

On boundedness and discontinuity of additive functions

by

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Let f be a real-valued function defined on the n -dimensional Euclidean space R_n ; f is said to be *additive* if it satisfies the functional equation

$$(1) \quad f(x+y) = f(x) + f(y)$$

for each $x, y \in R_n$. R. Ger and Marek Kuczma [6] have introduced the following classes of sets: A set $T \subset R_n$ belongs to the class \mathcal{B} if and only if each additive function upper-bounded on T is continuous. A set $T \subset R_n$ belongs to the class \mathcal{C} if and only if each additive function bounded (bilaterally) on T is continuous. It is known that $\mathcal{B} \neq \mathcal{C}$, see [6] or [10].

There are known various conditions upon T which are sufficient for $T \in \mathcal{B}$, resp. $T \in \mathcal{C}$, and also conditions upon T which are necessary for $T \in \mathcal{B}$, resp. $T \in \mathcal{C}$. For example, if T has a positive inner Lebesgue measure, then $T \in \mathcal{B}$, i.e. each additive function upper-bounded on T is continuous. This is the famous theorem of A. Ostrowski [18] generalized by Á. Császár [2] and S. Marcus [15]. Other results can be found in [8], [17], [14], [11], [12], [16], [4], [5], [6], [10], [9]. However, none of those results gives a condition which is both sufficient and necessary for $T \in \mathcal{B}$, resp. $T \in \mathcal{C}$. And the following is the problem of M. Kuczma [13]: give a characterization of the members of the classes \mathcal{B} and \mathcal{C} . The present paper is devoted to this problem.

The set R_n can be interpreted as a vector space over the field Q of rational numbers, and each additive function $f: R_n \rightarrow R$ (R denotes the set R_1) as a morphism from the vector space R_n to the vector space R , since each such additive function is Q -linear, i.e. $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, for $\alpha, \beta \in Q$, $x, y \in R_n$, see [1]. Hence it is natural to apply some methods and results from vector analysis to the functional equation (1). Let E be a vector space over Q . Then each basis of E is referred to as a Hamel basis. A subset C of a vector space E is Q -convex if

$$(1-a)C + aC \subset C$$

whenever $a \in Q$, $0 \leq a \leq 1$. Let A be a subset of E and let x be a point

from A . Then A is Q -radial at the point x if for each $y \in E$ there is a positive real c_y such that $x + ay \in A$ whenever $|a| < c_y$, $a \in Q$.

Marcin E. Kuczma [10] has considered the subsets $T \subset R_n$ which are both Q -convex and Q -radial at some point. He has shown that such a set T belongs to \mathfrak{B} if and only if T contains an open sphere. In the present paper there is proved a similar result with the class \mathfrak{B} replaced by \mathfrak{C} (see Theorem 3 below) and this result is a key to the characterization of the class \mathfrak{C} (see Theorem 4 below). However, a similar characterization of the class \mathfrak{B} is impossible (see Remark 1 after Theorem 4). In the paper we use some ideas from the above-mentioned paper of Marcin E. Kuczma.

Finally, here are some remarks concerning the notation. In the sequel, the usual set-theoretic operations are denoted by \cup , \cap , \setminus (difference), \times . The symbols $+$, $-$, \cdot , \sum , denote the algebraic operations; they may be applied also to sets, e.g. $A+B$ denotes the set of all elements of the form $a+b$, $a \in A$, $b \in B$. Rational numbers are always denoted by Greek letters.

Theorem 1, which follows presently is an analogon of the famous Hahn-Banach theorem on linear functionals, applied to vector spaces over Q . See also [10]. First we prove the following

LEMMA. Let E be a vector space over Q , and let X and Y be subspaces of E such that $X \subset Y$ and X has codimension 1 in Y ; let C be a Q -convex subset of E which is Q -radial at 0 and symmetric with respect to 0 (i.e. $C = -C$); finally, let $f: X \rightarrow R$ be an additive function such that $|f(x)| \leq 1$ for $x \in X \cap C$. Then there exists an additive function $g: Y \rightarrow R$ which is an extension of f and is such that $|g(x)| \leq 1$ for $x \in Y \cap C$.

Proof. According to the supposition we may write

$$Y = X + Qy$$

where y is a point in $Y \setminus X$. Consider the following sets $U, V \subset X \times Q$:

$$U = \{(x, \xi): x \in X, \xi > 0, (x-y)/\xi \in C\},$$

$$V = \{(x, \xi): x \in X, \xi > 0, (x+y)/\xi \in C\}.$$

Since C is Q -radial at 0, for each $x \in X$ there is a number $\xi > 0$ such that $(x \pm y)/\xi \in C$. Hence U and V are non-empty sets.

Write

$$u = \sup \{f(x) - \xi: (x, \xi) \in U\},$$

$$u' = \inf \{f(x) + \xi: (x, \xi) \in U\},$$

$$v = \inf \{\xi - f(x): (x, \xi) \in V\},$$

$$v' = \sup \{-\xi - f(x): (x, \xi) \in V\}.$$

We are going to show that

$$(2) \quad u' \geq u, \quad v \geq v', \quad v \geq u, \quad \text{and} \quad u' \geq v'.$$

Suppose, on the contrary, that $u' < u$. Then there exist $(x_1, \xi_1) \in U$ and $(x_2, \xi_2) \in U$ such that $f(x_1) + \xi_1 < f(x_2) - \xi_2$; hence, by the Q -linearity of f ,

$$(3) \quad f\left(\frac{x_2 - x_1}{\xi_1 + \xi_2}\right) > 1.$$

On the other hand, $(x_1 - y)/\xi_1 \in C$, $(x_2 - y)/\xi_2 \in C$. Since C is Q -convex and symmetric with respect to 0, we have

$$x = \frac{x_2 - x_1}{\xi_1 + \xi_2} = \frac{\xi_2}{\xi_1 + \xi_2} \cdot \frac{x_2 - y}{\xi_2} - \frac{\xi_1}{\xi_1 + \xi_2} \cdot \frac{x_1 - y}{\xi_1} \in C.$$

Thus $x \in X \cap C$, which implies $|f(x)| \leq 1$, contrary to (3).

Similarly, let $v' > v$. Then there exist $(z_1, \eta_1) \in V$, $(z_2, \eta_2) \in V$ such that $-\eta_1 - f(z_1) > \eta_2 - f(z_2)$, and hence

$$(4) \quad f(z) > 1,$$

where $z = (z_2 - z_1)/(\eta_1 + \eta_2)$. On the other hand, $(z_1 + y)/\eta_1 \in C$, $(z_2 + y)/\eta_2 \in C$; hence from the Q -convexity and symmetry of C , similarly to the preceding case, it follows that $z \in C$. Thus $|f(z)| \leq 1$, contrary to (4).

If $v < u$, then there exist $(w_1, \xi_1) \in V$, $(w_2, \xi_2) \in U$ such that $\xi_1 - f(w_1) < f(w_2) - \xi_2$, or equivalently,

$$(5) \quad f(w) > 1,$$

where $w = (w_1 + w_2)/(\xi_1 + \xi_2)$. On the other hand, $(w_1 + y)/\xi_1 \in C$, $(w_2 - y)/\xi_2 \in C$; hence, by the Q -convexity of C , $w \in C$, and consequently $f(w) \leq 1$, contrary to (5).

Finally, assume that $u' < v'$. Then there are $(t_1, \tau_1) \in U$, $(t_2, \tau_2) \in V$ such that $f(t_1) + \tau_1 < -f(t_2) - \tau_2$, so

$$(6) \quad f(t) < -1,$$

where $t = (t_1 + t_2)/(\tau_1 + \tau_2)$. On the other hand, $(t_1 - y)/\tau_1 \in C$, $(t_2 + y)/\tau_2 \in C$; hence, by the Q -convexity of C , $t \in C$, and consequently $f(t) \geq -1$, contrary to (6).

Thus (2) holds. Hence there exists a real number c such that $v \geq c \geq v'$ and $u' \geq c \geq u$. Define a function g by

$$g(x) = f(x) \quad \text{for } x \in X, \quad g(y) = c,$$

and extend it by Q -linearity onto the whole of Y . It remains to verify that $|g(z)| \leq 1$, for each $z \in Y \cap C$.

Let $x + ay \in Y \cap C$, where $x \in X$; we may assume that $a \neq 0$. If $a > 0$, then $\left(\frac{x}{a}, \frac{1}{a}\right) \in V$; hence

$$\frac{1}{a}(1-f(x)) \geq v \geq c \quad \text{and} \quad \frac{1}{a}(-1-f(x)) \leq v' \leq c,$$

and so

$$-1 \leq g(x+ay) = f(x) + ac \leq 1.$$

If $a < 0$, then $\left(-\frac{x}{a}, -\frac{1}{a}\right) \in U$, whence

$$\frac{1}{a}(1-f(x)) \leq u \leq c \quad \text{and} \quad \frac{1}{a}(-1-f(x)) \geq u' \geq c,$$

and so again

$$-1 \leq g(x+ay) \leq 1.$$

Now we are able to formulate the following

THEOREM 1. *Let E be a vector space over Q , X a subspace of E , and C a Q -convex subset of E , Q -radial at 0 and symmetric with respect to 0. If $f: X \rightarrow R$ is an additive function which is bounded on the set $X \cap C$, then there exists an additive function $F: E \rightarrow R$ which is an extension of f and which is bounded on C .*

Proof. The proof of the theorem is based on the lemma proved above and on Zorn's lemma. It is much the same as the proof of the Hahn-Banach theorem and so we omit it. The reader is referred to [3], [19] or [10].

In Theorems 2 and 3, which follow, there are constructed certain discontinuous additive functions. Theorem 2 is devoted to functions defined on R and Theorem 3 is a generalization of Theorem 2 to functions defined on R_n .

THEOREM 2. *Let C be a Q -convex subset of the real line, Q -radial at 0 and symmetric with respect to 0. Then either C is an interval or there exists a discontinuous additive function $F: R \rightarrow R$ bounded (bilaterally) on C .*

Proof. Assume that C is not an interval. The set C is dense in itself, since if $z \in C$ then also each $az \in C$, for $a \in Q$, $|a| \leq 1$. From this and from the Q -convexity of C it follows that C contains no non-trivial interval. Hence there exist reals x and y such that

$$(7) \quad x \in C, \quad y \in C, \quad 0 < |x| < \frac{1}{2}|y|.$$

The proof of Theorem 2 is based on Theorem 1. We construct a discontinuous additive function g on the set $Qx + Qy$, bounded on the set $(Qx + Qy) \cap C$, and then we extend g to the desired function F .

For each integer $k \geq 2$ define a set $A_k \subset Q \times Q$ by

$$A_k = \{(\alpha, \beta); \alpha x + \beta y \in C, \alpha, \beta \in Q, \alpha > k\}.$$

It is easy to verify that if $(\alpha, \beta) \in A_k$, then $\beta \neq 0$; otherwise there would exist some $a > k > 1$ such that $ax \in C$ and hence, by the Q -convexity of C , $\frac{1}{a}(ax) = x \in C$, contrary to (7). Thus we can define the sets $B_k \subset Q$ as follows:

$$B_k = \left\{ \frac{\alpha}{\beta}; (\alpha, \beta) \in A_k \right\}.$$

If there exists a k such that $A_k = \emptyset$, define an additive function g as follows: $g(ax + \beta y) = a$. By the symmetry of C , if $(a'x + \beta'y) \in C$, then $|a'| \leq k$. Hence g is bounded on $C \cap (Qx + Qy)$. Now Theorem 1 guarantees the existence of an additive function F , bounded on C and such that F is an extension of g . Clearly F is discontinuous since, for each $\alpha \in Q$, $F(\alpha y) = g(\alpha y) = 0$.

Thus we may assume that $A_k \neq \emptyset$, for each k . We show that all numbers from B_k have the same sign. Indeed, if there are positive rationals $\alpha_1, \alpha_2, \beta_1, \beta_2$, such that $\alpha_1 x + \beta_1 y \in C$, $\alpha_2 x - \beta_2 y \in C$, $\alpha_1 > k \geq 2$, $\alpha_2 > k \geq 2$, and, say, $\beta_1 \geq \beta_2$ (in the case of $\beta_1 \leq \beta_2$ the proof is similar), then by the Q -convexity and symmetry of C we have

$$\frac{1}{2} \left(\frac{\beta_2}{\beta_1} (\alpha_1 x + \beta_1 y) + (\alpha_2 x - \beta_2 y) \right) = \frac{1}{2} \left(\frac{\beta_2 \alpha_1}{\beta_1} + \alpha_2 \right) x \in C,$$

whence $\frac{\beta_2 \alpha_1}{\beta_1} + \alpha_2 < 2$; on the other hand, $\alpha_2 > 2$, $\frac{\beta_2 \alpha_1}{\beta_1} > 0$, and so

$$\frac{\beta_2 \alpha_1}{\beta_1} + \alpha_2 > 2 \quad \text{— a contradiction.}$$

In the sequel we may assume without loss of generality that each set B_k contains positive numbers (otherwise it suffices to replace y by $-y$ in (7)).

We show that, for each $(\alpha, \beta) \in A_k$,

$$(8) \quad \frac{\alpha}{\beta} < 2.$$

Assume, on the contrary, that $\frac{\alpha}{\beta} \geq 2$ for some $(\alpha, \beta) \in A_k$. If $\beta \leq 1$, then

$$\begin{aligned} \beta y \in C \text{ and hence, by the } Q\text{-convexity and symmetry of } C, \frac{1}{2}((\alpha x + \beta y) - \beta y) \\ = \left(\frac{\alpha}{2}\right) x \in C; \text{ but } \alpha > n \geq 2, \text{ whence } \frac{2}{\alpha} \left(\frac{\alpha}{2} x\right) = x \in C, \text{ contrary to (7). If} \end{aligned}$$

$\beta > 1$, then $\frac{1}{2}\left(\frac{1}{\beta}(ax + \beta y) - y\right) = \left(\frac{1-\alpha}{2\beta}\right)x \in C$, which again implies $x \in C$ — a contradiction.

Since, for each $k \geq 2$, $B_k \supset B_{k+1}$, there exists a $\lim_{k \rightarrow \infty} (\sup B_k) = c$.

In view of (8), $0 \leq c \leq 2$. Define an additive function g on the set $Qx + Qy$ as follows: $g(ax + \beta y) = \alpha - \beta c$. We show that g is bounded on $(Qx + Qy) \cap C$. Let $(ax + \beta y) \in C$. Since C is symmetric, it suffices to consider the case of $\beta > 0$. Let $\varepsilon > 0$. For each integer $k \geq 2$ choose a pair $(\alpha_k, \beta_k) \in A_k$ such that $\left|\frac{\alpha_k}{\beta_k} - c\right| < \varepsilon$. Since $\lim_{k \rightarrow \infty} \alpha_k = +\infty$, from (8) we have $\lim_{k \rightarrow \infty} \beta_k = +\infty$.

Let m be an integer such that $\beta_m > \beta$. By the Q -convexity and symmetry of C we have $\frac{\beta}{\beta_m}(a_mx + \beta_my) = \left(\frac{\beta a_m}{\beta_m}x + \beta y\right) = \omega x + \beta y \in C$ and

$$(9) \quad -\varepsilon + \frac{\omega}{\beta} < c < \varepsilon + \frac{\omega}{\beta}.$$

By the Q -convexity and symmetry of C we have $\frac{1}{2}((ax + \beta y) - (\omega x + \beta y)) = \frac{\alpha - \omega}{2}x \in C$, whence

$$(10) \quad -2 < \alpha - \omega < 2.$$

From (9) and (10) it follows that

$$g(ax + \beta y) = \alpha - \beta c < \alpha + \beta \varepsilon - \omega < 2 + \beta \varepsilon,$$

and similarly

$$g(ax + \beta y) > \alpha - \beta \varepsilon - \omega > -2 - \beta \varepsilon.$$

Since ε is arbitrary, we get

$$-2 \leq g(ax + \beta y) \leq 2.$$

Thus g is bounded on the set $(Qx + Qy) \cap C$.

Finally, we show that g is discontinuous on $Qx + Qy$. Let $\eta > 0$. Choose a positive rational number $\delta < 2$ such that $|\delta - c| < \eta$. For each β such that $|\beta| < 1/\eta$ we have $|g(\beta \delta x + \beta y)| = |\beta \delta - \beta c| = |\beta||\delta - c| < 1/\eta \cdot \eta = 1$. On the other hand, since $0 < \delta < 2$, we get from (7) $|\delta x + y| > |y|/3$. Thus the set of those z from $Qx + Qy$ for which $|g(z)| < 1$ is dense in the interval $(-|y|/3\eta, |y|/3\eta)$. Since η is arbitrary, we conclude that there is a subset A of $Qx + Qy$ which is dense in R and is such that $|g(z)| < 1$ for $z \in A$. But $g(x) = 1$, whence g is a non-zero additive function. From this it follows that g is discontinuous on $Qx + Qy$.

Now Theorem 1 applied to the function g guarantees the existence of a desired extension F of g , q.e.d.

THEOREM 3. Let C be a Q -convex subset of the n -dimensional Euclidean space R_n , Q -radial at 0, and symmetric with respect to 0. Then either C contains a sphere centred at 0 or there exists a discontinuous additive function $F: R_n \rightarrow R$ bounded on C .

Proof. Let e_1, e_2, \dots, e_n be the usual orthonormal basis for R_n . If for each i , $1 \leq i \leq n$, $C_i = R \cdot e_i \cap C$ is an interval (which is, by the symmetry of C , centred at 0) let d be the length of the smallest interval C_i . Let (x_1, x_2, \dots, x_n) , where $x_i \in R$, be a point from R_n such that $|x_i| < d/n$, for each i . Then $nx_i \cdot e_i \in C_i \subset C$ and hence, by the Q -convexity of C ,

$$(x_1, x_2, \dots, x_n) = \frac{1}{n}(nx_1 \cdot e_1 + nx_2 \cdot e_2 + \dots + nx_n \cdot e_n) \in C,$$

and consequently C contains a sphere.

Thus we may assume that, for some i , C_i is not an interval. In this case, by Theorem 2, there exists a discontinuous additive function $f: R \cdot e_i \rightarrow R$, bounded on $R \cdot e_i \cap C = C_i$, which can be extended, by Theorem 1, to a discontinuous additive function $F: R_n \rightarrow R$ bounded on C , q.e.d.

In the next sections we shall use the following notation: If T is a subset of R_n , let $Q(T)$ denote the Q -convex hull of T .

Now we are able to prove the main result.

THEOREM 4. Let T be a subset of the n -dimensional Euclidean space R_n . Then each additive function $f: R_n \rightarrow R$ bounded on T is continuous if and only if the Q -convex hull of $T - T$ contains a sphere.

In other words, $T \in C$ if and only if $Q(T - T)$ contains a ball.

Proof. Assume that $Q(T - T)$ contains a certain sphere. Let f be bounded on T , i.e. let $|f(x)| \leq M$, for $x \in T$. Then f is also bounded on $T - T$ with the bounding constant $2M$. If $z \in Q(T - T)$, then $z = a_1 x_1 + \dots + a_m x_m$, where $x_i \in T - T$, $a_i \in Q$, $a_i > 0$, $a_1 + \dots + a_m = 1$. Hence

$$\begin{aligned} |f(z)| &= |a_1 f(x_1) + \dots + a_m f(x_m)| \leq a_1 |f(x_1)| + \dots + a_m |f(x_m)| \\ &\leq (a_1 + \dots + a_m) 2M = 2M; \end{aligned}$$

thus f is bounded on the set $Q(T - T)$ of positive inner Lebesgue measure and consequently f is continuous (see [15]).

Now assume that $Q(T - T)$ contains no sphere. Clearly $Q(T - T)$ is Q -convex and symmetric with respect to 0. If $Q(T - T)$ is also Q -radial at 0, then there exists, by Theorem 3, a discontinuous additive function $f: R_n \rightarrow R$, bounded on $Q(T - T)$. Let a be a fixed point from T . Since $T - a \subset Q(T - T)$, we conclude that f is bounded on $T - a$ and consequently f is also bounded on T .

Hence it remains to consider the case where $Q(T - T)$ is not Q -radial at 0. In this case $Q(T - T)$ cannot contain a Hamel basis. Indeed, if H is

a Hamel basis contained in $Q(T-T)$, then each element $w \in R_n$ can be written in the form $w \in \sum a_i b_i$ (finite sum) where $b_i \in H \cup -H \subset Q(T-T)$, $a_i > 0$. From the Q -convexity of $Q(T-T)$ it follows that $w/\sum a_i \in Q(T-T)$ and consequently $aw \in Q(T-T)$ for each positive $a \leq \sum a_i$. Thus $Q(T-T)$ is Q -radial at 0 — a contradiction. Hence $Q(T-T)$ does not contain any Hamel basis and so the vector subspace (over Q) spanned by $Q(T-T)$ cannot be the whole R_n . To finish our proof we use the following result of R. Ger and Marek Kuczma [6]: If $A \in \mathcal{C}$, then the vector subspace spanned by A is R_n . Hence $Q(T-T) \notin \mathcal{C}$ and, similarly to the preceding case, we conclude that also $T \notin \mathcal{C}$, q.e.d.

Remark 1. In connection with Theorem 4 one may expect that the sets from the class \mathcal{B} can be characterized as follows: A set T is in \mathcal{B} if and only if the Q -convex hull of T contains a sphere, or at least has the positive inner Lebesgue measure. However, this hypothesis is false as is shown on an example by Marcin E. Kuczma [10].

Remark 2. A set $A \subset R_n$ is called to be *midpoint convex* if for each $x, y \in A$, $\frac{1}{2}(x+y) \in A$. R. Ger and Marek Kuczma [6] have proved the following result: Let $T \subset R_n$, and let $J(T)$ denote the midpoint convex hull of T . If the set $J(T) - J(T)$ has a positive inner Lebesgue measure, then $T \in \mathcal{C}$. The authors have conjectured that this condition is not necessary for $T \in \mathcal{C}$. Their conjecture is true, as can be shown on a rather complicated example. In fact, there exists a midpoint convex symmetric set $T \in \mathcal{C}$, which has the zero inner measure. Consequently, Q -convexity in Theorem 4 cannot be replaced by midpoint convexity.

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