overline{\Omega}, \Lambda^{\ell+1})\big)}{\operatorname{im}\big(d : \mathscr{C}^\infty(\overline{\Omega}, \Lambda^{\ell-1}) \to \mathscr{C}^\infty(\overline{\Omega}, \Lambda^\ell)\big)}$$

of the de Rham complex (4.14) *has dimension b_ℓ and is isomorphic to $\mathscr{H}_\ell(\overline{\Omega})$. There is a splitting*

$$\ker\Big(d : \mathscr{C}^\infty(\overline{\Omega}, \Lambda^\ell) \to \mathscr{C}^\infty(\overline{\Omega}, \Lambda^{\ell+1})\Big) \;=\; d\mathscr{C}^\infty(\overline{\Omega}, \Lambda^{\ell-1}) \,\oplus\, \mathscr{H}_\ell(\overline{\Omega})\,.$$

(b) *The cohomology space with compact support*

$$\frac{\ker\big(d : \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell) \to \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell+1})\big)}{\operatorname{im}\big(d : \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1}) \to \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)\big)}$$

of the de Rham complex (4.15) *has dimension $\tilde{b}_\ell$ and is isomorphic to $\mathscr{H}_{\overline{\Omega},\ell}(\mathbb{R}^n)$. There is a splitting*

$$\ker\Big(d : \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell) \to \mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell+1})\Big) \;=\; d\mathscr{C}^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{\ell-1}) \,\oplus\, \mathscr{H}_{\overline{\Omega},\ell}(\mathbb{R}^n)\,.$$

Remark 4.12 All the results of this section remain valid when H^s is replaced by B^s_{pq} $(0 < p \le \infty, 0 < q \le \infty)$, or by F^s_{pq} $(0 < p < \infty, 0 < q \le \infty)$.
We make the following additional comments.
In Corollary 4.7, all that is required of v is that, in part (a), v be the restriction to Ω of a distribution (with compact support) on $\mathbb{R}^n$, while in part (b), v be a distribution on $\mathbb{R}^n$ with support in $\overline{\Omega}$. Indeed, it is well known that distributions with compact support are of finite order, so there exists then a finite index t such that v belongs to one of the spaces required in the corollary.
The dimension of the cohomology spaces $\mathscr{H}^s_\ell(\Omega)$ and $\mathscr{H}^s_{\overline{\Omega},\ell}$, defined using B^s_{pq} or F^s_{pq} in place of H^s, are still equal to b_l and $\tilde{b}_l$. $\triangle$

We conclude by mentioning that we have now proved Theorem 1.1, stated in the Introduction.

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Addresses:

MARTIN COSTABEL
IRMAR, Université de Rennes 1, Campus de Beaulieu
35042 Rennes Cedex, France
E-mail: martin.costabel@univ-rennes1.fr

ALAN MCINTOSH
Centre for Mathematics and its Applications, Australian National University
Canberra, ACT 0200, Australia
E-mail: alan@maths.anu.edu.au