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On bounded biorthogonal systems in some function spaces

by

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Abstract. In this paper biorthogonal systems in the space of continuous functions $C(K)$ (K an infinite metric compact) and in the space B_p, $1 < p < \infty$, of almost periodic Besicovitch functions are considered. It is shown that there is a separable subspace $F \subset C(K)^*$ for which there is no biorthogonal system x_n, f_n, $x_n \in C(K)$, $f_n \in C(K)^*$ with $\|x_n\| = \|f_n\| = 1$ and $[f_n]_1^\infty \supset F$. It is proved that under the continuum hypothesis there is a decomposition of the real line $R = \bigcup_n R_n$, $n \in N$, for which the system $e^{i\lambda t} \in B_p$, $\lambda \in R_n$, is equivalent to the standard basis of the Hilbert space $l_2(R_n)$ for arbitrary n.

Introduction. Let X be a Banach space, X^* its dual and I some set of indices. A system x_i, f_i, $i \in I$, $x_i \in X$, $f_i \in X^*$, is called *biorthogonal* if $f_i(x_j) = 0$ for $i \neq j$ and 1 for $i = j$. A biorthogonal system is called *fundamental* if the closed linear span $[x_i\colon i \in I]$ is equal to X, and *total* if for any element $x \in X$, $x \neq 0$, there is an index i such that $f_i(x) \neq 0$. A fundamental and total biorthogonal system is said to be a *Markushevich basis* (an *M-basis*). A biorthogonal system is *bounded by a number c* if $\sup_i \|x_i\| \|f_i\| \leqslant c$. It is known (cf. [10]) that for any separable Banach space X, any separable subspace $F \subset X^*$ and any $\varepsilon > 0$ there exists an M-basis x_n, f_n bounded by $1+\varepsilon$ with $[f_n]_1^\infty \supset F$. Although the question whether every separable Banach space has an M-basis bounded by 1 is still open, we show that in the result of [10] quoted above $\varepsilon > 0$ is essential in some sense. Let us formulate the exact statement. Let K be a metric compact and let $C(K)$ be the space of real continuous functions on K. Its dual is the space $M(K)$ of Borel measures on the set K with bounded variation. Let δ_t, $t \in K$, be the atomic measure defined by $\delta_t\{t\} = 1$, $\delta_t\{K \setminus t\} = 0$.

THEOREM 1. *Let $(t_n)_1^\infty$ be a dense set in a nice metric compact K. The space $C(K)$ fails to have a biorthogonal system x_n, f_n bounded by 1 for which $[f_n]_1^\infty \supset (\delta_{t_n})_1^\infty$.*

This answers in the negative a question from [16, problem 8.2b)], where it is written that the question was raised by A. Pełczyński. Not every Banach space has an M-basis [16, p. 691], but if it has an M-basis then it has a bounded one, too [12]. In particular, a weakly compactly generated (WCG in short) space, i.e. a space which is a closed linear span of its weakly compact subset, has an M-basis [16, p. 693]. Therefore it has a bounded M-basis. It will be shown that there exists a WCG space X (namely $X = C[0,1]+c_0[0,1] \subset l_\infty[0,1]$) for which $\sup_i \|x_i\| \|f_i\| \geq 2$ for every Markushevich basis x_i, f_i. We also present a simple proof of the nonexistence of universal elements in the class of countable Markushevich bases. This answers a question of N. J. Kalton [5].

Denote by B_p, $1 < p < \infty$, the space of almost periodic Besicovitch functions, i.e. the completion of the complex linear space spanned by the functions $e^{i\lambda t}$ of the real variable t where the parameter λ runs through $\boldsymbol{R}$ in the norm

$$\|x\| = \lim_{T\to\infty} \Big((2T)^{-1} \int_{-T}^{T} |x(t)|^p\,dt\Big)^{1/p}.$$

The system $x_\lambda = e^{i\lambda t}$, $f_\lambda(x) = \lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} x(t)\,e^{i\lambda t}\,dt$ forms a Markushevich basis in the space B_p. If $p=2$, it is a noncountable orthogonal basis in the Hilbert space B_2.

THEOREM 2. *Let us assume the continuum hypothesis. There exists a decomposition of the real line* $\boldsymbol{R} = \bigcup_{n=1}^{\infty} R_n$ *into a countable collection of subsets such that for any* n, *any finite set* $(\lambda_k \in R_n,\ k = 1, \ldots, l)$ *and any complex scalars* $(a_k)_1^l$

$$c\Big(\sum_{k=1}^{l} |a_k|^2\Big)^{1/2} \leq \Big\|\sum_{k=1}^{l} a_k x_{\lambda_k}\Big\| \leq C\Big(\sum_{k=1}^{l} |a_k|^2\Big)^{1/2},$$

where the norm is taken in the space B_p *and the constants* c, C *depend on* p *only. Moreover, there are uniformly bounded projections* $B_p \to [x_\lambda\colon \lambda \in R_n]$ *parallel to subspaces* $[x_\lambda\colon \lambda \notin R_n]$.

In the first section all Banach spaces are assumed real, in the second they are complex. Many intermediate results are formulated in a nonmaximal generality. We use the following notation: $B(X)$ and $S(X)$ are the unit ball and the unit sphere of the normed space X respectively, $\operatorname{lin} M$ is the linear span of the set M and $M^\perp$ is the annihilator of M.

1. Spaces of continuous functions. A subspace $F \subset X^*$ is said to be *λ-norming*, $0 < \lambda \leq 1$, if for its Dixmier characteristic we have

$$r(F) = \inf \sup\{|f(x)|\colon f \in B(F)\} = \lambda,$$

where the infimum is taken over all $x \in S(X)$. The characteristic of the subspace F equals to the greatest scalar r such that the weak* closure of the ball $B(F)$ contains the ball $rB(X^*)$ of radius r [2]. The following statement is almost evident.

LEMMA 1. *Let* F *and* G *be subspaces of* X^*, $r(F) = \lambda$ *and*

$$\varrho(F, G) = \sup \inf\{\|f-g\|\colon g \in B(G)\} \leq \varepsilon,$$

where the supremum is taken over all $f \in B(F)$. *Then* $r(G) \geq \lambda - \varepsilon$.

LEMMA 2. *Let* X *be a separable Banach space, let* $g \in S(X^*)$ *be a functional and* $H \subset X^*$ *a* 1*-norming subspace. Suppose that* $\|h+ag\| = \|h\|+|a|$ *for any* $h \in H$ *and* $a \in \boldsymbol{R}$. *Let* f_0 *be a functional from* $B(X^*)$ *such that* $\|f_0 - g\| \leq \varepsilon$ *for some* $0 < \varepsilon < 1/2$ *and let* $F \subset H$ *be a subspace such that the sum* $F + \operatorname{lin} f_0$ *is* 1*-norming. Then the characteristic of* F *is not less than* $1-2\varepsilon$.

Proof. Let $x \in S(X)$. Since the subspace H is 1-norming, for every $\varepsilon_1 > 0$ there is an element $h \in (1-\varepsilon) S(H)$ with $h(x) \geq 1-\varepsilon-\varepsilon_1$. It is easy to see that $\varrho(F+\operatorname{lin} g, F+\operatorname{lin} f_0) \leq \varepsilon$. Therefore by Lemma 1 the characteristic of the subspace $F+\operatorname{lin} g$ is not less than $1-\varepsilon$. Hence there is a sequence $f_n + a_n g$, $\|f_n + a_n g\| \leq 1$, $f_n \in F$, $a_n \in \boldsymbol{R}$, weakly* convergent to the functional h. By the Hahn–Banach theorem, there exists an element $y \in S(X)$ with $h(y) \geq \|h\| - \varepsilon_1$ and $g(y) = 0$. Then

$$\begin{aligned}\underline{\lim}\,\|f_n\| &\geq \underline{\lim}\, f_n(y) = \lim (f_n + a_n g)(y) = h(y)\\ &\geq \|h\| - \varepsilon_1 = 1-\varepsilon-\varepsilon_1.\end{aligned}$$

Since $1 \geq \|f_n + a_n g\| = \|f_n\| + |a_n|$, we have $|a_n| \leq 1 - \|f_n\|$. Therefore $\overline{\lim}\,|a_n| \leq \varepsilon + \varepsilon_1$. Hence $\underline{\lim}\, f_n(x) \geq \lim (f_n + a_n g)(x) - \overline{\lim}\; a_n g(x) \geq h(x) - \overline{\lim}\,|a_n| \geq 1-\varepsilon-\varepsilon_1-\varepsilon-\varepsilon_1$. Since ε_1 is arbitrary, the characteristic of the subspace F is not less than $1-2\varepsilon$. ∎

LEMMA 3. *Let* K *be an infinite metric compact,* $t_n \in K$, $t_n \to t_0$, $t_n \neq t_0$. *Let* F *be a subspace of the hyperplane* $H = \{\mu \in M(K)\colon \mu\{t_0\} = 0\}$ *and let* μ_0 *be a measure on* K *such that* $\|\mu_0\| = 1$ *and* $\|\mu_0 - \delta_{t_0}\| \leq \varepsilon$ *for some* $0 < \varepsilon < 1/2$. *If the subspace* $F + \operatorname{lin} \mu_0$ *is* 1*-norming, then the characteristic of* F *is not less than* $1-2\varepsilon$.

Proof. It is easy to see that for any $h \in H$ and any $a \in \boldsymbol{R}$

$$\begin{aligned}\|h + a\delta_{t_0}\| &= \operatorname{Var}(h + a\delta_{t_0})(K \setminus t_0) + |(h + a\delta_{t_0})\{t_0\}|\\ &= \|h\| + |a|.\end{aligned}$$

Since $H \supset (\delta_{t_n})_1^\infty$, the subspace H is 1-norming. All the conditions of Lemma 2 are also satisfied if we set $g = \delta_{t_0}$ and $f_0 = \mu_0$. This proves the lemma. ∎

THEOREM 3. *Let* K *be an infinite metric compact,* $t_n \in K$, $t_n \to t_0$, $t_n \neq t_0$. *Let* x_n, μ_n, $x_n \in C(K)$, $\mu_n \in M(K)$, *be a biorthogonal sequence such that* $[\mu_n]_1^\infty$ *is a* 1*-norming subspace. If* $\delta_{t_0} \in [\mu_n]_1^\infty$, *then* $\sup_n \|x_n\| \|\mu_n\| > 1$.

Proof. Suppose that

(1) $$\sup_n \|x_n\| \, \|\mu_n\| = 1;$$

without loss of generality we can assume that $\|x_n\| = \|\mu_n\| = 1$. Let $0 < \varepsilon < 1/2$ and $\delta_{t_0} \in [\mu_n]_1^\infty$. Then for some $\mu_0 = \sum_{n=1}^{n_0} a_n \mu_n$, $\|\mu_0\| = 1$, we have $\|\delta_{t_0} - \mu_0\| < \varepsilon$. This means that

$$\varepsilon > \operatorname{Var}(\delta_{t_0} - \mu_0)(K) = |(\delta_{t_0} - \mu_0)\{t_0\}| + \operatorname{Var} \mu_0 (K \setminus t_0)$$
$$\geqslant |(\delta_{t_0} - \mu_0)\{t_0\}|.$$

From this it follows that $|\mu_0\{t_0\}| > 1 - \varepsilon$ and $\operatorname{Var} \mu_0(K \setminus t_0) < \varepsilon$.

We show that $\mu_n\{t_0\} = 0$ for $n > n_0$. Suppose that $\mu_n\{t_0\} = b \neq 0$ for some $n > n_0$ (it can be assumed that $b > 0$). Then $\operatorname{Var} \mu_n (K \setminus t_0) = 1 - b$. Hence

$$\left\| \mu_n - \frac{b}{\mu_0\{t_0\}} \mu_0 \right\| = \operatorname{Var}\left(\mu_n - \frac{b}{\mu_0\{t_0\}} \mu_0\right)(K \setminus t_0) + \left| \left(\mu_n - \frac{b}{\mu_0\{t_0\}} \mu_0\right)\{t_0\} \right|$$
$$\leqslant \operatorname{Var} \mu_n (K \setminus t_0) + \left| \frac{b}{\mu_0\{t_0\}} \mu_0 (K \setminus t_0) \right|$$
$$\leqslant 1 - b + \frac{b}{1-\varepsilon} < 1.$$

But the condition (1) implies that for any n and $\mu_0 \in \operatorname{lin}(\mu_k\colon k \neq n)$, $\|\mu_n - \mu_0\| \geqslant (\mu_n - \mu_0)(x_n) = \mu_n(x_n) = 1$. Therefore $[\mu_n]_{n_0+1}^\infty$ belongs to the hyperplane $H = \{\mu \in M(K)\colon \mu\{t_0\} = 0\}$. Set $F = [\mu_n]_{n_0+1}^\infty + ([\mu_n]_1^{n_0} \cap H)$. The subspace F is contained in H and $F + \operatorname{lin} \mu_0 = [\mu_n]_1^\infty$ is a 1-norming subspace. All the conditions of Lemma 3 are valid, hence the characteristic of F is greater than $1 - 2\varepsilon$. Therefore

$$\mu_0 \in \mathrm{cl}^* [\mu_n]_{n_0+1}^\infty + ([\mu_n]_1^{n_0} \cap H),$$

where cl* means the weak* closure. Hence

$$\sum_{n=1}^{n_0} a_n \mu_n = \mu_0 = \mu + \sum_{n=1}^{n_0} b_n \mu_n,$$

where $\mu \in \mathrm{cl}^* [\mu_n]_{n_0+1}^\infty$ and $\sum_{n=1}^{n_0} b_n \mu_n \in H$. Since $\mu_0 \notin H$, this implies that $[\mu_n]_1^{n_0} \cap \mathrm{cl}^* [\mu_n]_{n_0+1}^\infty \neq 0$. This contradicts the biorthogonality of the system x_n, μ_n. Thus the condition (1) cannot be true. ■

Proof of Theorem 1. It is sufficient to note that if t_n is a dense subset of the compact K then $[\delta_{t_n}]_1^\infty$ will be a 1-norming subspace. ■

We recall that an M-basis x_n, f_n is called *shrinking* if $[f_n]_1^\infty = X^*$. The space c of all convergent sequences is isometric to $C(\bar{N})$, where $\bar{N}$ is the one-point compactification of the natural numbers N. Its dual is the space $l_1(\bar{N})$ of all absolutely summing sequences and its bidual is the space $l_\infty(\bar{N})$ of all bounded sequences.

COROLLARY 1. *The space $c = C(\bar{N})$ has no shrinking M-basis bounded by* 1.

COROLLARY 2. *The space $l_1(\bar{N})$ has no fundamental biorthogonal sequence x_n, f_n bounded by 1 such that $f_n \in C(\bar{N}) \subset l_\infty(\bar{N})$.*

Indeed, the sequence f_n, x_n would then be a 1-norming biorthogonal system in $C(\bar{N})$ bounded by 1 and $[x_n]_1^\infty \ni \delta_n(f) = f(n)$. This contradicts Theorem 3. ■

A (Schauder) basis x_n with biorthogonal functionals f_n is called an *Auerbach basis* if $\|x_n\| \, \|f_n\| = 1$ for any n. A basis x_n, f_n of the space $C(K)$ is called *interpolating with nodes* t_n if for any n we have $\left(\sum_{i=1}^n f_i(x) x_i\right)(t_m) = x(t_m)$ for $m = 1, \ldots, n$. The closed linear span of the functionals f_n biorthogonal to an interpolating basis x_n is equal exactly to $[\delta_{t_n}]_1^\infty$ [15, p. 11]. Theorem 1 implies immediately

COROLLARY 3. *Let K be a nice metric compact and t_n a dense subset of K. The space $C(K)$ has no interpolating Auerbach basis with nodes t_n.*

It seems that the answer to the following question is unknown: has the space $C[0, 1]$ a Markushevich basis bounded by 1 ? But it is not difficult to construct a fundamental biorthogonal system bounded by 1 in this space. We give such a construction without proof.

Let a_n, t_n, τ_n, b_n, $n \in N$, be numbers such that for any n, $0 < a_n < t_n < \tau_n < b_n < a_{n+1} < 1$ and $a_n \to 1$. Let $x_0(t) \equiv 1$; for $n > 0$, let $x_n(t)$ be the polygonal function with nodes 0, a_n, t_n, τ_n, b_n, 1, $x_n(a_n) = x_n(0) = x_n(b_n) = x_n(1) = 0$, $x_n(t_n) = 1$, $x_n(\tau_n) = -1$. The functionals $f_0(x) = x(1)$, $f_n(x) = (x(t_n) - x(\tau_n))/2$ are biorthogonal to x_n. For any $m \geqslant 1$ let $y_m(t)$ be the polygonal function with nodes 0, a_m, t_m, τ_m, b_m, 1, $y_m(0) = y_m(a_m) = y_m(b_m) = y_m(1) = 0$, $y_m(t_m) = y_m(\tau_m) = 1$. We denote by Z the set of continuous functions vanishing at all points t_n, τ_n (hence at 1 too) and having at any point of $[0, 1]$ the absolute value of the right and left derivatives less than or equal to 1. Let $(z_m)_1^\infty$ be a dense sequence in the set Z. We shall label the sequence $(x_n, f_n, n$ odd) with two indices: $(x_m^k, f_m^k)_{m,k=1}^\infty$, and the sequence $(x_n, f_n, n$ even, $n > 0)$ also with two indices: $(x_n, f_n, n$ even, $n > 0) = (\tilde{x}_m^k, \tilde{f}_m^k)_{m,k=1}^\infty$, but the even elements will be labelled so that, for every fixed m, if $x_n = \tilde{x}_m^k$, $x_{n'} = \tilde{x}_m^{k'}$ and $k' > k$, then $n' > n$, and if $x_n = \tilde{x}_m^1$, then $n > m$. Put $u_m^k = x_m^k + z_m$, $\tilde{u}_m^k = x_m^k + y_m$. Then the system $(x_0, f_0) \cup (u_m^k, \tilde{u}_m^k, f_m^k, \tilde{f}_m^k)_{m,k=1}^\infty$ is a fundamental biorthogonal sequence in the space $C[0, 1]$, bounded by 1.

Remark. B. Godun showed the existence of a (not weakly compactly generated) Banach space with a fundamental biorthogonal system but without a fundamental biorthogonal system bounded by 1. We give an example of a WCG space X in which for every Markushevich basis $(x_i, f_i\colon i \in I)$

$$\sup_i \|x_i\| \|f_i\| \geqslant 2.$$

Recall that the weak* sequential closure of a set $F \subset X^*$ is defined to be the collection $F_{(1)}$ of all limits of sequences in F that weakly* converge in X^*. By induction, the weak* sequential closure of order α is defined to be $F_{(\alpha)} = \bigcup_{\beta<\alpha} (F_{(\beta)})_{(1)}$ for any ordinal α.

Example. Let $l_\infty[0, 1]$ be the space of all bounded functions on the segment $[0, 1]$ with supremum norm and let $c_0[0, 1]$ be its subspace consisting of functions $x(t)$ having a countable support and such that for some numbering t_n of this support, $x(t_n) \to 0$. The space $c_0[0, 1]$ is weakly compactly generated [1, p. 143], hence so is the space $X = c_0[0, 1] + C[0, 1] \subset l_\infty[0, 1]$ [1, p. 154]. Let $(x_i, f_i\colon i \in I)$ be some M-basis of the space X. Since the subspace $F = [f_i\colon i \in I] \subset X^*$ is total, for the first noncountable ordinal ω_1 we have $X^* = F_{(\omega_1)} = \bigcup_{\alpha<\omega_1} F_{(\alpha)}$ [11, p. 50]. By induction it is easy to verify that for any countable ordinal α the subspace $F_{(\alpha)}$ is contained in the subspace $G = \bigcup \mathrm{cl}^*[f_j\colon j \in J]$ where the union is taken over all countable subsets $J \subset I$. Therefore $G = X^*$. The annihilator $c_0[0, 1]^\perp \subset X^*$ is dual to the separable quotient space $X/c_0[0, 1] \simeq C[0, 1]$, hence weakly* separable; let $(g_n)_1^\infty$ be a weakly* dense sequence in $c_0[0, 1]^\perp$. Then, for some countable subset $J_n \subset I$, $\mathrm{cl}^*[f_j\colon j \in J_n] \ni g_n$, hence $c_0[0, 1]^\perp \subset \mathrm{cl}^*[f_j\colon j \in \bigcup_n J_n]$. Thus there exists a countable subset $J \subset I$ for which $c_0[0, 1]^\perp \subset \mathrm{cl}^*[f_j\colon j \in J]$ and $C[0, 1] \subset [x_j\colon j \in J]$. Let $i_0 \notin J$. Then $x_{i_0} \in c_0[0, 1]$, $f_{i_0} \in C[0, 1]^\perp$ and

$$\|x_{i_0}\| \|f_{i_0}\| = \|x_{i_0}\|/\mathrm{dist}(x_0, f_{i_0}^\perp) \geqslant \|x_{i_0}\|/\mathrm{dist}(x_{i_0}, C[0, 1]).$$

It is very easy to check that for $x \in c_0[0, 1]$

$$\mathrm{dist}(x, C[0, 1]) \leqslant \|x\|/2.$$

Thus

$$\|x_{i_0}\| \|f_{i_0}\| \geqslant 2. \ \blacksquare$$

A Markushevich basis $(x_i, f_i\colon i \in I)$ is called *universal* in the class of Markushevich bases of the same cardinality as I if for every M-basis $(y_j, g_j\colon j \in J)$ with $\mathrm{card}\, J = \mathrm{card}\, I$ there exist a subset $I_1 \subset I$ and a map $\varphi\colon J \to I_1$ for which the linear embedding, mapping y_j to $x_{\varphi(j)}$, is an isomorphism.

Theorem 4. *The class of countable M-bases has no universal element.*

Proof. Let X be a separable Banach space with a universal Markushevich basis $(x_n, f_n)_1^\infty$. Put $F = [f_n]_1^\infty \subset X^*$. Then for some countable ordinal α the weak* sequential closure $F_{(\alpha)}$ of the subspace F of order α will coincide with X^* (see for example [4]). On the other hand, for any countable ordinal β there exist a separable Banach space Y and a total subspace $G \subset Y^*$ such that $G_{(\beta)} \neq Y^*$ [4]. It is known that in the space Y there exists an M-basis $(y_k, g_k)_1^\infty$ with $g_k \in G$ [16, p. 224]. Let $(x_{n(k)}) \subset (x_n)$ be a subset equivalent to (y_k) and $T\colon Y \to X$ an isomorphism which determines this correspondence. Then $T^* F \subset [g_k]_1^\infty$ hence

$$Y^* = T^* X^* = T^*(F_{(\alpha)}) \subset ([g_k]_1^\infty)_{(\alpha)} \subset G_{(\alpha)} \subset G_{(\beta)} \neq Y^*$$

if $\beta > \alpha$. Contradiction. $\blacksquare$

2. Spaces of almost periodic functions. The *density character* of a Banach space X (written dens X) is the smallest cardinal $\mathfrak{m}$ for which X has a dense subset of cardinality $\mathfrak{m}$.

Definition 1. Let X be a Banach space and α_0 the first ordinal of cardinality dens X. A *projective resolution* of the identity operator I is defined to be a set of uniformly bounded projections $P_\alpha\colon X \to X$, $\omega \leqslant \alpha \leqslant \alpha_0$, where ω is the first infinite ordinal, such that for $\omega \leqslant \alpha$, $\beta \leqslant \alpha_0$ we have

1) $P_\alpha P_\beta = P_\beta P_\alpha = P_{\min(\alpha,\beta)}$;
2) $P_\alpha X = [P_{\gamma+1} X\colon \gamma < \alpha]$;
3) dens $P_\alpha X \leqslant \bar{\bar{\alpha}}$ ($\bar{\bar{\alpha}}$ is the cardinality of the ordinal α) and $P_{\alpha_0} = I$.

Put

$$X_\omega = P_\omega X \quad \text{and} \quad X_\alpha = (P_{\alpha+1} - P_\alpha) X$$

for $\omega < \alpha < \alpha_0$. A projective resolution is said to be *unconditional* if the following property is satisfied:

($\mathscr{P}$) There exists a constant K, called an *unconditional constant* of the projective resolution P_α, for which

$$\Big\| \sum_{k=1}^{n} \varepsilon_k x_k \Big\| \leqslant K \Big\| \sum_{k=1}^{n} x_k \Big\|$$

for every finite choice $x_1, \ldots, x_n$, $x_k \in X_{\alpha_k}$, $\alpha_k \neq \alpha_l$ when $k \neq l$, and every choice of signs $\varepsilon_k = \pm 1$.

Definition 2. Let X be a Banach space and α_0 the first ordinal of cardinality dens X. A transfinite sequence of closed subspaces $X_\alpha \subset X$ is called an *unconditional decomposition* of the space X if

1) dens $X_\alpha \leqslant \bar{\bar{\alpha}}$, $[X_\alpha\colon \omega \leqslant \alpha < \alpha_0] = X$,
2) condition ($\mathscr{P}$) is satisfied.

The number K is called an unconditional constant of the decomposition X_α. From condition $(\mathscr{P})$ it follows that the condition $(\mathscr{P})$ remains true if in place of $\varepsilon_k = \pm 1$ we write $\varepsilon_k = 0$ or 1. Hence there exist projections P_α: $X \to [X_\beta\colon \beta < \alpha]$ parallel to the subspaces $[X_\beta\colon \beta \geqslant \alpha]$ constructed for the unconditional decomposition X_α which are all bounded by the unconditional constant K and form an unconditional projective resolution.

Obviously, if X_α is an unconditional decomposition then any transfinite sequence $x_\alpha \in X_\alpha$, $x_\alpha \neq 0$, will be an (uncountable) unconditional basic sequence in the sense that for any finite choice $x_{\alpha_1}, \ldots, x_{\alpha_n}$, any scalars $(a_k)_1^n$ and any signs ε_k,

$$\Big\|\sum_{k=1}^{n} \varepsilon_k a_k x_k\Big\| \leqslant K \Big\|\sum_{k=1}^{n} a_k x_k\Big\|.$$

A subsequence $(P_{\alpha_\beta}\colon \omega \leqslant \beta \leqslant \beta_0)$ of a projective resolution is said to be a *subresolution* if it is a projective resolution itself. Any subresolution of an unconditional projective resolution is unconditional too; moreover, its unconditional constant is not greater than the initial one.

LEMMA 4. *Let a space X of density character $\aleph_1$ be isomorphic to the l_p-sum $\bigoplus_{l_p} \sum_{n=1}^{\infty} X_n$, $1 < p < \infty$, where every space X_n has an unconditional decomposition X_α^n, with the unconditional constants all bounded by a number K. Then X has an unconditional decomposition.*

Proof. Since the property of having an unconditional decomposition is preserved by isomorphisms, we shall suppose $X = \bigoplus_{l_p} \sum_{n=1}^{\infty} X_n$. Then, for any finite choice $(x_{\alpha_i}^m)_{i=1,\ldots,k_m}^{m=1,\ldots,n}$, $x_{\alpha_i}^m \in X_{\alpha_i}^m$ and any signs ε_i^m

$$\begin{aligned} \Big\|\sum_{i,m} \varepsilon_i^m x_{\alpha_i}^m\Big\| &= \Big(\Big\|\sum_{i=1}^{k_1} \varepsilon_i^1 x_{\alpha_i}^1\Big\|^p + \Big\|\sum_{i=1}^{k_2} \varepsilon_i^2 x_{\alpha_i}^2\Big\|^p + \ldots + \Big\|\sum_{i=1}^{k_n} \varepsilon_i^n x_{\alpha_i}^n\Big\|^p\Big)^{1/p} \\ &\leqslant K \Big(\Big\|\sum_{i=1}^{k_1} x_{\alpha_i}^1\Big\|^p + \Big\|\sum_{i=1}^{k_2} x_{\alpha_i}^2\Big\|^p + \ldots + \Big\|\sum_{i=1}^{k_n} x_{\alpha_i}^n\Big\|^p\Big)^{1/p} \\ &= K \Big\|\sum_{i,m} x_{\alpha_i}^m\Big\|. \end{aligned} \tag{2}$$

We arrange X_α^n into one transfinite sequence $(X_\alpha\colon \omega \leqslant \alpha \leqslant \omega_1)$. Property $(\mathscr{P})$ follows from inequality (2). Since the density character of each subspace X_α equals $\aleph_0$, this is the unconditional decomposition. ∎

LEMMA 5. *The space $L_p\{-1, 1\}^{\omega_1}$ has an unconditional decomposition; here $1 < p < \infty$ and $\{-1, 1\}^{\omega_1}$ is the ω_1-th power of the dyadic set with the standard cylindrical σ-algebra and measure.*

Proof follows as a matter of fact by inspecting the paper [3]. The space $L_p\{-1, 1\}^{\omega_1}$ is the set of complex functions of variables $\vec{t} = (t_1, \ldots, t_\alpha, \ldots)$, $\alpha < \omega_1$, each variable taking the values ± 1. Put $r_0(\vec{t}) \equiv 1$, $r_\alpha(\vec{t}) = t_\alpha$ and $w_{\alpha_1\ldots\alpha_n} = r_{\alpha_1}(\vec{t})\, r_{\alpha_2}(\vec{t}) \ldots r_{\alpha_n}(\vec{t})$. The notation is not accidental here. If we fix a sequence $(\alpha_k)_1^\infty$, then $(r_{\alpha_k})_{k=1}^\infty$ is equivalent to the Rademacher sequence $(r_k)_1^\infty$ in the space $L_p[0, 1]$ and $w_{\alpha_{k_1}\ldots\alpha_{k_n}}$ to the Walsh sequence $w_{k_1\ldots k_n} = r_{k_1} \cdot r_{k_2} \cdot \ldots \cdot r_{k_n}$. Put

$$X_\omega = [r_0, w_{\alpha_1\ldots\alpha_n}\colon \alpha_i \leqslant \omega,\ n = 1, 2, \ldots]$$

and for $\alpha > \omega$

$$X_\alpha = [w_{\alpha_1\ldots\alpha_n}\colon \alpha_1 = \alpha,\ \alpha_i < \alpha \text{ if } i > 1,\ n = 1, 2, \ldots].$$

The subspaces X_α form an unconditional decomposition: this follows in fact from the unconditionality of the Haar basis in $L_p[0, 1]$, more exactly, from the unconditionality of the finite-dimensional decomposition $X_n = [w_{n,i_1,\ldots,i_{n-1}}\colon i_k < n]$ in the space $L_p[0, 1]$ [3].

LEMMA 6. *Let $(e_i\colon i \in I)$ be a Markushevich basis in the space $X = L_p(\mu)$, μ a finite measure, $1 < p < \infty$, $\operatorname{card} I = \aleph_1$. Then there exist an unconditional resolution $(P_\beta\colon \omega \leqslant \beta \leqslant \omega_1)$ and a decomposition of the index set $I = \bigcup I_\beta$, $\omega \leqslant \beta < \omega_1$, such that for any $\omega \leqslant \beta < \omega_1$, $X_\beta = [e_i\colon i \in I_\beta]$.*

Proof. By the Maharam theorem [7] the space X is isomorphic to $\bigoplus_{l_p} \sum_{n=1}^{\infty} L_p\{-1, 1\}^{\gamma_n}$ where γ_n either $= \omega_1$ or $\leqslant \omega$. From Lemma 5 it follows that the space $L_p\{-1, 1\}^{\omega_1}$ has an unconditional resolution; for $\gamma_n \leqslant \omega$ the space $L_p\{-1, 1\}^{\gamma_n}$ has the trivial projective resolution $P_\omega = I$. Hence, by Lemma 4, the space X has an unconditional projective resolution $(P_\alpha\colon \omega \leqslant \alpha \leqslant \omega_1)$. Besides, since e_i is a Markushevich basis of the reflexive space X, by using it we can construct a projective resolution $P'_\alpha\colon X \to X$, $\omega \leqslant \alpha \leqslant \omega_1$, for which there exists a splitting $I = \bigcup I'_\alpha$, $\omega \leqslant \alpha < \omega_1$, into countable subsets such that for every $\alpha > \omega$, $(P'_{\alpha+1} - P'_\alpha) X = [e_i\colon i \in I'_\alpha]$ (and $P'_\omega X = [e_i\colon i \in I'_\omega]$) (see, for example, [12]).

It now remains to apply Theorem 1 and Corollary 2 from [13] to obtain a subresolution $(P_{\alpha_\beta}\colon \omega \leqslant \beta \leqslant \omega_1)$ of P_α with $P_{\alpha_\beta} = P'_{\alpha_\beta}$. ∎

Remark. Instead of the Maharam theorem and Lemma 4 we can apply Lindenstrauss' result [14] from which it follows that the space $L_p(\mu)$, μ a finite measure, of density character $\aleph_1$ is isomorphic to $L_p\{-1, 1\}^{\omega_1}$.

LEMMA 7. *Let $(e_i\colon i \in I)$ be a (perhaps uncountable) unconditional basic sequence in the space $L_p(S, \sigma, \mu)$, μ a finite measure, $1 < p < \infty$, for which $|e_i(s)| \equiv 1$ on the set S for any i. Then there exist numbers c, C depending only upon p and the unconditional basic constant K of the sequence e_i such that for* *every finite choice* $(i_k\colon k = 1, \ldots, l)$ *and complex scalars* $(a_k)_1^l$

$$c\Big(\sum_{k=1}^{l} |a_k|^2\Big)^{1/2} \leqslant \Big\|\sum_{k=1}^{l} a_k e_{i_k}\Big\| \leqslant C\Big(\sum_{k=1}^{l} |a_k|^2\Big)^{1/2}. \tag{3}$$

Proof. This result is essentially known. Its proof is a simple modification of an idea of Orlicz [8]. We write Khintchine's inequality [6, p. 66] in a convenient way: There exist constants d, D depending only upon p such that for any sequence $(x_k)_1^l$ from $L_p(S, \sigma, \mu)$ and any $s \in S$

$$d\Big(\sum_{k=1}^{l} |x_k(s)|^2\Big)^{1/2} \leqslant \int_0^1 \Big|\sum_{k=1}^{l} r_k(u)\, x_k(s)\Big|\, du \tag{4}$$

$$\leqslant \Big(\int_0^1 \Big|\sum_{k=1}^{l} r_k(u)\, x_k(s)\Big|^p du\Big)^{1/p} \leqslant D\Big(\sum_{k=1}^{l} |x_k(s)|^2\Big)^{1/2},$$

where $r_k(u)$, $u \in [0, 1]$, is the Rademacher sequence. Since the integrals in the second and third terms of (4) are simply finite sums and L_p is a Köthe functional space, all terms in (4) belong to the space $L_p(S, \sigma, \mu)$ when s runs through the set S. Utilizing the monotonicity of the norm in the space L_p we have

$$d\Big\|\Big(\sum_{k=1}^{l} |x_k(s)|^2\Big)^{1/2}\Big\| \leqslant \Big\|\int_0^1 \Big|\sum_{k=1}^{l} r_k(u)\, x_k(s)\Big|\, du\Big\|$$

$$\leqslant \int_0^1 \Big\|\sum_{k=1}^{l} r_k(u)\, x_k(s)\Big\|\, du \leqslant \Big(\int_0^1 \Big\|\sum_{k=1}^{l} r_k(u)\, x_k(s)\Big\|^p du\Big)^{1/p}$$

$$= \Big[\int_0^1 \Big(\int_S \Big|\sum_{k=1}^{l} r_k(u)\, x_k(s)\Big|^p d\mu\Big) du\Big]^{1/p}$$

(change the order of integration)

$$= \Big\|\Big(\int_0^1 \Big|\sum_{k=1}^{l} r_k(u)\, x_k(s)\Big|^p du\Big)^{1/p}\Big\|$$

$$\leqslant D\Big\|\Big(\sum_{k=1}^{l} |x_k(s)|^2\Big)^{1/2}\Big\|.$$

Put $a_k e_{i_k}$ in place of x_k. Since $|e_{i_k}(s)| \equiv 1$ we obtain

$$d\Big(\sum_{k=1}^{l} |a_k|^2\Big)^{1/2} \leqslant \int_0^1 \Big\|\sum_{k=1}^{l} r_k(u)\, a_k e_{i_k}\Big\|\, du \leqslant D\Big(\sum_{k=1}^{l} |a_k|^2\Big)^{1/2}.$$

Write the middle term in detail:

$$\int_0^1 \Big\|\sum_{k=1}^{l} r_k(u)\, a_k e_{i_k}\Big\|\, du = \sum_{m=1}^{2^l} \frac{1}{2^l} \Big\|\sum_{k=1}^{l} \varepsilon_m^k a_k e_{i_k}\Big\|,$$

where $\varepsilon_m^k = \pm 1$. But for any choice of signs ε_k

$$K^{-1}\Big\|\sum_{k=1}^{l} \varepsilon_k a_k e_{i_k}\Big\| \leqslant \Big\|\sum_{k=1}^{l} a_k e_{i_k}\Big\| \leqslant K\Big\|\sum_{k=1}^{l} \varepsilon_k a_k e_{i_k}\Big\|.$$

Hence

$$K^{-1}\int_0^1 \Big\|\sum_{k=1}^{l} r_k(u)\, a_k e_{i_k}\Big\|\, du \leqslant \Big\|\sum_{k=1}^{l} a_k e_{i_k}\Big\| \leqslant K\int_0^1 \Big\|\sum_{k=1}^{l} r_k(u)\, a_k e_{i_k}\Big\|\, du.$$

Therefore (3) is valid with constants $c = dK^{-1}$, $C = DK$. ■

THEOREM 5. *Suppose the space* $X = L_p(S, \sigma, \mu)$, $1 < p < \infty$, dens $X = \aleph_1$, μ *a finite measure, has an M-basis* $(e_i\colon i \in I)$ *such that* $\forall i\ |e_i(s)| \equiv 1$. *Then there exists a splitting of the index set* $I = \bigcup_{n=1}^{\infty} I_n$ *into countably many subsets such that for every n, every finite choice* $(i_k \in I_n\colon k = 1, \ldots, l)$ *and complex scalars* $(a_k)_1^l$

$$c\Big(\sum_{k=1}^{l} |a_k|^2\Big)^{1/2} \leqslant \Big\|\sum_{k=1}^{l} a_k e_{i_k}\Big\| \leqslant C\Big(\sum_{k=1}^{l} |a_k|^2\Big)^{1/2};$$

moreover, the constants c, C *depend only upon* p *and* μ.

Proof. Let P_β be the unconditional projective resolution constructed in Lemma 6 for the M-basis e_i. Since X_β is separable, each set I_β is countable: $I_\beta = (i_n^\beta)_{n=1}^\infty$. Put $I_n = \{i_n^\beta\colon \omega \leqslant \beta < \omega_1\}$ for every n. Each set $\{e_i\colon i \in I_n\}$ is an uncountable unconditional basic sequence; moreover, unconditional basic constants of the sequences $(e_i\colon i \in I_n)$ are bounded by the unconditional constant of the projective resolution P_β. To finish the proof it remains to use Lemma 4. ■

Remark 1. Let the conditions of Theorem 4 be satisfied and suppose the M-basis e_i is an orthonormal system in the sense of inner product, i.e. biorthogonal to e_i are the functionals defined by the formula $f_i(x) = \int_S x(s)\, e_i(s)\, d\mu$. Then there exist a constant b depending only upon p and $\mu(S)$ such that the projections $P_n\colon X \to [e_i\colon i \in I_n]$ parallel to the subspaces $[e_i\colon i \notin I_n]$ satisfy $\|P_n\| \leqslant b$.

The proof is standard. Let first $p \geqslant 2$. Then

$$\|P_n x\|_p \leqslant C\|P_n x\|_2 \leqslant C\|P_n\|_2 \|x\|_2 = C\|x\|_2 \leqslant C\mu(S)^{1/2-1/p}\|x\|_p.$$

Hence $\|P_n\|_p \leqslant b = C\mu(S)^{1/2-1/p}$. The case $p < 2$ reduces to the preceding one by passing to the dual space. ■

Remark 2. Specifically, the Walsh functions $w_{\alpha_1 \ldots \alpha_n}$ in the space $L_p\{-1, 1\}^{\omega_1}$, described in the proof of Lemma 5, satisfy all the conditions of Remark 1 (see [3]).

Proof of Theorem 2. It is known that there exist a measurable space

(S, σ, μ) with the finite measure μ and a map φ: $\boldsymbol{R} \to S$ for which the operator I: $L_p(\mu) \to B_p$ defined by $(Ix)(t) = x(\varphi(t))$, $t \in \boldsymbol{R}$, $x \in L_p(\mu)$, is an isometry (see, for example, [9, Chapter 1]). It is obvious that $|x(s)| \equiv 1$ iff $|(Ix)(t)| \equiv 1$. Since we assume the continuum hypothesis, the space $L_p(\mu)$ has the density character $\aleph_1$, the inverse images $I^{-1}(x_\lambda)$ form an M-basis in $L_p(\mu)$ and $|I^{-1}(x_\lambda)| \equiv 1$. Therefore using Theorem 5 we obtain the required splitting of the real line $\boldsymbol{R} = \bigcup_{n=1}^{\infty} R_n$. The same observations as in Remark 1 prove the boundedness of the projections P_n: $B_p \to [x_\lambda: \lambda \in R_n]$. It is sufficient to consider the inner product

$$(x, y) = \lim_{T \to \infty} (2T)^{-1} \int_{-T}^{T} x(t)\, y(t)\, dt$$

with respect to which the system x_λ is orthogonal. ■

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