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We give an arithmetic characterization which allow us to determine algorithmically when the semigroup ring associated to a simplicial affine semigroup is Buchsbaum. This characterization is based on a test performed on the Apéry sets of the extremal rays of the semigroup. We use this method to obtain the cardinality of minimal presentations for semigroups with minimal Apéry set.

Introduction.

Let $S = \langle n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+m} \rangle \subseteq \mathbb{N}^r$ be a simplicial affine semigroup, that is $\mathsf{L}_{\mathbb{Q}_0^+}(S) = \mathsf{L}_{\mathbb{Q}_0^+}(\{n_1, \ldots, n_r\})$, where $\mathsf{L}_{\mathbb{Q}_0^+}(A) = \{\sum q_i a_i \mid q_i \in \mathbb{Q}_0^+ \text{ and } a_i \in A\}$. We assume that the elements $n_1, \ldots, n_r$ are linearly independent (otherwise S can be embedded in $\mathbb{N}^s$ with $s < r$). This enables us to suppose that, up to isomorphism, $n_i = \alpha_i e_i$ with $\alpha_i \in \mathbb{N} \setminus \{0\}$ (as usual, e_i denotes the element in $\mathbb{N}^r$ all of whose coordinates are equal to zero except the i-th which is equal to one). We will refer to $n_1, \ldots, n_r$ as the extremal rays of S.

Let $K[S] = \bigoplus_{s \in S} Ky^s$ be the semigroup ring associated to S. We say that S is **Cohen-Macaulay** if the ring $K[S]$ is Cohen-Macaulay. The same stands for the notions of **Gorenstein**, **Buchsbaum** and **complete intersection** semigroup. In [10] the authors gave a characterization of the Cohen-Macaulay and Gorenstein property for simplicial affine semigroups in terms of the Apéry sets of its extremal rays. In that paper we also studied the form and cardinality of a minimal system of generators of the defining ideals of this type of semigroup rings. The mentioned paper was inspired mostly in the characterization given by Goto, Suzuki and Watanabe in [5] and in the generalization given in [16] by Trung and Hoa. Here we focus our attention on Buchsbaum semigroups. There are a lot of papers devoted to the study of the structure of arithmetically Buchsbaum monomial curves (see for instance [1, 7, 12, 15]). Using as a starting point the characterizations given by Trung in [14] and by Kamoi in [8], we present an alternative characterization of the Buchsbaum property in Theorem 5 (compare with Theorem 1.1, page 230, in [13]). This result is used later to achieve Theorem 9 which is the main result of this paper and presents an arithmetical characterization of the Buchsbaum property for simplicial affine semigroups

in terms of the Apéry sets of their extremal rays. This main theorem provides us with a procedure for deciding whether or not a given simplicial affine semigroup is Buchsbaum. Finally these results are also used to give the exact cardinality of a minimal presentation of a Buchsbaum simplicial affine semigroup with minimal Apéry sets (using the notation in [**4**], these are Buchsbaum simplicial affine semigroups with maximal embedding dimension, and what we count here is the number of elements in a minimal system of generators of the defining ideal of the semigroup ring associated to the given monoid; see this reference for an explicit expression of the Hilbert polynomial for this semigroup ring). The number of elements of a minimal presentation for this kind of monoids is obtained from the Apéry sets of its extremal rays and in this way this result generalizes the bound given for Cohen-Macaulay simplicial affine semigroups with maximal codimension presented in [**10**].

1. A characterization of Buchsbaum semigroups.

For every $k \in \mathbb{N}$, define

$$S_k = \{x \in \mathbb{N}^r \mid \text{there exists } 1 \leq i < j \leq r$$
$$\text{such that } x + kn_i \in S, x + kn_j \in S\}.$$

The characterizations given here for Buchsbaum semigroups are based on the following result.

Proposition 1. *The following conditions are equivalent.*
  (i) *S is a Buchsbaum semigroup.*
 (ii) *$S_2 + (S \setminus \{0\}) \subseteq S$.*
(iii) *For every $1 \leq i < j \leq r$ and $u, v \in S$, if $v + 2n_j = u + 2n_i$ then $v + (S \setminus \{0\}) \subseteq 2n_i + S$.*
 (iv) *For every $x \in S$, if $x - 2n_i, x - 2n_j \in S$, for some $i \neq j \in \{1, \ldots, r\}$, then $x + n_k - (2n_i + 2n_j) \in S$ for all $k \in \{1, \ldots, r + m\}$.*

Proof. The equivalence between (i) and (ii) appears in Lemma 3 of [**14**]. The conditions (i) and (iii) are equivalent by Proposition 2.3 of [**8**]. Finally (iii) if and only if (iv) follows easily taking $x = v + 2n_j = u + 2n_i$. □

There are several characterizations of the Cohen-Macaulay property similar to Proposition 1. Next we give one of them.

Proposition 2. *The semigroup S is a Cohen-Macaulay semigroup if and only if $S_k = S$ for every $k \in \mathbb{N} \setminus \{0\}$.*

Proof. The fact that $S_1 = S$ is equivalent to the Cohen-Macaulay property for S is part of Corollary 4.4 in [**16**]. Besides, once $S_1 = S$, one can prove that $S_k = S$ for all $k \geq 2$. □

In order to reformulate the Buchsbaum property for simplicial affine semi-groups, we have to introduce some notation. The **Apéry set** of an element n of S is the set

$$S(n) = \{s \in S \mid s - n \notin S\}.$$

The subgroup of $\mathbb{Z}^r$ generated by $\{n_1, \dots, n_r\}$ is denoted by $\mathsf{G}(\{n_1, \dots, n_r\})$. Let $\overline{S}$ be the set of elements x in $\mathbb{Z}^r$ such that $x + n_i \in S$ for all $i \in \{1, \dots, r+m\}$. Note that this set is a semigroup that contains S and that

$$\overline{S} = \{x \in \mathbb{N}^r \mid x + n_i \in S \text{ for all } i \in \{1, \dots, r+m\}\}.$$

Since $\overline{S} \subseteq \mathbb{N}^r$ and $\mathsf{L}_{\mathbb{Q}_0^+}(n_1, \dots, n_r) = (\mathbb{Q}_0^+)^r$, then $\overline{S}$ is simplicial as well.

As a consequence of the following result, $\overline{S}$ is finitely generated.

Lemma 3. *Every submonoid T of $\mathbb{N}^r$ containing $\{n_1, \dots, n_r\}$ is finitely generated.*

Proof. It is easy to see that since T is simplicial, every element t in T can be written as $t = \sum_{i=1}^r a_i n_i + w$ for some $a_i \in \mathbb{N}$ and $w \in \bigcap_{i=1}^r \mathrm{T}(n_i)$. If we want to demonstrate that T is finitely generated, it suffices to prove that the set $\bigcap_{i=1}^r \mathrm{T}(n_i)$ has a finite number of elements. For proving this, define in $\bigcap_{i=1}^r \mathrm{T}(n_i)$ the following equivalence relation:

$$x \sim y \text{ if } x - y \in \mathsf{G}(\{n_1, \dots, n_r\}).$$

Since there are at most $\alpha_1 \cdots \alpha_r$ elements in $\mathbb{N}^r$ modulo $\mathsf{G}(\{n_1, \dots, n_r\})$, there is a finite number of $\sim$-classes in $\bigcap_{i=1}^r \mathrm{T}(n_i)$. If we show that for every $x = (x_1, \dots, x_r) \in \bigcap_{i=1}^r \mathrm{T}(n_i)$, its $\sim$-class $[x]$ is finite, then we conclude the proof. Set $m = (x_1 \bmod \alpha_1, \dots, x_r \bmod \alpha_r)$. Clearly, for every element $y \in [x]$, there exists $a_1^y, \dots, a_r^y \in \mathbb{N}$ such that $y = \sum_{i=1}^r a_i^y n_i + m$. If there exists $y, z \in [x]$ for which $(a_1^y, \dots, a_r^y) < (a_1^z, \dots, a_r^z)$, then $z = y + \sum_{i=1}^r (a_i^z - a_i^y) n_i \notin \bigcap_{i=1}^r \mathrm{T}(n_i)$, which contradicts $z \in \bigcap_{i=1}^r \mathrm{T}(n_i)$. Hence the set of elements $A = \{(a_1^y, \dots, a_r^y) \mid y \in [x]\}$ is a set of incomparable elements with respect to the usual partial order in $\mathbb{N}^r$ (product order). Using Dickson's lemma, it follows that there exists a finite number of elements in A and thus a finite number of elements in $[x]$. $\qquad\square$

The following result indicates a connection between S_k and $\overline{S}$.

Lemma 4.

$$S \subseteq \overline{S} \subseteq S_1 \subseteq (\overline{S})_1 \subseteq S_2 \subseteq (\overline{S})_2.$$

Proof. Follows easily from the definitions of S_k and $\overline{S}$. $\qquad\square$

The next statement justifies the definition of the semigroup $\overline{S}$.

Theorem 5. *The semigroup S is Buchsbaum if and only if $\overline{S}$ is Cohen-Macaulay.*

Proof. Necessity. By Proposition 1, we have $S_2 + (S \setminus \{0\}) \subseteq S$, whence $S_2 \subseteq \overline{S}$. Since the opposite inclusion always holds, $S_2 = \overline{S}$ and by Lemma 4, it follows that $\overline{S} = (\overline{S})_1$, which by Proposition 2 implies that $\overline{S}$ is Cohen-Macaulay.

Sufficiency. By Proposition 2, $\overline{S} = (\overline{S})_2$. From Lemma 4 it follows that $S_2 = \overline{S}$, which by the definition of $\overline{S}$ leads to $S_2 + (S \setminus \{0\}) \subseteq S$. Proposition 1 asserts that S is Buchsbaum. □

From the characterization of the Cohen-Macaulay property, if S is Cohen-Macaulay and $x + n_i$, $x + n_j$ are in S, for $i \neq j \in \{1, \dots, r\}$, then is $x \in S$. Thus if S is Cohen-Macaulay and $r \geq 2$, then $S = \overline{S}$. (The case $r = 1$ is the numerical case and all numerical semigroups are Cohen-Macaulay and Buchsbaum.)

As an easy consequence of Theorem 5 we obtain the following remark (the proof is left to the reader).

Corollary 6. *Let T be a simplicial affine Cohen-Macaulay semigroup minimally generated by $\{n_1, \dots, n_r, n_{r+1}, \dots, n_{r+m}\}$ and A be a nonempty subset of $\{n_{r+1}, \dots, n_{r+m}\}$. Then $T \setminus A$ is Buchsbaum but not Cohen-Macaulay.*

We illustrate this with an example.

Example 7. Let S be the semigroup finitely generated by

$$\{(2, 0), (0, 2), (1, 1)\}.$$

Clearly S is Cohen-Macaulay (it is even a complete intersection). The semigroup $S \setminus \{(1, 1)\}$ can be generated by

$$\{(2, 0), (0, 2), 2(1, 1), 3(1, 1), (2, 0) + (1, 1), (0, 2) + (1, 1)\},$$

whence $S \setminus \{(1, 1)\} = \langle (2, 0), (0, 2), (3, 1), (1, 3) \rangle$ is Buchsbaum but not Cohen-Macaulay.

2. How to determine whether a simplicial affine semigroup is Buchsbaum.

As we did in the previous section, we define in $\bigcap_{i=1}^r \mathrm{S}(n_i)$ the following binary relation:

$$x \sim y \text{ if } x - y \in \mathsf{G}(\{n_1, \dots, n_r\}).$$

The next result is used in [**10**] for giving a procedure for determining whether a simplicial affine semigroup is Cohen-Macaulay.

Proposition 8. *The following statements are equivalent.*
  (i) *S is a Cohen-Macaulay semigroup.*
  (ii) *For any $s \in S$ and $i \neq j \in \{1, \dots, r\}$, if $s - n_i$ and $s - n_j$ are in S then $s - (n_i + n_j)$ also belongs to S.*

(iii) *For every element $s \in S$ there exists a unique element $(a_1, \dots, a_r) \in \mathbb{N}^r$ and a unique element w in $\bigcap_{i=1}^r \mathrm{S}(n_i)$ such that $s = \sum_{i=1}^r a_i n_i + w$.*
(iv) *For every $x, y \in \bigcap_{i=1}^r \mathrm{S}(n_i)$, if $x - y \in \mathsf{G}(\{n_1, \dots, n_r\})$, then $x = y$.*
(v) *For every $x \in \bigcap_{i=1}^r \mathrm{S}(n_i)$, $[x] = \{x\}$.*

Proof. The equivalence between (i)-(iv) appears in [**10**]. Condition (v) is a reformulation of (iv). □

The next proposition shows what happens in the Buchsbaum case.

Theorem 9. *The affine semigroup S is Buchsbaum if and only if, for every $x \in \bigcap_{i=1}^r \mathrm{S}(n_i)$, if $\#[x] \geq 2$, then there exists $m \in \overline{S}$ such that $[x] = \{m + n_1, \dots, m + n_r\}$.*

Proof. Necessity. Let us assume that $\#[x] \geq 2$. By Theorem 5, $\overline{S}$ is Cohen-Macaulay (recall that $\overline{S}$ is a simplicial affine semigroup whose extremal rays are the extremal rays of S). Proposition 8 ensures that there exist unique $m \in \bigcap_{i=1}^r \overline{\mathrm{S}}(n_i)$ and $(a_1, \dots, a_r) \in \mathbb{N}^r$ such that $x = \sum_{i=1}^r a_i n_i + m$. In particular, this implies that $x - m \in \mathsf{G}(\{n_1, \dots, n_r\})$, whence $y - m \in \mathsf{G}(\{n_1, \dots, n_r\})$ for all $y \in [x]$. We show that in this case m cannot be in S. If this were not the case, then $a_1 = \cdots = a_r = 0$, since $x \in \bigcap_{i=1}^r \mathrm{S}(n_i)$. Thus $m = x$. Recall that $\#[x] \geq 2$ and hence there exists $y \in [x] \setminus \{x\}$. Using once more Proposition 8, there exist $m' \in \bigcap_{i=1}^r \overline{\mathrm{S}}(n_i)$ and $(b_1, \dots, b_r) \in \mathbb{N}^r$ such that $y = \sum_{i=1}^r b_i n_i + m'$. It follows that $m' - m = (x - m) + (y - x) + (m' - y) \in \mathsf{G}(\{n_1, \dots, n_r\})$. Condition (v) of Proposition 8 ensures that $[m] = \{m\}$, which leads to $m = m'$. Therefore $b_1 = \cdots = b_r = 0$, since $y \in [x] \subseteq \bigcap_{i=1}^r \mathrm{S}(n_i)$. This means that $y = m = x$, which contradicts $y \neq x$.

We show next that $\{m + n_1, \dots, m + n_r\} \subseteq [x]$. Since $m \in \overline{S}$, we get that $m + n_k \in S$ for all $1 \leq k \leq r$. The affine semigroup S is simplicial and for this reason there exists $(c_1, \dots, c_r) \in \mathbb{N}^r$ and $w \in \bigcap_{i=1}^r \mathrm{S}(n_i)$ such that $m + n_1 = \sum_{i=1}^r c_i n_i + w$ (observe that this forces w to be in $[x]$). In addition, $w \in \overline{S}$ and $\overline{S}$ is Cohen-Macaulay, which by Proposition 8 implies that there exist $(d_1, \dots, d_r) \in \mathbb{N}^r$ and $m' \in \bigcap_{i=1}^r \overline{\mathrm{S}}(n_i)$ such that $w = \sum_{i=1}^r d_i n_i + m'$. As before, we can deduce that $m = m'$. It follows $m + n_1 = (c_1 + d_1)n_1 + \cdots + (c_r + d_r)n_r + m$. From Proposition 8 we get that $c_1 + d_1 = 1$ and that $c_2 + d_2 = \cdots = c_r + d_r = 0$. This leads to $w = m + n_1 \in [x]$. Similarly it is shown that $m + n_i \in [x]$ for all $i \in \{2, \dots, r\}$.

For the opposite inclusion, take $y \in [x]$. Then $y \in S \subseteq \overline{S}$. By the same argument used above, there exists $(a_1, \dots, a_r) \in \mathbb{N}^r$ for which $y = \sum_{i=1}^r a_i n_i + m$. The fact that $y \in S$ implies that $\sum_{i=1}^r a_i \geq 1$ and $y \in \bigcap_{i=1}^r \mathrm{S}(n_i)$ forces $\sum_{i=1}^r a_i = 1$. Hence $y = m + n_i$ for some $i \in \{1, \dots, r\}$.

Sufficiency. Define

$$A = \left\{ m_{[x]} \mid [x] \in \bigcap_{i=1}^r \mathrm{S}(n_i)/\sim \right\},$$

$$\text{where } m_{[x]} = \begin{cases} x & \text{if } \#[\text{x}] = 1, \\ m & \text{if } [\text{x}] = \{\text{m} + \text{n}_1, \dots, \text{m} + \text{n}_\text{r}\}. \end{cases}$$

From the definition of A, any two of its elements are incongruent modulo $\mathsf{G}(\{n_1, \dots, n_r\})$. If we prove that $\bigcap_{i=1}^r \overline{\mathsf{S}}(n_i) \subseteq A$, we get that $\bigcap_{i=1}^r \overline{\mathsf{S}}(n_i)$ fulfills the same condition, which by Proposition 8 means that $\overline{S}$ is Cohen-Macaulay and by Theorem 5 that S is Buchsbaum. Thus it suffices to show that $\bigcap_{i=1}^r \overline{\mathsf{S}}(n_i) \subseteq A$. Take $x \in \bigcap_{i=1}^r \overline{\mathsf{S}}(n_i)$. Then $x \in \overline{S}$, whence $x + n_1, x + n_2 \in S$. Since S is simplicial, there exist $(c_1, \dots, c_r), (d_1, \dots, d_r) \in \mathbb{N}^r$ and $w, w' \in \bigcap_{i=1}^r \mathsf{S}(n_i)$ such that $x + n_1 = \sum_{i=1}^r c_i n_i + w$ and $x + n_2 = \sum_{i=1}^r d_i n_i + w'$. It follows that $w' \in [w]$. By the definition of A, there exists $m \in A$ for which $w = m$ or $w = m + n_i$ for some $i \in \{1, \dots, r\}$ and $m' \in A$ such that $w' = m'$ or $w' = m' + n_j$ for some $j \in \{1, \dots, r\}$. In any case, since $w' \in [w]$, we have $m = m'$. Thus both $x + n_1$ and $x + n_2$ can be written as $x + n_1 = \sum_{i=1}^r a_i n_i + m$ and $x + n_2 = \sum_{i=1}^r b_i n_i + m$ for some $(a_1, \dots, a_r), (b_1, \dots, b_r) \in \mathbb{N}^r$, which this leads to

$$a_1 n_1 + (a_2 + 1)n_2 + a_3 n_3 + \cdots + a_r n_r = (b_1 + 1)n_1 + b_2 n_2 + \cdots + b_r n_r.$$

Since $\{n_1, \dots, n_r\}$ is a basis of $\mathbb{Q}^r$, we get that $a_1 = b_1 + 1$, which implies that $a_1 \geq 1$. Hence $x = (a_1 - 1)n_1 + a_2 n_2 + \cdots + a_r n_r + m$. In addition, $m \in \overline{S}$ and $x \in \bigcap_{i=1}^r \overline{\mathsf{S}}(n_i)$, which forces x to be equal to m. $\qquad \square$

If we know $\bigcap_{i=1}^r \mathsf{S}(n_i)$, then we can check for every $x \in \bigcap_{i=1}^r \mathsf{S}(n_i)$, whether $\#[x] = 1$ or $[x] = \{m + n_1, \dots, m + n_r\}$ for some $m \in \overline{S}$. If this is not the case, then S is not Buchsbaum. In [3, 10] an algorithm for computing the set $\bigcap_{i=1}^r \mathsf{S}(n_i)$ is presented. This idea is based on the fact that

$$\bigcap_{i=1}^r \mathsf{S}(n_i) \subseteq \left\{ \sum_{i=1}^m \gamma_{r+i} n_{r+i} \mid \gamma_{r+i} < c_{r+i} \text{ for all } i \in \{1, \dots, m\} \right\},$$

where $c_{r+i} = \min\{k \in \mathbb{N} - \{0\} : k n_{r+i} \in \langle n_1, n_2, \dots, n_r \rangle\} \leq \alpha_1 \cdots \alpha_r$.

Thus Theorem 9, together with the algorithm for computing $\bigcap_{i=1}^r \mathsf{S}(n_i)$, constitutes a method for deciding whether a simplicial affine semigroup is Buchsbaum.

Example 10. Let $S = \langle (2, 0), (0, 1), (1, 2), (3, 1) \rangle$. We compute $\mathsf{S}((2, 0)) \cap \mathsf{S}((0, 1))$ as explained in [10] and obtain

$$\mathsf{S}((2, 0)) \cap \mathsf{S}((0, 1)) = \{(0, 0), (1, 2), (3, 1)\}.$$

Observe that

$$[(0, 0)] = \{(0, 0)\}, \quad [(1, 2)] = \{(1, 2), (3, 1)\} = \{(1, 1) + (2, 0), (1, 1) + (0, 1)\}.$$

Taking $m = (1, 1)$, $m + (2, 0), m + (0, 1), m + (1, 2), m + (3, 1) \in S$ and hence $m \in \overline{S}$. By Theorem 9, S is Buchsbaum but not Cohen-Macaulay, since $\#[(1, 2)] \neq 1$ (Proposition 8).

The condition $\#[x] \in \{1, r\}$ is not sufficient for S to be Buchsbaum, as the following example shows.

Example 11. Let $S = \langle (2, 0), (0, 2), (3, 1), (1, 3), (1, 2) \rangle$. Using the procedure presented in [**10**] to compute $\bigcap_{i=1}^{r} \mathrm{S}(n_i) = \mathrm{S}((2, 0)) \cap \mathrm{S}((0, 2))$, we get

$$\mathrm{S}((2, 0)) \cap \mathrm{S}((0, 2)) = \{(0, 0), (3, 1), (1, 3), (1, 2), (4, 3), (2, 5)\}.$$

It follows that

$$[(0, 0)] = \{(0, 0)\}, \; [(3, 1)] = \{(3, 1), (1, 3)\},$$

$$[(1, 2)] = \{(1, 2)\}, \; [(4, 3)] = \{(4, 3), (2, 5)\}.$$

By looking at $[(3, 1)]$, the only possible candidate to be m is $(1, 1)$. However, $m + (1, 2) = (2, 3) \notin S$, which by Theorem 9 implies that S is not Buchsbaum, since $m \notin \overline{S}$.

3. Buchsbaum semigroups with minimal Apéry set.

In the sequel we assume that $\{n_1, \dots, n_r, n_{r+1}, \dots, n_{r+m}\}$ is a minimal system of generators of S. By the definition of $\bigcap_{i=1}^{r} \mathrm{S}(n_i)$, this implies that $\{n_{r+1}, \dots, n_{r+m}\}$ is included in $\bigcap_{i=1}^{r} \mathrm{S}(n_i) \setminus \{0\}$. We say that S has **minimal Apéry set** if

$$\{0, n_{r+1}, \dots, n_{r+m}\} = \bigcap_{i=1}^{r} \mathrm{S}(n_i).$$

Here we transfer a result known for Cohen-Macaulay simplicial affine semigroups fulfilling this condition to the Buchsbaum case. To this end, we need to recall some basic concepts in order to fix notation.

Let φ be the map defined by

$$\varphi : \mathbb{N}^{r+m} \to S, \; \varphi(a_1, \dots, a_{r+m}) = \sum_{i=1}^{r+m} a_i n_i,$$

and denote its kernel congruence by σ. Then S is isomorphic to $\mathbb{N}^{r+m}/\sigma$.

We say that ρ is a **minimal system of generators** of σ if ρ generates σ and its cardinal is minimal among the cardinal of the sets generating σ. In this case we also say that ρ is a **minimal presentation** of S. It can be shown that $\#\rho \geq r + m - r = m$ (see [**6**]).

Let $n \in S - \{0\}$. Define the graph G_n as the graph whose vertices are

$$\mathrm{V}(G_n) = \{n_i \mid n - n_i \in S, i \in \{1, \dots, r + m\}\}$$

and whose edges are

$$\mathrm{E}(G_n) = \{\overline{n_i n_j} \mid n - (n_i + n_j) \in S, i, j \in \{1, \dots, r + m\}, i \neq j\}.$$

Define ρ_n as follows.

1) If G_n is connected, then $\rho_n = \emptyset$.

2) If G_n is not connected and $G_n^1, \ldots, G_n^t$ are the connected compo-
nents of G_n, then choose a vertex $n_{j_i} \in \mathrm{V}(G_n^i)$ and an element $\alpha_i^n = (a_1^i, \ldots, a_{r+m}^i) \in \mathbb{N}^{r+m}$ such that $\varphi(\alpha_i^n) = n$ and $a_{j_i}^i \neq 0$; define

$$\rho_n = \{(\alpha_2^n, \alpha_1^n), \ldots, (\alpha_t^n, \alpha_1^n)\}.$$

Take $\rho = \bigcup_{n \in S} \rho_n$. Then ρ is a minimal system of generators of σ (this follows from a straightforward generalization presented in [**3**, **11**] of the results given in [**9**]). Furthermore, every minimal system of generators of σ has the same cardinality.

Example 12. Let

$$S = \langle (2, 0), (0, 1), (1, 2), (3, 1) \rangle \subseteq \mathbb{N}^2.$$

The elements $n \in S$ for which G_n is not connected are $(3, 2)$, $(6, 2)$, $(4, 3)$ and $(2, 4)$.

Graph	Connected components	Relators
$G_{(3,2)}$	$\{(2,0), (1,2)\}, \{(0,1), (3,1)\}$	$e_1 + e_3 = e_2 + e_4$
$G_{(6,2)}$	$\{(2,0), (0,1)\}, \{(3,1)\}$	$3e_1 + 2e_2 = 2e_4$
$G_{(4,3)}$	$\{(2,0), (0,1)\}, \{(1,2), (3,1)\}$	$2e_1 + 3e_2 = e_3 + e_4$
$G_{(2,4)}$	$\{(2,0), (0,1)\}, \{(1,2)\}$	$e_1 + 4e_2 = 2e_3$

Hence

$$\begin{aligned}
\rho = \quad &\{((1,0,1,0),(0,1,0,1)), ((3,2,0,0),(0,0,0,2)), \\
&((2,3,0,0),(0,0,1,1)), ((1,4,0,0),(0,0,2,0))\}
\end{aligned}$$

is a minimal presentation of S.

In [**10**] the authors show that if S is a Cohen-Macaulay simplicial affine semigroup with minimal Apéry set (there called with maximal codimension), then $\#\rho = m(m+1)/2$. Moreover, this property characterizes Cohen-Macaulay simplicial affine semigroup with minimal Apéry set. Let us see what happens in the Buchsbaum case.

Theorem 13. *Let S be a Buchsbaum simplicial affine semigroup with minimal Apéry set. Let $\sim$ be the equivalence relation defined over $\bigcap_{i=1}^r \mathrm{S}(n_i)$ as before and $\lambda = \#\{[x] \in \bigcap_{i=1}^r \mathrm{S}(n_i)/\sim : \#[x] = r\}$. For every minimal system of presentation ρ of S,*

$$\#\rho = \frac{m(m+1)}{2} + \lambda \frac{r(r-1)}{2}.$$

Proof. As we have indicated before, every minimal system of generators has the same cardinality. Hence it suffices to count the elements belonging to $\rho = \bigcup_{n \in S} \rho_n$. For doing this, we must know which are the elements in S fulfilling that G_n is not connected. If $n \in S$ and G_n is not connected, then this graph must contain a connected component with some of its vertices lying in $\{n_1, \ldots, n_r\}$, otherwise $n - n_i \notin S$ for all $i \in \{1, \ldots, r\}$ and thus $n \in \bigcap_{i=1}^r \mathrm{S}(n_i) = \{0, n_{r+1}, \ldots, n_{r+m}\}$, contradicting that $\{n_1, \ldots, n_{r+m}\}$ is a minimal system of generators of S. For the rest of the proof and for a given $n \in S$ such that G_n is not connected, we fix G_n^1 (defined in the description of ρ given above) as one of these connected components of G_n fulfilling that some of its vertices are contained in $\{n_1, \ldots, n_r\}$. From the construction of ρ, for every $n \in S$ and every component of G_n other than G_n^1, we get a new element in ρ. It follows that in order to count the cardinality of ρ, we only have to decide how many connected components different from the fixed G_n^1's are in all the possible non-connected graphs G_n's. We first count those connected components in all the possible non-connected graphs not having vertices in $\{n_1, \ldots, n_r\}$ (these are of course different from any G_n^1) and then we will count those connected components having some vertices in $\{n_1, \ldots, n_r\}$ and different from the fixed G_n^1's.

Take $n \in S$ such that G_n is not connected and contains a connected component C whose vertices belong to $\{n_{r+1}, \ldots, n_{r+m}\}$. Then n can be expressed as $n = \sum_{i=1}^m a_i n_{r+i}$ with $(a_1, \ldots, a_m) \in \mathbb{N}^m$. Observe that $\sum_{i=1}^m a_i \geq 2$, since otherwise $n \in \bigcap_{i=1}^r \mathrm{S}(n_i)$. We claim that $\sum a_i = 2$. If this were not the case, then there would exist $i, j, k \in \{1, \ldots, m\}$ (maybe not different) such that $n = n_{r+i} + n_{r+j} + n_{r+k} + s$, for some $s \in \langle n_{r+1}, \ldots, n_{r+m} \rangle$. Since $n_{r+i} + n_{r+j} \notin \bigcap_{i=1}^r \mathrm{S}(n_i)$, there exists $l \in \{1, \ldots, r\}$ such that $n_{r+i} + n_{r+j} - n_l \in S$. However, this leads to $n - (n_{r+k} + n_l) \in S$, which implies that n_l is a vertex of C, contradicting $\mathrm{V}(C) \subseteq \{n_{r+1}, \ldots, n_{r+m}\}$. Hence n must be of the form $n = n_{r+i} + n_{r+j}$ with $i, j \in \{1, \ldots, m\}$. Conversely, since $n_{r+i} + n_{r+j} \notin \bigcap_{i=1}^r \mathrm{S}(n_i)$ for all $i, j \in \{1, \ldots, m\}$, each element of this form yields an element in ρ. In this way we collect $m(m+1)/2$ elements in ρ.

Now we determine for which $n \in S$ the graph G_n has at least two connected components containing vertices belonging to $\{n_1, \ldots, n_r\}$ (recall that one of these was taken to be G_n^1). If n fulfills this condition, then there must exist $i, j \in \{1, \ldots, r\}$ such that $n - n_i, n - n_j \in S$ and $n - (n_i + n_j) \notin S$ (n_i and n_j are in different connected components of G_n). Since S is Buchsbaum, Theorem 5 ensures that $\overline{S}$ is Cohen-Macaulay. The elements $n - n_i$, $n - n_j$ belong to S, which implies that they belong to $\overline{S}$ and by Proposition 8, we obtain that $n - (n_i + n_j) \in \overline{S}$. This leads to $n - (n_i + n_j) = m \in \overline{S} \setminus S$. As we did in the proof of Theorem 9, it is easy to show that $\{m + n_1, \ldots, m + n_r\} \subseteq \bigcap_{i=1}^r \mathrm{S}(n_i) = \{0, n_{r+1}, \ldots, n_{r+m}\}$. Since $n - n_i = m + n_j$ and $n - n_j = m + n_i$, there exists $s, t \in \{1, \ldots, m\}$ such that

$n = n_{r+t} + n_j = n_{r+s} + n_i = (m+n_i) + n_j = (m+n_j) + n_i$, which implies that $n_{r+s} \in [n_{r+t}]$ and thus $\#[n_{r+t}] = r = \#[n - n_i]$. In addition, $m - n_k \notin \overline{S}$ for all $k \in \{1, \ldots, r\}$, since otherwise $(m - n_k) + n_k = m$ should belong to S. Hence $m \in \bigcap_{i=1}^{r} \overline{S}(n_i)$. By Proposition 8, if there exists $m' \in \overline{S} \setminus S$ and $i', j' \in \{1, \ldots, r\}$ such that $n = (m + n_i) + n_j = (m' + n_i') + n_j'$, then m must be equal to m' and $\{i, i\} = \{i', j'\}$. This implies that in this case there are exactly two connected components of G_n with some of its vertices in $\{n_1, \ldots, n_r\}$ (and this yields a new element in ρ). Thus for a fixed m, we get as many new elements in ρ as elements of the form $(m + n_i) + n_j$ we can write with $i < j$. This makes $r(r-1)/2$ new elements in ρ. Moreover, for each element $x \in \bigcap_{i=1}^{r} S(n_i)$ such that $\#[x] = r$, we get an element m as before. It follows that we obtain $\lambda r(r-1)/2$ elements in ρ corresponding to the graphs having at least two connected components (and therefore exactly two) with some of its vertices lying in $\{n_1, \ldots, n_r\}$.

We conclude that $\#\rho = m(m+1)/2 + \lambda r(r-1)/2$. $\square$

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