

Research Article

On Bounded Boundary and Bounded Radius Rotations

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We establish a relation between the functions of bounded boundary and bounded radius rotations by using three different techniques. A well-known result is observed as a special case from our main result. An interesting application of our work is also being investigated.

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1. Introduction

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. We say that $f \in A$ is subordinate to $g \in A$, written as $f < g$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in E with $w(0) = 0$ and $|w(z)| < 1$ ($z \in E$), such that $f(z) = g(w(z))$. In particular, when g is univalent, then the above subordination is equivalent to $f(0) = g(0)$ and $f(E) \subseteq g(E)$.

For any two analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in E), \quad (1.2)$$

the convolution (Hadamard product) of f and g is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in E). \quad (1.3)$$

We denote by $S^*(\alpha)$, $C(\alpha)$, ($0 \leq \alpha < 1$), the classes of starlike and convex functions of order α , respectively, defined by

$$\begin{aligned} S^*(\alpha) &= \left\{ f \in A: \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, z \in E \right\}, \\ C(\alpha) &= \{ f \in A: zf'(z) \in S^*(\alpha), z \in E \}. \end{aligned} \quad (1.4)$$

For $\alpha = 0$, we have the well-known classes of starlike and convex univalent functions denoted by S^* and C , respectively.

Let $P_k(\alpha)$ be the class of functions $p(z)$ analytic in the unit disc E satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{p(z) - \alpha}{1 - \alpha} \right| d\theta \leq k\pi, \quad (1.5)$$

where $z = re^{i\theta}$, $k \geq 2$, and $0 \leq \alpha < 1$. For $\alpha = 0$, we obtain the class P_k introduced in [1]. Also, for $p \in P_k(\alpha)$, we can write $p(z) = (1 - \alpha)q_1(z) + \alpha$, $q_1 \in P_k$. We can also write, for $p \in P_k(\alpha)$,

$$p(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + (1 - 2\alpha)ze^{-it}}{1 - ze^{-it}} d\mu(t), \quad z \in E, \quad (1.6)$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(t) = 2\pi, \quad \int_0^{2\pi} |d\mu(t)| \leq k\pi. \quad (1.7)$$

For (1.6) together with (1.7), see [2]. Since $\mu(t)$ has a bounded variation on $[0, 2\pi]$, we may write $\mu(t) = A(t) - B(t)$, where $A(t)$ and $B(t)$ are two non-negative increasing functions on $[0, 2\pi]$ satisfying (1.7). Thus, if we set $A(t) = ((k/4) + (1/2))\mu_1(t)$ and $B(t) = ((k/4) - (1/2))\mu_2(t)$, then (1.6) becomes

$$\begin{aligned} p(z) &= \left(\frac{k}{4} + \frac{1}{2} \right) \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + (1 - 2\alpha)ze^{-it}}{1 - ze^{-it}} d\mu_1(t) \\ &\quad - \left(\frac{k}{4} - \frac{1}{2} \right) \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + (1 - 2\alpha)ze^{-it}}{1 - ze^{-it}} d\mu_2(t). \end{aligned} \quad (1.8)$$

Now, using Herglotz-Stieltjes formula for the class $P(\alpha)$ and (1.8), we obtain

$$p(z) = \left(\frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) p_2(z), \quad z \in E, \quad (1.9)$$

where $P(\alpha)$ is the class of functions with real part greater than α and $p_i \in P(\alpha)$, for $i = 1, 2$.

We define the following classes:

$$\begin{aligned} R_k(\alpha) &= \left\{ f: f \in A \text{ and } \frac{zf'(z)}{f(z)} \in P_k(\alpha), 0 \leq \alpha < 1 \right\}, \\ V_k(\alpha) &= \left\{ f: f \in A \text{ and } \frac{(zf'(z))'}{f'(z)} \in P_k(\alpha), 0 \leq \alpha < 1 \right\}. \end{aligned} \quad (1.10)$$

We note that

$$f \in V_k(\alpha) \iff zf' \in R_k(\alpha). \quad (1.11)$$

For $\alpha = 0$, we obtain the well-known classes R_k and V_k of analytic functions with bounded radius and bounded boundary rotations, respectively. These classes are studied by Noor [3–5] in more details. Also it can easily be seen that $R_2(\alpha) = S^*(\alpha)$ and $V_2(\alpha) = C(\alpha)$.

Goel [6] proved that $f \in C(\alpha)$ implies that $f \in S^*(\beta)$, where

$$\beta = \beta(\alpha) = \begin{cases} \frac{4^\alpha(1-2\alpha)}{4-2^{2\alpha+1}}, & \alpha \neq \frac{1}{2}, \\ \frac{1}{2 \ln 2}, & \alpha = \frac{1}{2}, \end{cases} \quad (1.12)$$

and this result is sharp.

In this paper, we prove the result of Goel [6] for the classes $V_k(\alpha)$ and $R_k(\alpha)$ by using three different methods. The first one is the same as done by Goel [6], while the second and third are the convolution and subordination techniques.

2. Preliminary Results

We need the following results to obtain our results.

Lemma 2.1. *Let $f \in V_k(\alpha)$. Then there exist $s_1, s_2 \in S^*(\alpha)$ such that*

$$f'(z) = \frac{(s_1(z)/z)^{(k/4)+(1/2)}}{(s_2(z)/z)^{(k/4)-(1/2)}}, \quad z \in E. \quad (2.1)$$

Proof. It can easily be shown that $f \in V_k(\alpha)$ if and only if there exists $g \in V_k$ such that

$$f'(z) = (g'(z))^{1-\alpha}, \quad z \in E, \text{ see [2]}. \quad (2.2)$$

From Brannan [7] representation form for functions with bounded boundary rotations, we have

$$g'(z) = \frac{\left(\frac{g_1(z)}{z}\right)^{\left(\frac{k}{4}\right)^+ \left(\frac{1}{2}\right)}}{\left(\frac{g_2(z)}{z}\right)^{\left(\frac{k}{4}\right)^- \left(\frac{1}{2}\right)}}, \quad g_i \in S^*, i = 1, 2. \quad (2.3)$$

Now, it is shown in [8] that for $s_i \in S^*(\alpha)$, we can write

$$s_i(z) = z \left[\frac{g_i(z)}{z} \right]^{1-\alpha}, \quad g_i \in S^*, i = 1, 2. \quad (2.4)$$

Using (2.3) together with (2.4) in (2.2), we obtain the required result. \square

Lemma 2.2 (see [9]). *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$, and $\Psi(u, v)$ be a complex-valued function satisfying the conditions:*

- (i) $\Psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$,
- (ii) $(1, 0) \in D$ and $\operatorname{Re} \Psi(1, 0) > 0$,
- (iii) $\operatorname{Re} \Psi(iu_2, v_1) \leq 0$, whenever $(iu_2, v_1) \in D$ and $v_1 \leq -(1/2)(1 + u_2^2)$.

If $h(z) = 1 + c_1z + \dots$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $\operatorname{Re} \Psi(h(z), zh'(z)) > 0$ for $z \in E$, then $\operatorname{Re} h(z) > 0$ in E .

Lemma 2.3. *Let $\beta > 0$, $\beta + \gamma > 0$, and $\alpha \in [\alpha_0, 1)$, with*

$$\alpha_0 = \max \left\{ \frac{\beta - \gamma - 1}{2\beta}, \frac{-\gamma}{\beta} \right\}. \quad (2.5)$$

If

$$\left\{ h(z) + \frac{zh'(z)}{\beta h(z) + \gamma} \right\} < \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad (2.6)$$

then

$$h(z) < Q(z) < \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad (2.7)$$

where

$$Q(z) = \frac{1}{\beta G(z)} - \frac{\gamma}{\beta}, \quad (2.8)$$

$$G(z) = \int_0^1 \left[\frac{1-z}{1-tz} \right]^{2\beta(1-\alpha)} t^{\beta+\gamma-1} dt = \frac{{}_2F_1(2\beta(1-\alpha), 1, \beta+\gamma+1; z/(z-1))}{(\beta+\gamma)},$$

${}_2F_1$ denotes Gauss hypergeometric function. From (2.7), one can deduce the sharp result that $h \in P(\beta)$, with

$$\beta = \beta(\alpha, \beta, \gamma) = \min \operatorname{Re} Q(z) = Q(-1). \quad (2.9)$$

This result is a special case of the one given in [10, page 113].

3. Main Results

By using the same method as that of Goel [6], we prove the following result. We include all the details for the sake of completeness.

3.1. First Method

Theorem 3.1. *Let $f \in V_k(\alpha)$. Then $f \in R_k(\beta)$, where $\beta = \beta(\alpha)$ is given by (1.12). This result is sharp.*

Proof. Since $f \in V_k(\alpha)$, we use Lemma 2.1, with relation (1.11) to have

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} &= \left(\frac{k}{4} + \frac{1}{2}\right) \frac{zs_1'(z)}{s_1(z)} - \left(\frac{k}{4} - \frac{1}{2}\right) \frac{zs_2'(z)}{s_2(z)} \\ &= \left(\frac{k}{4} + \frac{1}{2}\right) \frac{(zf_1'(z))'}{f_1'(z)} - \left(\frac{k}{4} - \frac{1}{2}\right) \frac{(zf_2'(z))'}{f_2'(z)}, \end{aligned} \quad (3.1)$$

where $s_i \in S^*(\alpha)$ and $f_i \in C(\alpha)$, $i = 1, 2$.

Therefore, from (2.4), we have

$$\frac{zf'(z)}{f(z)} = \left(\frac{k}{4} + \frac{1}{2}\right) \frac{z[g_1(z)/z]^{1-\alpha}}{\int_0^z [g_1(\phi)/\phi]^{1-\alpha} d\phi} - \left(\frac{k}{4} - \frac{1}{2}\right) \frac{z[g_2(z)/z]^{1-\alpha}}{\int_0^z [g_2(\phi)/\phi]^{1-\alpha} d\phi}, \quad (3.2)$$

that is,

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= \left(\frac{k}{4} + \frac{1}{2}\right) \left[\int_0^z \left[\frac{z}{\phi} \right]^{1-\alpha} \left[\frac{g_1(\phi)}{g_1(z)} \right]^{1-\alpha} \frac{d\phi}{z} \right]^{-1} \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right) \left[\int_0^z \left[\frac{z}{\phi} \right]^{1-\alpha} \left[\frac{g_2(\phi)}{g_2(z)} \right]^{1-\alpha} \frac{d\phi}{z} \right]^{-1}, \end{aligned} \quad (3.3)$$

where we integrate along the straight line segment $[0, z]$, $z \in E$.

Writing

$$\frac{zf'(z)}{f(z)} = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} + \frac{1}{2}\right)p_2(z), \quad (3.4)$$

and using (3.3), we have

$$p_i(z) = \left[\int_0^z \left[\frac{z}{\phi} \right]^{1-\alpha} \left[\frac{g_i(\phi)}{g_i(z)} \right]^{1-\alpha} \frac{d\phi}{z} \right]^{-1}, \quad (3.5)$$

where $p_i(0) = 1$ and hence by [11] we have

$$\left| p_i(z) - \frac{1+r^2}{1-r^2} \right| \leq \frac{2r}{1-r^2}, \quad |z| = r, z \in E. \quad (3.6)$$

Therefore,

$$\min_{f_i \in C(\alpha)} \min_{|z|=r} \operatorname{Re}[p_i(z)] = \min_{f_i \in C(\alpha)} \min_{|z|=r} |p_i(z)|. \quad (3.7)$$

Let $z = re^{i\theta}$ and $\phi = Re^{i\theta}$, $0 < R < r < 1$. For fixed z and ϕ , we have from (2.4)

$$\left| \frac{g_i(\phi)}{g_i(z)} \right| \leq \frac{R}{r} \left(\frac{1+r}{1+R} \right)^2. \quad (3.8)$$

Now, using (3.8), we have, for a fixed $z \in E$, $|z| = r$,

$$\left| \int_0^z \left[\frac{z}{\phi} \right]^{1-\alpha} \left[\frac{g_i(\phi)}{g_i(z)} \right]^{1-\alpha} \frac{d\phi}{z} \right| \leq \int_0^r \left(\frac{1+r}{1+R} \right)^{2(1-\alpha)} \frac{dR}{r}. \quad (3.9)$$

Let

$$T(r) = \int_0^r \left(\frac{1+r}{1+R} \right)^{2(1-\alpha)} \frac{dR}{r}, \quad (3.10)$$

with $R = rt$, $0 < t < 1$, we have

$$T(r) = \int_0^1 \left(\frac{1+r}{1+rt} \right)^{2(1-\alpha)} dt. \quad (3.11)$$

By differentiating we note that

$$T'(r) = 2(1 - \alpha) \int_0^1 \frac{(1-t)}{(1+rt)^2} \left(\frac{1+r}{1+rt} \right)^{(1-2\alpha)} dt > 0, \quad (3.12)$$

and therefore $T(r)$ is a monotone increasing function of r and hence

$$\begin{aligned} \max_{0 \leq r \leq 1} T(r) &= T(1) = 2^{2(1-\alpha)} \int_0^1 \frac{dt}{(1+t)^{2(1-\alpha)}} \\ &= \begin{cases} \frac{(2-4^{(1-\alpha)})}{(2\alpha-1)}, & \text{if } \alpha \neq \frac{1}{2} \\ 2 \ln 2, & \text{if } \alpha = \frac{1}{2}. \end{cases} \end{aligned} \quad (3.13)$$

By letting

$$\beta(\alpha) = \min \left[\left[\int_0^z \left[\frac{z}{\phi} \right]^{1-\alpha} \left[\frac{g_i(\phi)}{g_i(z)} \right]^{1-\alpha} \frac{d\phi}{z} \right]^{-1}, \quad z \in E, \right] \quad (3.14)$$

for all $g_i(z) \in S^*$, we obtain the required result from (3.7), (3.13), and (3.14).

Sharpness can be shown by the function $f_0 \in V_k(\alpha)$ given by

$$\frac{(zf_0'(z))'}{f_0'(z)} = \left(\frac{k}{4} + \frac{1}{2} \right) \left(\frac{1 - (1-2\alpha)z}{1+z} \right) - \left(\frac{k}{4} - \frac{1}{2} \right) \left(\frac{1 + (1-2\alpha)z}{1-z} \right). \quad (3.15)$$

It is easy to check that $f_0 \in R_k(\beta)$, where β is the exact value given by (1.12). \square

3.2. Second Method

Theorem 3.2. Let $f \in V_k(\alpha)$. Then $f \in R_k(\beta)$, where

$$\beta = \frac{1}{4} \left[(2\alpha - 1) + \sqrt{4\alpha^2 - 4\alpha + 9} \right]. \quad (3.16)$$

Proof. Let

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= (1-\beta)p(z) + \beta \\ &= (1-\beta) \left[\left(\frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) p_2(z) \right] + \beta \end{aligned} \quad (3.17)$$

$p(z)$ is analytic in E with $p(0) = 1$. Then

$$\frac{(zf'(z))'}{f'(z)} = (1-\beta)p(z) + \beta + \frac{(1-\beta)zp'(z)}{(1-\beta)p(z) + \beta} \quad (3.18)$$

that is,

$$\begin{aligned} \frac{1}{1-\alpha} \left[\frac{(zf'(z))'}{f'(z)} - \alpha \right] &= \frac{1}{1-\alpha} \left[(1-\beta)p(z) + \beta - \alpha + \frac{(1-\beta)zp'(z)}{(1-\beta)p(z) + \beta} \right] \\ &= \frac{(\beta-\alpha)}{1-\alpha} + \frac{(1-\beta)}{1-\alpha} \left[p(z) + \frac{(1/(1-\beta))zp'(z)}{p(z) + (\beta/(1-\beta))} \right]. \end{aligned} \quad (3.19)$$

Since $f \in V_k(\alpha)$, it implies that

$$\frac{(\beta-\alpha)}{1-\alpha} + \frac{(1-\beta)}{1-\alpha} \left[p(z) + \frac{(1/(1-\beta))zp'(z)}{p(z) + (\beta/(1-\beta))} \right] \in P_k, \quad z \in E. \quad (3.20)$$

We define

$$\varphi_{a,b}(z) = \frac{1}{1+b} \frac{z}{(1-z)^a} + \frac{b}{1+b} \frac{z}{(1-z)^{1+a}}, \quad (3.21)$$

with $a = 1/(1-\beta)$, $b = \beta/(1-\beta)$. By using (3.17) with convolution techniques, see [5], we have that

$$\frac{\varphi_{a,b}(z)}{z} * p(z) = \left(\frac{k}{4} + \frac{1}{2} \right) \left[\frac{\varphi_{a,b}(z)}{z} * p_1(z) \right] - \left(\frac{k}{4} - \frac{1}{2} \right) \left[\frac{\varphi_{a,b}(z)}{z} * p_2(z) \right] \quad (3.22)$$

implies

$$p(z) + \frac{azp'(z)}{p(z)+b} = \left(\frac{k}{4} + \frac{1}{2} \right) \left[p_1(z) + \frac{azp'_1(z)}{p_1(z)+b} \right] - \left(\frac{k}{4} - \frac{1}{2} \right) \left[p_2(z) + \frac{azp'_2(z)}{p_2(z)+b} \right]. \quad (3.23)$$

Thus, from (3.20) and (3.23), we have

$$\frac{(\beta-\alpha)}{1-\alpha} + \frac{(1-\beta)}{1-\alpha} \left[p_i(z) + \frac{azp'_i(z)}{p_i(z)+b} \right] \in P, \quad i = 1, 2. \quad (3.24)$$

We now form the functional $\Psi(u, v)$ by choosing $u = p_i(z)$, $v = zp'_i(z)$ in (3.24) and note that the first two conditions of Lemma 2.2 are clearly satisfied. We check condition (iii) as follows:

$$\begin{aligned}
 \operatorname{Re}[\varphi(iu_2, v_1)] &= \frac{1}{1-\alpha} \left[(\beta - \alpha) + \operatorname{Re} \left(\frac{v_1}{iu_2 + (\beta/(1-\beta))} \right) \right] \\
 &= \frac{1}{1-\alpha} \left[(\beta - \alpha) + \frac{v_1(\beta/(1-\beta))}{u_2^2 + (\beta/(1-\beta))^2} \right] \\
 &\leq \frac{1}{1-\alpha} \left[(\beta - \alpha) - \frac{1}{2} \frac{(1+u_2^2)(\beta/(1-\beta))}{u_2^2 + (\beta/(1-\beta))^2} \right] \\
 &= \frac{2(\beta - \alpha)(u_2^2 + (\beta/(1-\beta))^2) - (1+u_2^2)(\beta/(1-\beta))}{2(u_2^2 + (\beta/(1-\beta))^2)(1-\alpha)} \tag{3.25} \\
 &= \frac{[2(\beta - \alpha)(\beta^2/(1-\beta)^2) - (\beta/(1-\beta))] + (2\beta - 2\alpha - (\beta/(1-\beta)))u_2^2}{2(u_2^2 + (\beta/(1-\beta))^2)(1-\alpha)} \\
 &= \frac{A + Bu_2^2}{2C}, \quad 2C > 0,
 \end{aligned}$$

where

$$\begin{aligned}
 A &= \frac{\beta}{(1-\beta)^2} [2(\beta - \alpha)\beta - (1-\beta)], \\
 B &= \frac{1}{1-\beta} (2(\beta - \alpha)(1-\beta) - \beta), \\
 C &= (1-\alpha) \left(u_2^2 + \left(\frac{\beta}{1-\beta} \right)^2 \right) > 0.
 \end{aligned} \tag{3.26}$$

The right-hand side of (3.25) is negative if $A \leq 0$ and $B \leq 0$. From $A \leq 0$, we have

$$\beta = \beta(\alpha) = \frac{1}{4} \left[(2\alpha - 1) + \sqrt{4\alpha^2 - 4\alpha + 9} \right], \tag{3.27}$$

and from $B \leq 0$, it follows that $0 \leq \beta < 1$.

Since all the conditions of Lemma 2.2 are satisfied, it follows that $p_i \in P$ in E for $i = 1, 2$ and consequently $p \in P_k$ and hence $f \in R_k(\beta)$, where β is given by (3.16). The case $k = 2$ is discussed in [12]. \square

3.3. Third Method

Theorem 3.3. Let $f \in V_k(\alpha)$. Then $f \in R_k(\beta)$, where

$$\beta = \beta_1(\alpha, 1, 0) = \begin{cases} \frac{2\alpha - 1}{2 - 2^{2(1-\alpha)}}, & \text{if } \alpha \neq \frac{1}{2}, \\ \frac{1}{2 \ln 2}, & \text{if } \alpha = \frac{1}{2}. \end{cases} \quad (3.28)$$

Proof. Let

$$\frac{zf'(z)}{f(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) \frac{zs'_1(z)}{s_1(z)} - \left(\frac{k}{4} - \frac{1}{2}\right) \frac{zs'_2(z)}{s_2(z)}, \quad (3.29)$$

and let

$$\frac{zs'_i(z)}{s_i(z)} = p_i(z), \quad i = 1, 2. \quad (3.30)$$

Then p, p_i are analytic in E with $p(0) = 1, p_i(0) = 1, i = 1, 2$.

Logarithmic differentiation yields

$$\begin{aligned} \frac{(zf'(z))'}{f'(z)} &= p(z) + \frac{zp'(z)}{p(z)} = \left(\frac{k}{4} + \frac{1}{2}\right) \frac{(zs'_1(z))'}{s'_1(z)} - \left(\frac{k}{4} - \frac{1}{2}\right) \frac{(zs'_2(z))'}{s'_2(z)} \\ &= \left(\frac{k}{4} + \frac{1}{2}\right) \left(p_1(z) + \frac{zp'_1(z)}{p_1(z)}\right) - \left(\frac{k}{4} - \frac{1}{2}\right) \left(p_2(z) + \frac{zp'_2(z)}{p_2(z)}\right). \end{aligned} \quad (3.31)$$

Since $f \in V_k(\alpha)$, it follows that $(zs'_i)' / s'_i \in P(\alpha), z \in E$, or $s_i \in C(\alpha)$ for $z \in E$. Consequently,

$$\left(p_i(z) + \frac{zp'_i(z)}{p_i(z)}\right) \in P(\alpha), \quad (3.32)$$

where $zs'_i(z)/s_i(z) = p_i(z), i = 1, 2$. We use Lemma 2.3 with $\gamma = 0, \beta = 1 > 0, \alpha \in [0, 1)$, and $h = p_i$ in (3.32), to have $p_i \in P(\beta)$, where β is given in (3.28) and this estimate is best possible, extremal function Q is given by

$$Q(z) = \begin{cases} \frac{(1-2\alpha)z}{(1-z)[1-(1-z)^{1-2\alpha}]}, & \text{if } \alpha \neq \frac{1}{2}, \\ \frac{z}{(z-1)\log(1-z)}, & \text{if } \alpha = \frac{1}{2}, \end{cases} \quad (3.33)$$

see [10]. MacGregor [13] conjectured the exact value given by (3.28). Thus $s_i \in S^*(\beta)$ and consequently $f \in R_k(\beta)$, where the exact value of β is given by (3.28). \square

3.4. Application of Theorem 3.3

Theorem 3.4. Let g and h belong to $V_k(\alpha)$. Then $F(z)$, defined by

$$F(z) = \int_0^z \left(\frac{g(t)}{t} \right)^\mu \left(\frac{h(t)}{t} \right)^\eta dt, \quad (3.34)$$

is in the class $V_k(\delta)$, where $0 \leq \mu < \eta \leq 1$, $\delta = \delta(\alpha) = (1 - (\mu + \eta)(1 - \beta))$, and $\beta(\alpha)$ is given by (1.12).

Proof. From (3.34), we can easily write

$$\frac{(zF'(z))'}{F'(z)} = \mu \frac{zg'(z)}{g(z)} + \eta \frac{zh'(z)}{h(z)} + 1 - (\mu + \eta). \quad (3.35)$$

Since g and h belong to $V_k(\alpha)$, then, by Theorem 3.3, $zg'(z)/g(z)$ and $zh'(z)/h(z)$ belong to $P_k(\beta)$, where $\beta = \beta(\alpha)$ is given by (1.12). Using

$$\begin{aligned} \frac{zg'(z)}{g(z)} &= (1 - \beta)q_1(z) + \beta, & q_1 &\in P_k, \\ \frac{zh'(z)}{h(z)} &= (1 - \beta)q_2(z) + \beta, & q_2 &\in P_k, \end{aligned} \quad (3.36)$$

in (3.35), we have

$$\frac{1}{1 - \delta} \left[\frac{(zF'(z))'}{F'(z)} - \delta \right] = \frac{\mu}{\mu + \eta} q_1(z) + \frac{\eta}{\mu + \eta} q_2(z). \quad (3.37)$$

Now by using the well-known fact that the class P_k is a convex set together with (3.37), we obtain the required result. \square

For $\alpha = 0$, $\mu = 0$, and $\eta = 1$, we have the following interesting corollary.

Corollary 3.5. Let f belongs to $V_k(0)$. Then $F(z)$, defined by

$$F(z) = \int_0^z \frac{f(t)}{t} dt \quad (\text{Alexander's integral operator}), \quad (3.38)$$

is in the class $V_k(1/2)$.

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