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On buried points and phase transition points in the Julia sets concerning renormalization transformation

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Abstract

Considering a family of rational maps $T_{n\lambda}$ concerning renormalization transformation, we give a perfect description of buried points and phase transition points in the Julia set $J(T_{n\lambda})$. Furthermore, we prove that $J(T_{n\lambda})$ contains an open interval where all points are buried points for some parameters n and λ, which is according to the problem that Curry and Mayer proposed.

MSC: 37F10; 37F45

Keywords: buried point; phase transition; Julia set; renormalization transformation

1 Introduction

It is well known that the Julia set of a rational map is often a fractal. We say that a point in the Julia set is a buried point if it does not lie in the boundary of any Fatou component. If all points in some connected component of the Julia set are buried points, we say this component is called a buried component. Since McMullen [1] gave an example of a rational function with so-called buried points and buried components in its Julia set, much interest has been devoted to investigation of the geometrical and topological properties of buried points and buried components [2–7]. In fact, the existence of buried points and buried components shows that the Julia set has very complex topological properties. In this paper, we investigate the properties of buried points of the family of rational maps $T_{n\lambda}$ concerning renormalization transformation, where

$$T_{n\lambda}(z) = \left(\frac{z^2 + \lambda - 1}{2z + \lambda - 2} \right)^n \tag{1}$$

with two parameters $n \in \mathbb{N}$ ($n > 1$) and $\lambda \in \mathbb{R}$. This physical model can be derived from the limit distribution of zeroes of the partition function in the famous Yang-Lee theory [8–11]. In fact, some interesting relationships among the phase transitions, the critical exponents, the critical amplitudes and the shape of the Julia sets were found in [12]. After this, many works have been devoted to the Julia sets of the renormalization transformation [13–17].

In order to introduce our results, for any given natural number $n > 1$, we define the following constants:

$$\beta_n = 2 + \max_{-2 \leq t \leq 0} \frac{t^{2n} - 2t^{n+1} + 1}{t - 1}.$$

Figure 1 The Julia set $J(T_{5\lambda})$ for $\lambda = 1.824$. The white region in the real axis of this figure is indeed a buried open interval.
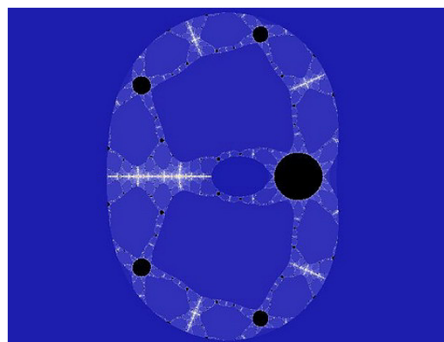
In [14], Qiao proved that $\beta_n \in (1, 2)$ when n is an even number and $\beta_n \in (2, 3)$ when n is an odd number. Moreover, he also proved that the equation $(t - 1)^{2n-1} - 2(t - 1)^n - t + 3 = 0$ has one unique real solution $\gamma_n \in (2, 3]$ when n is an even number. Furthermore, in this paper we prove the following.

Theorem 1 *Suppose that $T_{n\lambda}$ is defined as* (1), *we have*

(1) *if n is an even number, then $J(T_{n\lambda})$ contains buried points if and only if $\lambda \in (\beta_n, \gamma_n)$;*

(2) *if $n = 3$, then $J(T_{n\lambda})$ contains buried points if and only if $\lambda \in [3 - \sqrt{2}, \beta_n]$;*

(3) *if $n > 3$, then $J(T_{n\lambda})$ contains buried points if and only if $\lambda \in [\lambda_\alpha, \beta_n]$, where $\lambda_\alpha \in (1, 2)$ is an absolute constant.*

In [7], Curry and Mayer proposed some questions about buried points. One of these questions is the following:

Is there the set of buried points whose components are not either points, circles, or the irrational points of the Sierpinski curve?

In this paper, considering the Julia sets $J(T_{n\lambda})$, we prove that the set of buried points may contain an open interval of the real axis $\mathbb{R}$. In fact, we have the following.

Theorem 2 *Suppose that $n > 3$ is an odd number, then there exists $\lambda \in (1, 2)$ such that $J(T_{n\lambda})$ contains an open interval in which all points are buried points* (*see Figure* 1).

For this physical model, it is well known that there is a specific physical significance about the intersection points between the Julia set $J(T_{n\lambda})$ and the positive real axis $\mathbb{R}^+$ when λ is a positive integer. In fact, these intersection points are the locations of phase transitions of this model. A natural problem is how to distribute on $\mathbb{R}^+$ the locations of phase transitions? In the following we will show that the number of this kind of intersection points may be 1, 2, 3, or ∞. Denote by $\#(J(T_\lambda) \cap \mathbb{R}^+)$ the number of the intersection points of $J(T_\lambda)$ and $\mathbb{R}^+$. Then we have the following.

Theorem 3 *If λ is a positive integer, then*

(1) $\#(J(T_{n\lambda}) \cap \mathbb{R}^+) = \infty$ *for any even number $n \geq 2$ and $\lambda = 1$* (*see Figure* 2);

(2) $\#(J(T_{n\lambda}) \cap \mathbb{R}^+) = 3$ *for any odd number $n \geq 3$ and $\lambda = 1$* (*see the left figure in Figure* 3);

(3) $\#(J(T_{n\lambda}) \cap \mathbb{R}^+) = 2$ *for any positive $n \geq 2$ and $\lambda \in [2, 2 + \frac{1}{n-\sqrt[1]{2}-1})$* (*see the immediate figure in Figure* 3);

(4) $\#(J(T_{n\lambda}) \cap \mathbb{R}^+) = 1$ *for any positive $n \geq 2$ and $\lambda \in [2 + \frac{1}{n-\sqrt[1]{2}-1}, +\infty)$* (*see the right figure in Figure* 3).
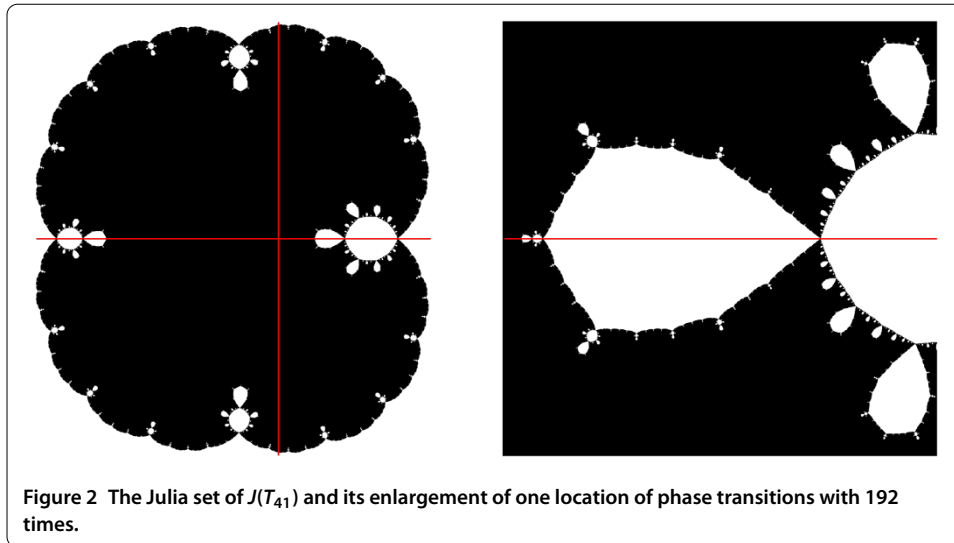
Figure 2 The Julia set of $J(T_{41})$ and its enlargement of one location of phase transitions with 192 times.
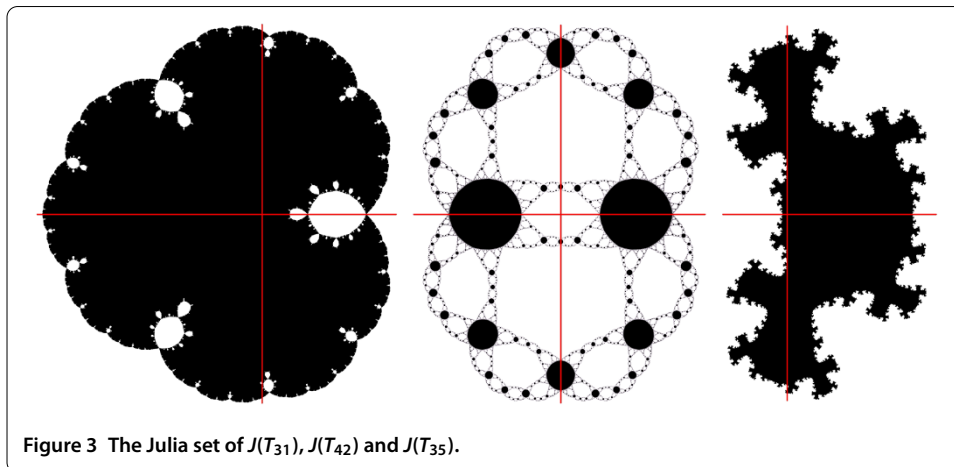

Figure 3 The Julia set of $J(T_{31})$, $J(T_{42})$ and $J(T_{35})$.

2 Some notations and preliminary results

Let f be a rational map with degree $d \geq 2$ from a complex sphere $\overline{\mathbb{C}}$ to itself. The notation f^k means the kth iteration of f. A point z is called a *critical point* if $f'(z) = 0$. A point z is called a *periodic point* if $f^k(z) = z$ for some $k \geq 1$, the minimal of such k is called the *period* of z. For a periodic point z_0 with periodic k, the *multiplier* of z_0 is defined as $(f^k)'(z_0)$. The periodic point z_0 is either *attracting, indifferent*, or *repelling* according to $|(f^k)'(z_0)| < 1$, $|(f^k)'(z_0)| = 1$, or $|(f^k)'(z_0)| > 1$. The *Julia set*, denoted by $J(f)$, is the closure of repelling periodic points. Its complement is called the *Fatou set*, denoted by $F(f)$. A component D of $F(f)$ is called completely invariant if $f(D) = f^{-1}(D) = D$. Moreover, if D is a completely invariant component, then $J(f) = \partial D$. Let $P(f)$ be the *post-critical set* of f, *i.e.*, the closure of the forward orbits of critical points. It plays a crucial rule in the study of complex dynamics. For the classical results in complex dynamics, see [18, 19] and [20].

In order to prove our theorems, we need the following lemmas.

Lemma 1 ([5, 6]) *Let f be a rational map f with $\deg f \geq 2$ and $J(f)$ connected and locally connected, then $J(f)$ contains buried points if and only if f^2 has no completely invariant Fatou component.*

Lemma 2 ([17]) *If $J(T_{n\lambda})$ is connected, then it is locally connected.*

Lemma 3 ([17]) *If $n \geq 3$ is an odd integer, then $T_{n\lambda}$ has only four real fixed points $q_1, q_2, 1, q_3$ ($q_1 < q_2 \leq 0$, $q_3 > 1$) for $\lambda \in [1, 2)$.*

Lemma 4 ([17]) *Suppose that q_2 is the fixed point stated in Lemma 3. If $n = 3$ and $\lambda \in (1, 2)$, then q_2 is an attracting fixed point. If $n > 3$ is an odd integer, then there exist two numbers λ_α and λ_β ($\lambda_\alpha < \lambda_\beta$) in $(1, 2)$ such that*

   (1) *q_2 is an attracting fixed point for $\lambda \in (1, \lambda_\alpha) \cup (\lambda_\beta, 2)$;*
   (2) *q_2 is a parabolic fixed point and $T'_{n\lambda}(q_2) = -1$ for $\lambda = \lambda_\alpha$ or $\lambda = \lambda_\beta$;*
   (3) *q_2 is a repelling fixed point for $\lambda \in (\lambda_\alpha, \lambda_\beta)$.*

Lemma 5 ([14] and [15])

   (1) *If $n \geq 2$ is an even integer, then $T_{n\lambda}^2$ contains a completely invariant Fatou component if and only if $\lambda \in (-\infty, \beta_n] \cup (\gamma_n, +\infty)$. Furthermore, $J(T_{n\lambda})$ is connected for $\lambda \in (\beta_n, \gamma_n)$.*
   (2) *If $n > 1$ is an odd integer, then $T_{n\lambda}^2$ contains a completely invariant Fatou component for $\lambda \in (-\infty, 1] \cup (\beta_n, +\infty)$. Furthermore, $T_{n\lambda}^2$ contains at most three Fatou periodic (not completely invariant) cycles for $\lambda \in [2, \beta_n]$.*

3 Proof of Theorem 1

For $\lambda \neq 0$, by (1), we have

$$T'_{n\lambda}(z) = \frac{2n(z-1)(z+\lambda-1)(z^2+\lambda-1)^{n-1}}{(2z+\lambda-2)^{n+1}}. \tag{2}$$

So $T_{n\lambda}$ has only four critical values $1, \infty, 0$ and $T_{n\lambda}(-\lambda+1) = (-\lambda+1)^n$.

By Lemma 1, Lemma 2 and Lemma 5, it is easy to see that $J(T_{n\lambda})$ contains buried points if and only if $\lambda \in (\beta_n, \gamma_n)$ for even integer n. Similarly, if $n > 1$ is an odd number, $J(T_{n\lambda})$ contains no buried point for $\lambda \in (-\infty, 1] \cup (\beta_n, +\infty)$, $J(T_{n\lambda})$ contains buried points for $\lambda \in [2, \beta_n]$. Hence we need only to investigate the case that $n > 1$ is an odd integer and $\lambda \in (1, 2)$.

Obviously, 1 and ∞ are two super-attracting fixed points of $T_{n\lambda}$ for $\lambda \in (1, \beta_n]$. In the following we denote by $A_\lambda(1)$ and $A_\lambda(\infty)$ two Fatou components of $T_{n\lambda}$ that contain 1 and ∞, respectively. It is easy to see that $T_{n\lambda}(x)$ is monotone increasing from q_1 to $-(\lambda-1)^n$ on $(q_1, 1-\lambda)$ and monotone decreasing from $-(\lambda-1)^n$ to $-\infty$ on $(1-\lambda, -\frac{\lambda}{2}+1)$, then $q_2 < T_{n\lambda}(1-\lambda) = -(\lambda-1)^n$. It follows that there exists a unique point $q_1^* \in (1-\lambda, -\frac{\lambda}{2}+1)$ such that $T_{n\lambda}(x) > q_1$ for $x \in (q_1, q_1^*)$ and $T_{n\lambda}(x) < q_1$ for $x \in (q_1^*, -\frac{\lambda}{2}+1)$. By a similar discussion, there exists a unique point $q_3^* \in (-\frac{\lambda}{2}+1, 1)$ such that $T_{n\lambda}(x) > q_3$ for $x \in (-\frac{\lambda}{2}+1, q_3^*)$ and $T_{n\lambda}(x) < q_1$ for $x \in (q_1^*, -\frac{\lambda}{2}+1)$. Hence we get

$$1 - \lambda < q_2 < (1-\lambda)^2 < q_1^* < 1 - \frac{1}{2}\lambda < q_3^* < 1 < q_3.$$

Furthermore,

$$[-\infty, q_1) \cup (q_3, \infty] \subset A_\lambda(\infty), \qquad (q_3^*, q_3) \subset A_\lambda(1), \qquad (q_1^*, q_3^*) \subset D\left(-\frac{\lambda}{2}+1\right) \tag{3}$$

and

$$T_{n\lambda}\big([q_1, q_1^*]\big) = \big[q_1, (1-\lambda)^n\big] \subset \big[q_1, q_1^*\big], \tag{4}$$

where $D(-\frac{\lambda}{2}+1)$ is the preimage of $A_\lambda(\infty)$ containing $-\frac{\lambda}{2}+1$. The fixed points q_1 and q_3 are repelled. Below we prove

$$q_1^* > 0, = 0, < 0 \quad \Leftrightarrow \quad \lambda \in \left(1, 2 - \frac{1}{1 + \sqrt[n-1]{2}}\right), = 2 - \frac{1}{1 + \sqrt[n-1]{2}}, \in \left(2 - \frac{1}{1 + \sqrt[n-1]{2}}, 2\right), \tag{5}$$

respectively.

In fact, set $x = \frac{\lambda-1}{2-\lambda}$, then

$$T_{n\lambda}^2(0) = T_{n\lambda}(0)\left(\frac{x^{2n-1} + \frac{1}{x+1}}{2x^n + \frac{1}{x+1}}\right)^n. \tag{6}$$

It can be verified that

$$\frac{x^{2n-1} + \frac{1}{x+1}}{2x^n + \frac{1}{x+1}} > 1, = 1, \text{ or } < 1$$

is equivalent to $x > \sqrt[n-1]{2}, = \sqrt[n-1]{2}, \text{ or } < \sqrt[n-1]{2}$, respectively. We get (5) since $T_{n\lambda}(0) < 0$.

(I) If $n = 3$ and $\lambda \in (1, 2 - \frac{1}{\sqrt[n-1]{2}+1})$. It is easy to verify that $(q_1, q_1^*) \subset A_\lambda(q_2)$ since $T_{3\lambda}(x) > x$ for $x \in (q_1, q_2)$. Hence $0 \in A_\lambda(q_2)$. Next we define

$$\Gamma(\lambda) = \left\{z : \left|z + \frac{\lambda}{2} - 1\right| = \frac{\lambda}{2}\right\} \cap \{z : \mathbf{Re\, z} \leq \mathbf{0}\}.$$

It follows that

$$T_{3\lambda}\big(\Gamma(\lambda)\big) = \big[-(\lambda-1)^n, 0\big] \subset \big(q_1, q_1^*\big) \subset A_\lambda(q_2).$$

Note that $1 - \lambda \in \Gamma(\lambda) \cap (q_1, q_1^*)$, then $\Gamma(\lambda) \subset A_\lambda(q_2)$. Since $T_{3\lambda}^{-1}(0) = \pm\sqrt{\lambda-1}i \in \Gamma(\lambda)$, then $A_\lambda(q_2)$ is completely invariant. This implies that $J(T_{3\lambda}) = \partial A_\lambda(q_2)$ and thus $J(T_{3\lambda})$ contains no buried point.

If $n = 3$ and $\lambda \in [2 - \frac{1}{\sqrt[n-1]{2}+1}, 2)$. By (3) and (5), it is easy to see that $A_\lambda(q_2)$ contains only one critical point $z = 1 - \lambda$ and $\deg(T_{n\lambda}|_{A_\lambda(q_2)}) = 2$. Hence $A_\lambda(q_2)$ cannot be completely invariant since $\deg(T_{n\lambda}) = 5$. Note that $A_\lambda(1)$ and $A_\lambda(\infty)$ are also not completely invariant, then $J(T_{3\lambda})$ must contain buried points by Lemma 1.

(II) If $n \geq 5$. Let λ_α and λ_β be two numbers stated in Lemma 4. Below we prove

$$\lambda_\alpha < 2 - \frac{1}{\sqrt[n-1]{2}+1} \tag{7}$$

and

$$\lambda_\beta > 2 - \frac{1}{\sqrt[n-1]{2}+1}. \tag{8}$$

By the proof of Proposition 3 in [17], we know that q_2 is attracting, parabolic, or repelling according to

$$(2n+1)t^{2n} - 2nt^{2n-1} - 2nt^{n+1} + 2(n-1)t^n + 1 > 0, = 0, \text{ or } < 0,$$

respectively, where $t = \sqrt[n]{x} \in (-1, 0)$ and $T_{n\lambda}(x) = x$. Set

$$P_n(t) = (2n+1)t^{2n} - 2nt^{2n-1} - 2nt^{n+1} + 2(n-1)t^n + 1.$$

Take $t_0^n = -\frac{1}{2(n-1)}$, it is easy to see that $-t_0 - t_0^{n-1} > \frac{1}{5}$ for $n \geq 5$. By a calculation, we have

$$P_n(t_0) = \frac{1}{n-1}\left(\frac{2n+1}{4(n-1)} - n\left(-t_0 - t_0^{n-1}\right)\right) < 0.$$

Note that $P_n(0) = 1$, then there exists $\alpha \in (t_0, 0)$ such that $P_n(\alpha) = 0$. We can easily deduce from $T_{n\lambda}(x) = x$ that

$$\lambda = \phi(t) = \frac{t^{2n} - 2t^{n+1} + 2t - 1}{t - 1}.$$

By the proof of Proposition 2 in [17], we know that $\phi(t)$ is monotone decreasing from $\phi(-1) = 2$ to $\phi(0) = 1$ on $(-1, 0)$. So we conclude that $\lambda_\alpha < \lambda_{t_0}$ since q_2 is a parabolic fixed point of $T_{n\lambda_\alpha}$ by Lemma 4.

Put $t_* = -t_0$, we have

$$\begin{aligned}
\lambda_{t_0} &= \frac{(t_0^n - 1)(t_0^n + 1 - 2t_0)}{t_0 - 1} \\
&= \frac{(2n-1)}{2(n-1)}\left(2 - \frac{(2n-1)}{2(n-1)(t_* + 1)}\right) \\
&= \frac{3}{2} - \left(\frac{(2n-1)^2}{4(n-1)^2(t_* + 1)} - \left(\frac{1}{n-1} + \frac{1}{2}\right)\right).
\end{aligned}$$

It can be proved that $\varphi(n) = (1 - \frac{4n-5}{2(n^2-1)})^n$ ($n \geq 5$) is monotone decreasing and it tends to $\frac{1}{e^2}$ when $n \to \infty$. It is obvious that $t_*^n = \frac{1}{2(n-1)} < \frac{1}{e^2}$ for $n \geq 5$. Hence $t_*^n < (1 - \frac{4n-5}{2(n^2-1)})^n$, i.e., $t_* < \frac{2n^2-4n+3}{2(n^2-1)}$. By a calculation, we have

$$\frac{(2n-1)^2}{4(n-1)^2(t_* + 1)} - \left(\frac{1}{n-1} + \frac{1}{2}\right) > 0.$$

So $\lambda_{t_0} < \frac{3}{2}$ and thus (7) is obvious.

Below we prove (8).

Take $t_1^n = -\frac{2}{3}$, it is easy to see that $t_1 \in (-1, -0.85)$ for $n \geq 5$. Furthermore, we have $P_n(t_1) < 0$. Note that $P_n(-1) = 3$, there exists $\beta \in (-1, t_1)$ such that $P_n(\beta) = 0$. Because $\lambda = \phi(t) = \frac{t^{2n} - 2t^{n+1} + 2t - 1}{t-1}$ is monotone decreasing from $\phi(-1) = 2$ to $\phi(0) = 1$ on $(-1, 0)$, it follows that $\lambda_\beta > \lambda_{t_1}$. Hence

$$\lambda_{t_1} = \frac{(t_1^n - 1)(t_1^n + 1 - 2t_1)}{t_1 - 1} = \frac{5}{3} \cdot \left(\frac{1}{3} - 2t_1\right) \cdot \frac{1}{1 - t_1} > \frac{5}{3}.$$

We get (8).

If $\lambda \in (1, \lambda_\alpha)$. By (5) and (7), we know $q_1^* > 0$. By the same discussion as used in the case $n = 3$, we can deduce that $A_\lambda(q_2)$ is completely invariant for $\lambda \in (1, \lambda_\alpha)$. This implies that $J(T_{n\lambda}) = \partial A_\lambda(q_2)$ and thus $J(T_{n\lambda})$ contains no buried point.

If $\lambda \in [\lambda_\alpha, \lambda_\beta]$, it is easy to see that $F(T_{n\lambda})$ contains at most another k-periodic cycle except for $A_\lambda(1)$ and $A_\lambda(\infty)$. By (3) and (4), we can deduce that $k \geq 2$ since q_2 is a repelling fixed point by Lemma 4 (in fact, the periodic Fatou components lie on both sides of q_2). Then $T_{n\lambda}^2$ cannot contain a completely invariant Fatou component. This shows that $J(T_{n\lambda})$ contains buried points.

If $\lambda \in (\lambda_\beta, 2)$. By (5), $q_1^* < 0$. By (3), $T_{n\lambda}(0) \in A_\lambda(\infty)$. It shows that $A_\lambda(q_2)$ is not completely invariant since it contains only one critical point $z = 1 - \lambda$ and $\deg(T_{n\lambda}^k|_{A_\lambda(q_2)}) = 2$, and thus $J(T_{n\lambda})$ contains buried points. It completes the proof of Theorem 1.

4 Proof of Theorem 2

Note that the post-critical set $P(T_{n\lambda}) \subset \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. By [17], the Julia set $J(T_{n\lambda})$ is connected for $\lambda \in (1, 2)$. By (3) and (4), we can prove that

$$\partial A_\lambda(\infty) \cap \overline{\mathbb{R}} = \{q_1, q_3\} \quad \text{and} \quad \partial A_\lambda(1) \cap \overline{\mathbb{R}} = \{q_3^*, q_3\}. \tag{9}$$

In fact, let Γ be the circle centered at $1 - \frac{1}{2}\lambda$ with radius $\frac{1}{2}\lambda$. Then Γ passes through critical points 1, $1 - \lambda$ and $\pm i\sqrt{\lambda - 1}$, and $T_{n\lambda}(\Gamma) = [(1 - \lambda)^n, 1] \subset \overline{\text{Int}(\Gamma)}$. We conclude that $A_\lambda(\infty) \cap \overline{\text{Int}(\Gamma)} = \emptyset$ and $\partial A_\lambda(\infty) \cap [1 - \lambda, 1] = \emptyset$. For $x \in (q_1, 1 - \lambda)$, there exists a positive integer k such that $T_{n\lambda}^k(x) \in (1 - \lambda, (1 - \lambda)^n]$. This implies $x \notin \partial A_\lambda(\infty)$. Hence $\partial A_\lambda(\infty) \cap \overline{\mathbb{R}} = \{q_1, q_3\}$.

Let Ω_0 be the component of $\mathbb{C} \setminus T_{n\lambda}^{-1}([-\infty, 0])$ containing 1. Then $1 - \frac{1}{2}\lambda \in \partial\Omega_0$ and $A_\lambda(1) \subset \Omega_0$. Therefore $\overline{A_\lambda(1)} \cap [q_1, q_1^*] = \emptyset$ and $\partial A_\lambda(1) \cap \overline{\mathbb{R}} = \{q_3^*, q_3\}$.

Obviously, $T_{n\lambda}^{-1}(A_\lambda(\infty))$ has only two components $A_\lambda(\infty)$ and $D(-\frac{\lambda}{2} + 1)$. By $T_{n\lambda} : \mathbb{R} \to \mathbb{R} \setminus ((1 - \lambda)^n, 1)$ and $\overline{D(-\frac{\lambda}{2} + 1)} \cap \mathbb{R} = [q_1^*, q_3^*] \subset ((1 - \lambda)^n, 1)$, we know that $\overline{W} \cap \mathbb{R} = \emptyset$ for every component W of $T_{n\lambda}^{-1}(D(-\frac{\lambda}{2} + 1))$. Similarly, by (3) and (4), we can deduce that $\overline{W_*} \cap \mathbb{R} = \emptyset$ for every component W_* of $T_{n\lambda}^{-1}(A_\lambda(1))$ except for $A_\lambda(1)$. Hence the closures of all pullbacks of W and W_* are disjoint with $\mathbb{R}$ since $T_{n\lambda}$ maps $\mathbb{R}$ into $\mathbb{R} \cup \{\infty\}$. By [17], we know that there exist at least two real numbers $\lambda_1, \lambda_2 \in (1, 2)$ such that $T_{n\lambda_1}$ and $T_{n\lambda_2}$ are Feigenbaum-like maps. Obviously, $T_{n\lambda_1}$ and $T_{n\lambda_2}$ contain only two Fatou periodic domains $A_\lambda(1)$ and $A_\lambda(\infty)$. By the above analysis and (4), we conclude that (q_1, q_1^*) lies in the Julia sets $J(T_{n\lambda_1})$ and $J(T_{n\lambda_2})$, and each point of this open interval does not belong to the boundaries of any Fatou components. This implies that (q_1, q_1^*) is a buried interval of $J(T_{n\lambda_1})$ and $J(T_{n\lambda_2})$ (in fact, by a calculation, there exists $\lambda_0 \in (1, 2)$ such that $1 - \lambda_0$ is strictly eventually periodic, then (q_1, q_1^*) is also a buried interval of $J(T_{n\lambda_0})$). It completes the proof of Theorem 2.

5 Proof of Theorem 3

Obviously, $T_{n1}(z) = (\frac{z^2}{2z-1})^n$. It has only four critical points 0, $\frac{1}{2}$, 1 and ∞. Let $\phi(z) = \frac{1}{z}$, then

$$S(z) = \phi^{-1} \circ T_{n1} \circ \phi(z) = z^n(2 - z)^n$$

is a polynomial. Note that $\phi(\infty) = 0$, then the immediate basin $A_\lambda(0)$ containing $z = 0$ of T_{n1} is completely invariant. Below we prove (1) and (2) of Theorem 3.

(1) If n is an even number. We consider the real fixed points of T_{n1}, *i.e.*, the set $\{x \in \mathbb{R} | T_{n1}(x) = (\frac{x^2}{2x-1})^n = x\}$. It is easy to see that $T_{n\lambda}(x)$ has no fixed point on $(-\infty, 0)$ since $T_{n\lambda}(x) \geq 0$ for $x \in \mathbb{R}$. By (2), $T_{n1}(x)$ is monotone increasing on $(0, \frac{1}{2})$ from 0 to $+\infty$. Note that $T_{n1}(\frac{1}{2}) = +\infty$ and $z = 1$ is an attracting fixed point, then there exists at least one fixed point $\alpha \in (0, \frac{1}{2})$ such that $T_{n1}(x) > x$ for $x \in (0, \alpha)$ and $(0, \alpha) \subset A_\lambda(0)$.

Next we claim that α is the unique fixed point in $(0, \frac{1}{2})$.

Otherwise, if there exists another fixed point γ in $(0, \frac{1}{2})$. Without loss of generality, we assume $\alpha < \gamma$, then $T_{n1}(x) > x$ or $T_{n1}(x) < x$ on (α, γ). Note that $T_{n1}(x)$ is monotone increasing on $(0, \frac{1}{2})$, then one of them (α or γ) is an attracting fixed point or a parabolic fixed point. Since $T_{n1}(0) = 0$, $T_{n1}(\frac{1}{2}) = +\infty$, $T_{n1}(1) = 1$ and $T_{n1}(\infty) = \infty$, then the immediate basin contains no critical point. But the Sullivan theorem [18] says that each immediate basin contains at least one critical point. It is a contradiction. Hence α is the unique fixed point on $(0, \frac{1}{2})$. Similarly, we can deduce that there also exists a unique fixed point β in $(1, +\infty)$. Hence T_{n1} has only four real fixed points 0, 1, α, β. α and β are two repelling fixed points.

Furthermore, by the monotonicity of T_{n1} on $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$, we know that there exist only two points $\beta_2 \in (\alpha, \frac{1}{2})$ and $\beta_1 \in (\frac{1}{2}, 1)$ such that $T_{n1}(\beta_2) = T_{n1}(\beta_1) = \beta$. Set $f(x) = T_{n1}(x) - \beta_2$. Note that $f(\alpha) < 0$ and $f(\beta_2) > 0$. By the monotonicity of $f(x)$, we know that there exists a unique point $\beta_3 \in (\alpha, \beta_2)$ such that $T_{n1}(\beta_3) = \beta_2$. Proceeding like this, we can obtain a strictly monotone decreasing sequence $\{\beta_m\} \subset (\alpha, \beta)$, which satisfies $T_{n1}(\beta_{m+1}) = \beta_m$ ($m = 2, 3, \ldots$). This implies $\#(J(T_{n1}) \cap \mathbb{R}^+) = \infty$ since β is a repelling fixed point.

(2) If $n > 1$ is an odd number. By Lemma 3, we know that T_{n1} has only four fixed points $q_1 < 0$, $q_2 = 0, 1$ and $q_3 > 1$. By a similar discussion as used in (3) in Theorem 1, we can deduce that there exist only two points $q_1^* \in (0, \frac{1}{2})$ and $q_3^* \in (\frac{1}{2}, 1)$ such that $T_{n1}(q_1^*) = q_1$ and $T_{n1}(q_3^*) = q_3$. Furthermore, we can deduce that

$$(q_1, q_1^*) \subset A_{n1}(0), \qquad (q_1^*, q_3^*) \subset A_\lambda^{-1}(\infty),$$

$$(q_3, q_3) \subset A_\lambda(1), \qquad (q_3, +\infty) \subset A_\lambda(\infty).$$

Hence we get $J(T_{n1}) \cap \mathbb{R}^+ = \{q_1^*, q_3^*, q_3\}$, *i.e.*, $\#(J(T_{n1}) \cap \mathbb{R}^+) = 3$.

(3) We know that $T_{n\gamma}$ has only one fixed point $q \in (1, +\infty)$. Obviously, q is a repelling fixed point and $(q, +\infty) \subset A_\lambda(\infty)$. In what follows we distinguish two cases to discuss.

(a) If $n \geq 2$ and $\lambda = 2$. $T_{n2}(z) = (\frac{z^2+1}{2z})^n$, T_{n2} has only six critical points ± 1, $\pm i$, 0, ∞ and $T_{n2}(\pm 1) = 1$, $T_{n2}(\pm i) = 0$, $T_{n2}(0) = \infty$, $T_{n2}(\infty) = \infty$. Considering the real fixed points of T_{n2}, $T_{n2}(x)$ has no fixed point on $(-\infty, 0)$. Note that $T_{n\lambda}(x)$ is monotone decreasing on $(0, 1)$ from $+\infty$ to 1, then $T_{n\lambda}(x)$ has no fixed point on $(0, 1)$.

Note that $T_{n\lambda}(x)$ is monotone decreasing on $(0, 1)$ from $+\infty$ to 1, then there exists a unique point $q_1 \in (0, 1)$ such that $T_{n2}(q_1) = q$. By Lemma 5, $[0, q_1) \subset D(-\frac{\lambda}{2} + 1)$ and $(q_1, q) \subset A_\lambda(1)$. We get $J(T_{n2}) \cap \mathbb{R}^+ = \{q_1, q\}$, *i.e.*, $\#(J(T_{n2}) \cap \mathbb{R}^+) = 2$.

(b) If $n > 2$ and $\lambda \in (2, 2 + \frac{1}{n - \sqrt[n]{1/2} - 1})$. By a similar discussion as (6) in Theorem 1, we can easily deduce that $T_{n\lambda}^2(0) > T_{n\lambda}(0)$. Then $T_{n\lambda}(0) > 1$. We conclude $T_{n\lambda}(0) > q$ since $T_{n\lambda}(x) < x$ for $x \in (1, q)$. By (2), $T_{n\lambda}(x)$ is monotone decreasing on $(0, 1)$. Then there exists a unique point $q_1 \in (0, 1)$ such that $T_{n1}(q_1) = q$. Note that $q_1 \in (0, 1)$, then $J(T_{n\lambda}) \cap \mathbb{R}^+ = \{q_1, q\}$, *i.e.*, $\#(J(T_{n\lambda}) \cap \mathbb{R}^+) = 2$.

(4) We also distinguish two cases to discuss.

(c) If $n = 2$ and $\lambda = 3$. We know that T_{23} has a unique repelling point $x_0 = 4$ on $(1, +\infty)$. It is easy to see that $T_{23}(x) < x$ for $x \in (1, 4)$ and $T_{23}(x) > x$ for $x \in (4, +\infty)$. Note that $T_{23}(0) = 4$ and $T_{23}(x)$ is monotone decreasing for $x \in (0, 1)$, then $T_{23}(x) \in (1, 4)$ for $x \in (0, 1)$. Then $(0, 4) \subset A_\lambda(1)$ and $(4, +\infty) \subset A_\lambda(\infty)$. We get $J(T_{23}) \cap \mathbb{R}^+ = \{x_0\}$, *i.e.*, $\#(J(T_{23}) \cap \mathbb{R}^+) = 1$.

(d) If $n > 2$ and $\lambda \in (2 + \frac{1}{n - \sqrt[n]{2} - 1}, +\infty)$. By a similar discussion as (6) in Theorem 1, we also get $T_{n\lambda}^2(0) < T_{n\lambda}(0)$. Hence $T_{n\lambda}(0) \in (1, q)$ since $T_{n\lambda}(0) > 1$. Note that $T_{n\lambda}(x)$ is monotone decreasing for $x \in [0, 1)$, then $T_{n\lambda}(x) \in (1, q)$ for $x \in [0, 1)$. This means $[0, q) \subset A_\lambda(1)$. We obtain $J(T_{n\lambda}) \cap \mathbb{R}^+ = \{q\}$, *i.e.*, $\#(J(T_{n\lambda}) \cap \mathbb{R}^+) = 1$. The proof of Theorem 3 is completed.

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