ON BROWNIAN MOTION, BOLTZMANN'S EQUATION, AND THE FOKKER-PLANCK EQUATION*

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Abstract. In order to describe Brownian motion rigorously, Boltzmann's integral equation must be used. The Fokker-Planck type of equation is only an approximation to the Boltzmann equation and its domain of validity is worth examining.

A treatment of the Brownian motion in velocity space of a particle with known initial velocity based on Boltzmann's integral equation is given. The integral equation, which employs a suitable scattering kernel, is solved and its solution compared with that of the corresponding Fokker-Planck equation. It is seen that when M/m, the mass ratio of the particles involved, is sufficiently high and the dispersion of the velocity distribution sufficiently great, the Fokker-Planck equation is an excellent description. Even when the dispersion is small, the first and second moments of the Fokker-Planck solution are reliable. The higher moments, however, are then in considerable error—an error which becomes negligible as the dispersion increases.

1. In the treatment of Brownian motion, it is customary to assume a Langevin equation and simple dynamical statistics of the individual collisions and then to deduce a Fokker-Planck equation describing the random motion of the heavy particle. The Fokker-Planck equation obtained is a second-order partial differential equation and the absence of higher-order differential terms is inferred directly from the above assumptions. As will be seen, the solution of this Fokker-Planck equation does not provide a completely satisfactory physical description. Consequently, the assumptions underlying the equation cannot be correct [1, 2, 3, 4] and the extent of their approximate validity comes under question.

That the solution of the Fokker-Planck equation is not a wholly satisfactory representation of Brownian motion may be seen in the following way. Consider a heavy particle known to have the velocity $\mathbf{v}_0$ at $t = 0$. For all subsequent time, there is a finite probability that the particle will have undergone no collision. It must, therefore, be expected that the probability density $w(\mathbf{v}, t)$** describing the stochastic motion in velocity space will always have a singular component of the form $f(t)\delta(\mathbf{v} - \mathbf{v}_0)$, where $\delta(\mathbf{v} - \mathbf{v}_0)$ is the Dirac delta-function. If one were to try to describe the motion by the Fokker-Planck equation

$$\frac{\partial w(\mathbf{v}, t)}{\partial t} = \frac{1}{2} D\nabla^2 w(\mathbf{v}, t) + \eta \nabla \cdot \{\mathbf{v} w(\mathbf{v}, t)\}, \tag{1}$$

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**The function $w(\mathbf{v}, 0)$ obeying (2) is often represented in the literature by $P_2(\mathbf{v}_0/\mathbf{v}; t)$, the probability density for velocity $\mathbf{v}$, t seconds after there is a known velocity $\mathbf{v}$.

subject to the initial condition

$$w(\mathbf{v}, 0) = \delta(\mathbf{v} - \mathbf{v}_0), \tag{2}$$

no such singular component would be available in the solution. The immediate disappearance of an initial singularity is, indeed, characteristic of all diffusion equations of finite order. Only by means of an integral equation can such a singularity be maintained.

All of this, of course, is in keeping with the fact that fundamental to the description of Brownian motion is Boltzmann's equation, an integral equation of the type desired [4]. If $A(\mathbf{v}', \mathbf{v})\, d\mathbf{v}$ is the probability per unit time that a particle with velocity $\mathbf{v}'$ will undergo a transition to a volume $d\mathbf{v}$ about $\mathbf{v}$, the Boltzmann equation describing the motion is

$$\frac{\partial w(\mathbf{v}, t)}{\partial t} = \int w(\mathbf{v}', t) A(\mathbf{v}', \mathbf{v})\, d\mathbf{v}' - w(\mathbf{v}, t) \int A(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}'. \tag{3}$$

This is simply an expression of the fact that the rate of change of the population of a cell in velocity space is the difference between the rate of departures from the cell and the rate of arrivals.

From this Boltzmann equation a corresponding Fokker-Planck equation may be derived. If Eq. (3) is multiplied by an arbitrary, but suitably behaved function $R(\mathbf{v})$, and integrated over $\mathbf{v}$,

$$\begin{aligned}\int R(\mathbf{v}) \frac{\partial w(\mathbf{v}, t)}{\partial t}\, d\mathbf{v} &= \iint R(\mathbf{v}) w(\mathbf{v}', t) A(\mathbf{v}', \mathbf{v})\, d\mathbf{v}'\, d\mathbf{v} \\ &\qquad - \iint R(\mathbf{v}) w(\mathbf{v}, t) A(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}\, d\mathbf{v}' \\ &= \iint \left\{ \sum_0^\infty \frac{(\mathbf{v} - \mathbf{v}')^{(n)}}{n!} \cdot \nabla'^{(n)} R(\mathbf{v}') \right\} w(\mathbf{v}', t) A(\mathbf{v}', \mathbf{v})\, d\mathbf{v}\, d\mathbf{v}' \\ &\qquad - \iint R(\mathbf{v}) w(\mathbf{v}, t) A(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}\, d\mathbf{v}' \\ &= \iint \left\{ \sum_1^\infty \frac{(\mathbf{v} - \mathbf{v}')^{(n)}}{n!} \cdot \nabla'^{(n)} R(\mathbf{v}') \right\} w(\mathbf{v}', t) A(\mathbf{v}', \mathbf{v})\, d\mathbf{v}'\, d\mathbf{v}.\end{aligned} \tag{4}$$

Here $(\mathbf{v} - \mathbf{v}')^{(n)} \cdot \nabla'^{(n)}$ is to be understood as the dot product of two n-th-rank tensors. Integrating by parts, one has

$$\int R(\mathbf{v}') \frac{\partial w(\mathbf{v}', t)}{\partial t}\, d\mathbf{v}' = \iint R(\mathbf{v}') \sum_0^\infty \frac{1}{n!} \nabla'^{(n)} \cdot \{(\mathbf{v}' - \mathbf{v})^{(n)} A(\mathbf{v}', \mathbf{v}) W(\mathbf{v}', t)\}\, d\mathbf{v}'\, d\mathbf{v}. \tag{5}$$

Since $R(\mathbf{v}')$ is an arbitrary function, the associated coefficients may be equated to yield

$$\frac{\partial w(\mathbf{v}, t)}{\partial t} = \sum_1^\infty \frac{1}{n!} \nabla^{(n)} \cdot \{A_n(\mathbf{v}) w(\mathbf{v}, t)\}, \tag{6}$$

where $A_n(\mathbf{v})$ is the tensor

$$A_n(\mathbf{v}) = \int (\mathbf{v} - \mathbf{v}')^{(n)} A(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}'. \tag{7}$$

Equations (3) and (6) are equivalent and provide an *exact* description of Brownian motion.

This treatment may be readily generalized to include Brownian motion in coordinate and velocity space.

2. In those treatments of Brownian motion based on Langevin's equation, moments higher than the second are found to vanish, and the Fokker-Planck equation (3) is obtained. As already noted, such an equation is certainly unsatisfactory when the dispersion is small. It would, therefore, be desirable to try to treat the Boltzmann equation directly. Plainly an exact kernel $A(\mathbf{v}, \mathbf{v}')$ is unavailable and its use is almost certainly not feasible. However, it is possible to introduce a kernel which provides a reasonably accurate description of the microscopic scattering process and which is, at the same time, amenable to treatment. Such a kernel is of the form $A(\mathbf{v}, \mathbf{v}') = \mathcal{A}(\mathbf{v}' - \gamma\mathbf{v})$, where γ is a dynamical damping parameter close in value to, but less than, one. Some justification for this form may be found along the following lines:

Let $B(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}'$ be the probability per unit time of a particle with initial velocity v making a transition to a volume element $d\mathbf{v}'$ about $\mathbf{v}'$, when all the particles with which the heavy particle collides are stationary. If the lighter particles have an equilibrium distribution $w(\mathbf{v}'')$, then

$$A(\mathbf{v}, \mathbf{v}') = \int B(\mathbf{v} - \mathbf{v}'', \mathbf{v}' - \mathbf{v}'')w(\mathbf{v}'')\, d\mathbf{v}''. \tag{8}$$

Since the particle under observation is very much heavier than the particles with which it collides, $B(\mathbf{v}, \mathbf{v}')$ is a highly localized function of $\mathbf{v}'$, centered roughly about $\gamma\mathbf{v}$ where again γ is very close to but less than unity. If $B(\mathbf{v}, \mathbf{v}')$ is assumed to have the form $B(\mathbf{v}' - \gamma\mathbf{v})$, then

$$\begin{aligned} A(\mathbf{v}, \mathbf{v}') &= \int B[\mathbf{v}' - \mathbf{v}'' - \gamma(\mathbf{v} - \mathbf{v}'')]w(\mathbf{v}'')\, d\mathbf{v}'' \\ &= \int B[\mathbf{v}' - \gamma\mathbf{v} - (1 - \gamma)\, \mathbf{v}'']w(\mathbf{v}'')\, d\mathbf{v}'', \end{aligned}$$

so that this will have the form of $\mathcal{A}(\mathbf{v}' - \gamma\mathbf{v})$.

Note that the form of $\mathcal{A}(\mathbf{v}' - \gamma\mathbf{v})$ implies that the mean free time τ of a heavy particle is independent of its velocity, since

$$\frac{1}{\tau(\mathbf{v})} = \int A(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}' = \int \mathcal{A}(\mathbf{v}' - \gamma\mathbf{v})\, d\mathbf{v}' = \int \mathcal{A}(\mathbf{v}'')\, d\mathbf{v}'', \tag{9}$$

a constant. This behavior is proper to Brownian motion where the heavy particle moves so slowly compared to the lighter particles that the mean relative velocity of the heavy particle does not vary significantly.

It would appear offhand that the functional form of $\mathcal{A}(\mathbf{v})$ could be chosen arbitrarily. However, this is not the case since $\mathcal{A}(\mathbf{v}, \mathbf{v}')$ must satisfy the equilibrium condition:

$$\omega(\mathbf{v}')A(\mathbf{v}', \mathbf{v}) = \omega(\mathbf{v})A(\mathbf{v}, \mathbf{v}') \tag{10}$$

where $\omega(\mathbf{v})$ is the equilibrium distribution of the heavy particles which the particle will ultimately assume. If it is also demanded that $\omega(\mathbf{v})$ depend only on $|\mathbf{v}|$, the two restrictions imply that $\mathcal{A}(\mathbf{v})$ must have the form $\mathcal{A}_0 \exp\{-\beta\mathbf{v}^2\}$ and that $\omega(\mathbf{v})$ must have the corresponding form $\omega_0 \exp\{-\beta(1 - \gamma^2)\mathbf{v}^2\}$, where $\mathcal{A}_0$ and ω_0 are constants (see Appendix 1). That the Gaussian character of the equilibrium distribution follows from the form of $\mathcal{A}(\mathbf{v} - \gamma\mathbf{v}')$ is reassuring.

Thus, the Boltzmann equation to be solved is

$$\frac{\partial w(\mathbf{v}, t)}{\partial t} = \mathfrak{A}_0 \int w(\mathbf{v}', t) \exp \{-\beta(\mathbf{v} - \gamma\mathbf{v}')^2\} \, d\mathbf{v}' - \frac{1}{\tau} w(\mathbf{v}, t), \tag{11}$$

where

$$\begin{aligned} \frac{1}{\tau} = \int \mathfrak{A}(\mathbf{v}' - \gamma\mathbf{v}) \, d\mathbf{v}' &= \mathfrak{A}_0 \int \exp \{-\beta(\mathbf{v}' - \gamma\mathbf{v})^2\} \, d\mathbf{v}' \\ &= \mathfrak{A}_0 \int \exp \{-\beta\mathbf{v}''^2\} \, d\mathbf{v}'' = \mathfrak{A}_0 \left(\frac{\pi}{\beta}\right)^{3/2}. \end{aligned} \tag{12}$$

Before discussing the solution of this equation, it is worth while to put down the corresponding Fokker-Planck equation. The first moment will be given by

$$\begin{aligned} A_1 &= \mathfrak{A}_0 \int (\mathbf{v} - \mathbf{v}') \exp \{-\beta(\mathbf{v}' - \gamma\mathbf{v})^2\} \, d\mathbf{v}' \\ &= \mathfrak{A}_0 \int [(\gamma\mathbf{v} - \mathbf{v}') + (1 - \gamma)\mathbf{v}] \exp \{-\beta(\mathbf{v}' - \gamma\mathbf{v})^2\} \, d\mathbf{v}' \\ &= \mathfrak{A}_0(1 - \gamma)\mathbf{v} \int \exp \{-\beta(\mathbf{v}' - \gamma\mathbf{v})^2\} \, d\mathbf{v}' \\ &= \mathfrak{A}_0\left(\frac{\pi}{\beta}\right)^{3/2}(1 - \gamma)\mathbf{v} = \left(\frac{1 - \gamma}{\tau}\right)\mathbf{v}. \end{aligned} \tag{13}$$

Similarly for the second moment,

$$\begin{aligned} A_2 &= \mathfrak{A}_0 \int (\mathbf{v} - \mathbf{v}')(\mathbf{v} - \mathbf{v}') \exp \{-\beta(\mathbf{v}' - \gamma\mathbf{v})^2\} \, d\mathbf{v}' \\ &= \mathfrak{A}_0 \int [(\mathbf{v}' - \gamma\mathbf{v})(\mathbf{v}' - \gamma\mathbf{v}) + (1 - \gamma)^2\mathbf{v}\mathbf{v}] \exp \{-\beta(\mathbf{v}' - \gamma\mathbf{v})^2\} \, d\mathbf{v}' \\ &= \frac{\mathfrak{A}_0}{2} \pi^{3/2}\beta^{-5/2}\epsilon + (1 - \gamma)^2\mathbf{v}\mathbf{v}\mathfrak{A}_0\left(\frac{\pi}{\beta}\right)^{3/2} \\ &= \frac{1}{2\beta\tau} \epsilon + \frac{(1 - \gamma)^2\mathbf{v}\mathbf{v}}{\beta\tau}, \end{aligned} \tag{14}$$

where ϵ is the unit tensor.

If the latter part of A_2 is ignored since $(1 - \gamma)^2$ is small and if the higher moments (whose effect will be small for $t \gg \tau$) are ignored, Eq. (1) is regained where now

$$D = \frac{1}{2\beta\tau} \quad \text{and} \quad \eta = \frac{1 - \gamma}{\tau}. \tag{15}$$

As is seen in Appendix 2, the solution of the Boltzmann equation (11) subject to condition (2) is given by

$$w_B(\mathbf{v}, t) = \left[\delta(\mathbf{v} - \mathbf{v}_0) + \sum_1^\infty \frac{1}{n!} \left(\frac{t}{\tau}\right)^n \left(\frac{\beta}{\pi\Delta_n}\right)^{3/2} \exp \left\{-\frac{\beta}{\Delta_n}(\mathbf{v} - \gamma^n\mathbf{v}_0)^2\right\}\right] \exp \left\{-\frac{t}{\tau}\right\}, \tag{16}$$

where

$$\Delta_n = \frac{1-\gamma^{2n}}{1-\gamma^2}. \tag{17}$$

The solution of the Fokker-Planck equation (1) subject to condition (2) may be taken directly from Wang and Uhlenbeck [1] and is given by

$$w_{FP}(\mathbf{v}, t) = \left[\frac{\eta}{\pi D(1 - \exp\{-2\eta t\})}\right]^{3/2} \exp\left\{\frac{-\eta(\mathbf{v} - \mathbf{v}_0 \exp\{-\eta t\})^2}{D(1 - \exp\{-2\eta t\})}\right\}. \tag{18}$$

Note that the singularity $\delta(\mathbf{v} - \mathbf{v}_0)$ is preserved in the solution of the integral equation but is not in the solution of the Fokker-Planck equation. For $t \gg \tau$, however, the delta-function ceases to play a prominent role.

From Eqs. (16), (17) one finds that the equilibrium distribution for the solution of the Boltzmann equation is given by

$$\omega_B(\mathbf{v}) = \lim_{t\to\infty} w_B(\mathbf{v}, t) = \left[\frac{\beta(1-\gamma^2)}{\pi}\right]^{3/2} \exp\{-\beta(1-\gamma^2)v^2\}. \tag{19}$$

For the Fokker-Planck equation,

$$\omega_{FP}(\mathbf{v}) = \left(\frac{\eta}{\pi D}\right)^{3/2} \exp\left\{-\frac{\eta}{D}v^2\right\}. \tag{20}$$

Inserting the values of η, D from (15), this becomes:

$$\omega_{FP}(\mathbf{v}) = \left(\frac{2(1-\gamma)\beta}{\pi}\right)^{3/2} \exp\{-2\beta(1-\gamma)v^2\}. \tag{21}$$

Plainly if γ is sufficiently close to one, then

$$(1-\gamma^2) = (1-\gamma)(1+\gamma) \simeq 2(1-\gamma)$$

and the two equilibrium distributions are identical.

It is also of interest to compare the manner in which the average velocity and the variance vary in time. These quantities are defined by

$$\langle\mathbf{v}\rangle(t) = \int \mathbf{v} w(\mathbf{v}, t)\, d\mathbf{v}$$

and

$$\sigma^2(t) = \int (v^2 - \langle\mathbf{v}\rangle^2) w(\mathbf{v}, t)\, d\mathbf{v}.$$

$\langle\mathbf{v}\rangle_B(t)$, $\langle\mathbf{v}\rangle_{FP}(t)$, $\sigma_B^2(t)$, and $\sigma_{FP}^2(t)$ may be computed directly from their respective equations of motion. Thus, if Eq. (1) is multiplied on both sides by $\mathbf{v}$ and integrated over $\mathbf{v}$, then

$$\frac{d\langle\mathbf{v}\rangle}{dt} = -\eta\langle\mathbf{v}\rangle, \qquad \text{so that} \qquad \langle\mathbf{v}\rangle_{FP}(t) = \mathbf{v}_0 \exp\{-\eta t\} = \mathbf{v}_0 \exp\left\{-\frac{1-\gamma}{\tau}t\right\}. \tag{22}$$

Similarly, multiplying by $\mathbf{v}^2$ and integrating, it is found that

$$\frac{d\langle\mathbf{v}^2\rangle}{dt} = 3D - 2\eta\langle\mathbf{v}^2\rangle, \tag{23}$$

which yields in turn

$$\frac{d\sigma^2}{dt} = \frac{d}{dt}(\langle \mathbf{v}^2 \rangle - \langle \mathbf{v} \rangle^2) = 3D - 2\eta\sigma^2; \tag{24}$$

and so

$$\sigma_{FP}^2(t) = \frac{3D}{2\eta}(1 - \exp\{-2\eta t\}) = \frac{3}{4\beta(1-\gamma)}\left(1 - \exp\left\{-\frac{2}{\tau}(1-\gamma)t\right\}\right). \tag{25}$$

The same procedure may be applied to the integral equation (11) to give

$$\langle \mathbf{v} \rangle_B(t) = \mathbf{v}_0 \exp\left\{-\left(\frac{1-\gamma}{\tau}\right)t\right\} \tag{26}$$

and

$$\langle \mathbf{v}^2 \rangle_B(t) = \frac{3}{2\beta(1-\gamma^2)}\left[1 - \exp\left\{-\left(\frac{1-\gamma^2}{\tau}\right)t\right\}\right] + \mathbf{v}_0^2 \exp\left\{-\left(\frac{1-\gamma^2}{\tau}\right)t\right\}. \tag{27}$$

Correspondingly, one finds that

$$\begin{aligned}\sigma_B^2(t) = \frac{3}{2\beta(1-\gamma^2)}&\left[1 - \exp\left\{-\frac{(1-\gamma^2)t}{\tau}\right\}\right] \\ &+ \mathbf{v}_0^2\left[\exp\left\{-\left(\frac{1-\gamma^2}{\tau}\right)t\right\} - \exp\left\{-2\left(\frac{1-\gamma}{\tau}\right)t\right\}\right].\end{aligned} \tag{28}$$

These same results could also have been obtained from the solutions (16) and (18), but the computations are more tedious.

It is seen that $\langle \mathbf{v}_B \rangle(t)$ and $\langle \mathbf{v}_{FP} \rangle(t)$ are identical and that $\sigma_B^2(t)$ and $\sigma_{FP}^2(t)$ are nearly identical. Indeed, if the smaller term in A_2 had not been ignored in obtaining the corresponding Fokker-Planck approximation, $\sigma_B^2(t)$ and $\sigma_{FP}^2(t)$ would have been precisely the same. For consider the Boltzmann equation in its differential form:

$$\frac{\partial w}{\partial t} = \sum_1^\infty \nabla^{(n)} \cdot (A_n w).$$

The above procedure yields

$$\frac{d\langle \mathbf{v}^2 \rangle}{dt} = \int \mathbf{v}^2 \nabla^{(1)} \cdot (A_1 w)\, d\mathbf{v} + \int \mathbf{v}^2 \nabla^{(2)} \cdot (A_2 w)\, d\mathbf{v},$$

since all integrals involving higher moments vanish when integration by parts is carried out. Moreover, from the choice of $A(\mathbf{v}, \mathbf{v}')$, the two integrals are simple functions of $\langle \mathbf{v}^2 \rangle$ and the above differential equation does determine $\langle \mathbf{v}^2 \rangle(t)$. The same procedure applied to the Fokker-Planck equation can only yield the same result, because all contributing terms are present.

It is seen then that the validity of the Fokker-Planck approximation is excellent when γ is sufficiently close to one. For the ordinary domain of Brownian motion this will certainly be the case. For the elastic collision of hard spheres, for example, it is easily found that

$$\langle \delta \mathbf{v} \rangle = \frac{-4}{3}\frac{m}{M+m}\mathbf{v},$$

where $\langle \delta \mathbf{v} \rangle$ is the mean change in velocity suffered by a particle of mass M and velocity v in a single collision with particles of mass m. Then

$$\frac{d\langle \mathbf{v} \rangle}{dt} \simeq \frac{\langle \delta \mathbf{v} \rangle}{\tau} = \frac{-4}{3} \frac{m}{(M + m)\tau} \mathbf{v},$$

so that, from Eqs. (15) and (22)

$$\eta = \frac{(1 - \gamma)}{\tau} \simeq \frac{4}{3} \frac{m}{(M + m)\tau}.$$

$1 - \gamma$ then is given by $4/3\ m/(M + m)$ and for typical Brownian motion will be extremely small.

If it were possible to treat the exact kernel $A(\mathbf{v}, \mathbf{v}')$, one would still expect to find excellent agreement between the Fokker-Planck and Boltzmann solutions for $t \gg \tau$. Even when $t \sim \tau$, the first and second moments of the Fokker-Planck equation should be reliable. But for $t \sim \tau$, higher-order moments would be in considerable error. However, for $t \gg \tau$, these errors will become entirely negligible.

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Appendix 1

The Restrictions on $\mathcal{A}(\mathbf{v})$ and $\omega(\mathbf{v})$ Imposed by Equilibrium

Denoting the rectangular components of $\mathbf{v}$ by v_i , $i = 1, 2, 3$, and letting

$$\psi(v_1 , v_2 , v_3) = \ln \mathcal{A}(\mathbf{v}),$$

the equilibrium relation (10) can be written in the form

$$\ln \omega(\mathbf{v}') + \psi(v_1 - \gamma v_1' , v_2 - \gamma v_2' , v_3 - \gamma v_3') \\ = \ln \omega(\mathbf{v}) + \psi(v_1' - \gamma v_1 , v_2' - \gamma v_2 , v_3' - \gamma v_3). \tag{1-1}$$

Taking the partial derivative of Eq. (1–1) with respect to v_i , v_j' , and noting that

$$\frac{\partial^2}{\partial v_i \, \partial v_j'} \ln \omega(\mathbf{v}') = \frac{\partial^2}{\partial v_i \, \partial v_j'} \ln \omega(\mathbf{v}) = 0,$$

it is seen that

$$-\gamma \psi_{ij}(v_1 - \gamma v_1' , v_2 - \gamma v_2' , v_3 - \gamma v_3') = -\gamma \psi_{ij}(v_1' - \gamma v_1 , v_2' - \gamma v_2 , v_3' - \gamma v_3), \tag{1-2}$$

where

$$\psi_{ij} = \frac{\partial^2}{\partial v_i \, \partial v_j} \psi(v_1 , v_2 , v_3).$$

Setting $v_i' = \gamma v_i$, $v_i'' = (1 - \gamma^2)v_i$ in (1–2), it is further observed that

$$\psi_{ij}(v_1'', v_2'', v_3'') = \psi_{ij}(0, 0, 0) = -\beta_{ij} ,$$

where β_{ij} is a constant. Hence $\psi(v_1 , v_2 , v_3)$ must be of the form

$$\psi(v_1 , v_2 , v_3) = -\sum_{i=1}^{3} \sum_{j=1}^{3} \beta_{ij}v_iv_j + \sum_{1}^{3} \alpha_i v_i + \Delta,$$

where α_i and Δ are constants. Inserting this result into $(1 - 1)$, one finds that

$$\ln \omega(\mathbf{v}) = -\sum_{j=1}^{3} \sum_{j=1}^{3} \beta_{ij}(1 - \gamma^2)v_iv_j + \sum_{1}^{3} \alpha_i(1 + \gamma)v_i + \Delta'. \tag{1-3}$$

But, since the distribution of small particles is assumed to be isotropic, one has

$$\omega(\mathbf{v}) = \omega(| \mathbf{v} |) = \omega((v_1^2 + v_2^2 + v_3^2)^{1/2}). \tag{1-4}$$

The only possible way $(1 - 3)$ can satisfy this condition is for

$$\beta_{ij} = \beta\delta_{ij} , \qquad \alpha_i = 0. \tag{1-5}$$

Hence

$$\mathfrak{A}(v) = \mathfrak{A}_0 \exp \{-\beta v^2\}, \qquad \omega(v) = \omega_0 \exp \{-\beta(1 - \gamma^2)v^2\},$$

where $\mathfrak{A}_0$ and ω_0 are constants.

Appendix 2

Solution of the Boltzmann Equation

It is desired to solve Eq. (11) subject to the condition (2). Two methods will be given. One procedure is to introduce the Fourier transform of $w(\mathbf{v}, t)$, i.e.,

$$T_w(\mathbf{k}, t) = \int \exp \{i\mathbf{k}\cdot\mathbf{v}\} w(\mathbf{v}, t) \, d\mathbf{v}$$

with

$$T_w(\mathbf{k}, 0) = \int \exp \{i\mathbf{k}\cdot\mathbf{v}\} \; \delta(\mathbf{v} - \mathbf{v}_0) \, d\mathbf{v} = \exp \{i\mathbf{k}\cdot\mathbf{v}_0\}.$$

Taking the Fourier transform of Eq. (11) one obtains the following equation for $T_w(\mathbf{k}, t)$:

$$\frac{\partial}{\partial t} T_w(\mathbf{k}, t) = A(\mathbf{k})T_w(\gamma\mathbf{k}, t) - \frac{1}{\tau} T_w(\mathbf{k}, t), \tag{2-1}$$

where

$$A(\mathbf{k}) = \int \exp\{i\mathbf{k}\cdot\mathbf{v}\}\mathfrak{A}(\mathbf{v})\,d\mathbf{v} = \mathfrak{A}_0 \int \exp\{i\mathbf{k}\cdot\mathbf{v} - \beta v^2\}\,d\mathbf{v}$$

$$= \mathfrak{A}_0\left(\frac{\pi}{\beta}\right)^{3/2} \exp\left\{-\frac{k^2}{4\beta}\right\} = \frac{1}{\tau}\exp\left\{-\frac{k^2}{4\beta}\right\}.$$

It is now convenient to introduce the Laplace transform

$$L_w(\mathbf{k}, s) = \int_0^\infty \exp\{-st\}T_w(\mathbf{k}, t)\,dt;$$

then

$$\int_0^\infty \exp\{-st\}\frac{\partial}{\partial t}T_w(\mathbf{k}, t)\,dt = -T_w(\mathbf{k}, 0) + sL_w(\mathbf{k}, s) = -\exp\{i\mathbf{k}\cdot\mathbf{v}_0\} + sL_w(\mathbf{k}, s).$$

Thus, taking the Laplace transform of (2-1), one obtains the equation for $L_w(\mathbf{k}, s)$,

$$-\exp\{i\mathbf{k}\cdot\mathbf{v}_0\} + sL_w(\mathbf{k}, s) = \frac{1}{\tau}\exp\left\{-\frac{k^2}{4\beta}\right\}L_w(\gamma\mathbf{k}, s) - \frac{1}{\tau}L_w(\mathbf{k}, s).$$

This may be rearranged to give

$$L_w(\mathbf{k}, s) = \frac{1}{s + \tau^{-1}}\exp\{i\mathbf{k}\cdot\mathbf{v}_0\} + \frac{1}{\tau}\frac{1}{s + \tau^{-1}}\exp\left\{-\frac{k^2}{4\beta}\right\}L_w(\gamma\mathbf{k}, s). \tag{2-2}$$

The finite difference equation (2-2) may be solved by the following procedure: Replace $\mathbf{k}$ by $\gamma\mathbf{k}$. This yields

$$L_w(\gamma\mathbf{k}, s) = +\frac{1}{s + \tau^{-1}}\exp\{i\gamma\mathbf{k}\cdot\mathbf{v}_0\} + \frac{1}{\tau}\frac{1}{s + \tau^{-1}}\exp\left\{-\frac{\gamma^2k^2}{4\beta}\right\}L_w(\gamma^2\mathbf{k}, s). \tag{2-3}$$

Equation (2-3) may be used to eliminate $L_w(\gamma\mathbf{k}, s)$ from (2-2), yielding

$$\begin{aligned} L_w(k, s) = \frac{1}{s + \gamma^{-1}}\exp\{i\mathbf{k}\cdot\mathbf{v}_0\} &+ \frac{1}{\tau}\frac{\exp\{i\gamma\mathbf{k}\cdot\mathbf{v}_0 - k^2/4\beta\}}{(s + \tau^{-1})^2} \\ &+ \frac{1}{\tau}\frac{\exp\{-\gamma^2k^2/4\beta\}}{s + \tau^{-1}}L_w(\gamma^2\mathbf{k}, s). \end{aligned} \tag{2-4}$$

Replacing $\mathbf{k}$ by $\gamma^2\mathbf{k}$ in (2-2), the resulting equation may be used to eliminate $L_w(\gamma^2\mathbf{k}, s)$ from (2-4). Continuing in this fashion yields the solution

$$L_w(\mathbf{k}, s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{\tau^n}\frac{\exp\{i\gamma^n\mathbf{k}\cdot\mathbf{v}_0 - (k^2/4\beta)\Delta_n\}}{(s + \tau^{-1})^{n+1}},$$

where

$$\Delta_n = \frac{1 - \gamma^{2n}}{1 - \gamma^2}.$$

It is to be noted that the series is absolutely convergent.

$T_w(\mathbf{k}, t)$ may readily be obtained from $L_w(\mathbf{k}, s)$ by taking the inverse Laplace transform. Using a Bromwich contour it is apparent that

$$\begin{aligned}
T_w(\mathbf{k}, t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} L_w(\mathbf{k}, s)\, ds \\
&= \frac{1}{2\pi i}\int_{Br} e^{st} L_w(\mathbf{k}, s)\, ds \\
&= \text{Residue at } s = -\frac{1}{\tau} \text{ of } L_w(\mathbf{k}, s) \\
&= \sum_0^\infty (-1)^n \frac{\exp\{i\gamma^n \mathbf{k}\cdot\mathbf{v}_0 - (k^2/4\beta)\Delta_n\}}{\tau^n} \\
&\qquad\qquad \times \left[\text{Residue at } s = -\frac{1}{\tau} \text{ of } \frac{e^{st}}{(s+\tau^{-1})^{n+1}}\right] \\
&= \sum_0^\infty (-1)^n \frac{\exp\{i\gamma^n \mathbf{k}\cdot\mathbf{v}_0 - (k^2/4\beta)\Delta_n\}}{\tau^n} \frac{(-t)^n}{n!} \exp\left\{-\frac{t}{\tau}\right\} \\
&= \left[\sum_0^\infty \frac{1}{n!}\left(\frac{t}{\gamma}\right)^n \exp\{i\gamma^n \mathbf{k}\cdot\mathbf{v}_0 - (k^2/4\beta)\Delta_n\}\right] \exp\left\{-\frac{t}{\tau}\right\}.
\end{aligned} \tag{2-5}$$

The inverse Fourier transform may now be performed and this yields

$$\begin{aligned}
w(\mathbf{v}, t) &= \frac{1}{(2\pi)^3}\int \exp\{-i\mathbf{k}\cdot\mathbf{v}\} T_w(\mathbf{k}, t)\, d\mathbf{k} \\
&= \frac{\exp\{-t/\tau\}}{(2\pi)^3} \sum_0^\infty \frac{1}{n!}\left(\frac{t}{\tau}\right)^n \int \exp\left\{i(\gamma^n \mathbf{v}_0 - \mathbf{v})\cdot\mathbf{k} - \frac{k^2}{4\beta}\Delta_n\right\} d\mathbf{k} \\
&= \exp\left\{-\frac{t}{\tau}\right\}\left[\delta(v - v_0) + \sum_1^\infty \frac{1}{n!}\left(\frac{t}{\tau}\right)^n \left(\frac{\beta}{\pi\Delta_n}\right)^{3/2} \exp\left\{-\frac{\beta}{\Delta_n}(\mathbf{v} - \gamma^n \mathbf{v}_0)^2\right\}\right].
\end{aligned} \tag{2-6}$$

This solution may be verified by substitution.

Equation (11) may also be solved in the following way [4]. Consider the sequence of equations,

$$\begin{aligned}
\frac{\partial w_0(\mathbf{v}, t)}{\partial t} &= \frac{-w_0(\mathbf{v}, t)}{\tau} \\
\frac{\partial w_1(\mathbf{v}, t)}{\partial t} &= \frac{-w_1(\mathbf{v}, t)}{\tau} + \int w_0(\mathbf{v}', t)\mathfrak{A}(\mathbf{v} - \gamma\mathbf{v}')\, d\mathbf{v}' \\
&\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial w_n(\mathbf{v}, t)}{\partial t} &= \frac{-w_n(\mathbf{v}, t)}{\tau} + \int w_{n-1}(\mathbf{v}', t)\mathfrak{A}(\mathbf{v} - \gamma\mathbf{v}')\, d\mathbf{v}', \text{ etc.,}
\end{aligned} \tag{2-7}$$

subject to the initial conditions

$$\begin{cases} w_0(\mathbf{v}, 0) = \delta(\mathbf{v} - \mathbf{v}_0) \\ w_i(\mathbf{v}, 0) = 0, \qquad i \neq 0. \end{cases} \tag{2-8}$$

Plainly,

$$w(\mathbf{v}, t) = \sum_0^\infty w_n(\mathbf{v}, t) \tag{2-9}$$

satisfies Eq. (11) and the condition $w(\mathbf{v}, 0) = \delta(\mathbf{v} - \mathbf{v}_0)$.

Then

$$w_0(\mathbf{v}, t) = \delta(\mathbf{v} - \mathbf{v}_0) \exp\{-t/\tau\}$$

and

$$w_n(\mathbf{v}, t) = \exp\left\{-\frac{t}{\tau}\right\} \int_0^t \exp\left\{\frac{s}{\tau}\right\} \int w_{n-1}(\mathbf{v}'', s)\mathfrak{A}(\mathbf{v} - \gamma\mathbf{v}'')\, d\mathbf{v}''\, ds \tag{2-10}$$

satisfy the equations and one need only evaluate the sequence of functions, $w_n(v, t)$. It is seen from this last equation that if $w_{n-1}(\mathbf{v}, t)$ is a product of a function of $\mathbf{v}$ and a function of t, $w_n(\mathbf{v}, t)$ is also such a product. Since w_0 has such a form, all our w_n decompose in this way. Assume $w_n(\mathbf{v}, t) = U_n(\mathbf{v})g_n(t)$. Then

$$g_n(t) = \exp\left\{-\frac{t}{\tau}\right\} \int_0^t \exp\left\{\frac{s}{\tau}\right\} g_{n-1}(s)\, ds$$

and

$$U_n(\mathbf{v}) = \mathfrak{A}_0 \int U_{n-1}(\mathbf{v}') \exp\{-\beta(\mathbf{v} - \gamma\mathbf{v}')^2\}\, d\mathbf{v}'. \tag{2-11}$$

It is now easy to see that

$$g_n(t) = \frac{t^n}{n!} \exp\left\{-\frac{t}{\tau}\right\}. \tag{2-12}$$

$U_n(\mathbf{v})$ has the form $\alpha_n \exp\{-\beta_n(\mathbf{v} - \boldsymbol{\delta}_n)^2\}$, and α_n, β_n, $\boldsymbol{\delta}_n$ are connected by recursion relations derived from Eq. (2-11), stating

$$\alpha_{n+1} \exp\{-\beta_{n+1}(v - \boldsymbol{\delta}_{n+1})^2\} = \mathfrak{A}_0 \int \alpha_n \exp\{-\beta_n(\mathbf{v}' - \boldsymbol{\delta}_n)^2\} \exp\{-\beta(\mathbf{v} - \gamma\mathbf{v}')^2\}\, d\mathbf{v}'.$$

This gives

$$\begin{aligned}
\alpha_{n+1} &= \left[\frac{\pi}{\beta_n + \gamma^2\beta}\right]^{3/2} \mathfrak{A}_0\alpha_n \qquad \text{with} \qquad & \alpha_1 &= \mathfrak{A}_0 \\
& & \beta_1 &= B \\
\beta_{n+1} &= \frac{\beta\beta_n}{\beta_n + \gamma^2\beta} & \delta_1 &= v_0 \\
\boldsymbol{\delta}_{n+1} &= \gamma\boldsymbol{\delta}_n\,.
\end{aligned} \tag{2-13}$$

$$\begin{aligned}
\therefore \boldsymbol{\delta}_n &= \gamma^n\mathbf{v}_0 \\
\beta_n &= \frac{\beta}{\Delta_n} \qquad \text{where} \qquad \Delta_n = \frac{\gamma^{2n} - 1}{\gamma^2 - 1} \\
\alpha_n &= \frac{1}{\tau^n}\left(\frac{\beta}{\pi\Delta_n}\right)^{3/2}.
\end{aligned} \tag{2-14}$$

When these are substituted into the series of Eq. (2-9), the solution is again obtained.