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ON C^*-ALGEBRAS ASSOCIATED TO RIGHT LCM SEMIGROUPS

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ABSTRACT. We initiate the study of the internal structure of C^*-algebras associated to a left cancellative semigroup in which any two principal right ideals are either disjoint or intersect in another principal right ideal; these are variously called right LCM semigroups or semigroups that satisfy Clifford's condition. Our main findings are results about uniqueness of the full semigroup C^*-algebra. We build our analysis upon a rich interaction between the group of units of the semigroup and the family of constructible right ideals. As an application we identify algebraic conditions on S under which $C^*(S)$ is purely infinite and simple.

1. INTRODUCTION

In recent years, C^*-algebras associated to semigroups have received much attention due to the range of new examples and interesting applications that they encompass. One such application is to the connections between operator algebras and number theory, which have grown deeper since Cuntz's work in [6] on the C^*-algebra $\mathcal{Q}_{\mathbb{N}}$ associated to the affine semigroup over the natural numbers $\mathbb{N} \rtimes \mathbb{N}^\times$. Laca and Raeburn [12] continued the analysis of C^*-algebras associated to $\mathbb{N} \rtimes \mathbb{N}^\times$ by examining the Toeplitz algebra $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$, including an analysis of its KMS structure. Cuntz, Deninger and Laca [7] have since examined the KMS structure of Toeplitz-type C^*-algebras associated to $ax+b$-semigroups $R \rtimes R^\times$ of rings of integers R in number fields.

Li has recently defined C^*-algebras associated to left cancellative semigroups S with identity and initiated a study of when certain naturally arising $*$-homomorphisms are injective [19, 20]. The reduced C^*-algebra $C_r^*(S)$ associated to S is defined by means of the left regular representation of S on the Hilbert space $\ell^2(S)$. The full C^*-algebra $C^*(S)$ is defined to be the universal C^*-algebra generated by isometries and projections, subject to certain relations which are imposed by the regular representation. For certain classes of semigroups, the canonical isomorphism between the full and reduced semigroup C^*-algebras was established in [19, 20, 27].

In [2], the authors studied the full semigroup C^*-algebra arising from an algebraic construction called a Zappa-Szép product of semigroups. The resulting semigroups display ordering features similar to the quasi-lattice ordered semigroups introduced

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by Nica [26], but by contrast contain a non-trivial group of units. These semigroups were called right LCM (for least common multiples) in [2], and we shall henceforth use this terminology, but mention that in [17, §4.1] and [27] these are known as semigroups that satisfy Clifford's condition. The class of right LCM semigroups is pleasantly large and includes quasi-lattice ordered semigroups, certain semidirect products of semigroups, and also semigroups that model self-similar group actions; see [2, 16, 18].

In the present work we begin a study of the internal structure of C^*-algebras associated to right LCM semigroups. The main thrust of our work is that when S is a right LCM semigroup one may unveil the internal structure of $C^*(S)$ and answer questions about its uniqueness by carefully analysing the relationship between the group of units S^* and the constructible right ideals of S.

The problem of finding good criteria for injectivity of $*$-homomorphisms on $C^*(S)$ and in particular of deciding uniqueness of such C^*-algebras is at the moment not settled in the generality of left cancellative semigroups. A powerful method to prove injectivity of $*$-representations was developed by Laca and Raeburn in [11, Theorem 3.7] for $C^*(S)$ with (G, S) quasi-lattice ordered. Their work recast Nica's C^*-algebras associated to quasi-lattice ordered groups in [26] by viewing them as C^*-crossed products by semigroups of endomorphisms. Based on this realisation, they adapted a technique introduced by Cuntz in [5] which involved expecting onto a diagonal subalgebra.

There are new technical obstacles to be overcome when dealing with a semigroup S that has a non-trivial group of units. In particular, not all of Laca and Raeburn's programme can be carried through beyond the case of quasi-lattice ordered pairs. One challenge is that the diagonal subalgebra of $C^*(S)$, denoted $\mathcal{D}$ in [19], may be too small to accommodate the range of a conditional expectation from $C^*(S)$; cf. an observation made in [27]. Furthermore, generating isometries in $C^*(S)$ that correspond to elements from the group of units S^* give rise to unitaries. These unitaries together with the generating projections from $\mathcal{D}$ yield two new subalgebras of $C^*(S)$ whose role in explaining the structure of $C^*(S)$ is yet to be fully understood.

Our initial approach was to push to the fullest extent the Laca-Raeburn strategy to an arbitrary right LCM semigroup S, with or without an identity. It soon became evident that the presence of non-trivial units in S^* makes it unlikely that [11, Theorem 3.7] will extend in the greatest generality to right LCM semigroups. However, by carefully analysing the action of the group of units S^* on the constructible right ideals $\mathcal{J}(S)$ of S we are able to identify conditions on S which ensure that injectivity of $*$-homomorphisms on $C^*(S)$ can be characterised on $\mathcal{D}$. This approach has lead us to find conditions on a right LCM semigroup S which ensure that $C^*(S)$ is purely infinite and simple. The examples we have of such semigroups belong to a class of semidirect products $G \rtimes_\theta P$ of a group G by an injective endomorphic action θ of a semigroup P. C^*-algebras associated to such semidirect products where $P = \mathbb{N}$ were studied by Cuntz and Vershik in [9] and by Vieira in [30]. Our $C^*(G \rtimes_\theta P)$ may be interpreted as higher dimensional versions of those C^*-algebras. We mention that K-theory and internal structure of C^*-algebras associated to $ax + b$-semigroups of certain integral domains were analysed recently by Li; see [21].

The organisation of the paper is as follows. In Section 2 we collect some standard results about semigroups. We also introduce our conventions on semidirect product semigroups and identify an abstract characterisation of the examples of interest $G \rtimes_\theta P$. Section 3 contains an introduction to right LCM semigroups and their associated full and reduced C^*-algebras. Since we do not assume that S necessarily contains an identity element, we explain how the definitions of $C_r^*(S)$ and $C^*(S)$ from [19] can be adapted to this, slightly more general, situation. In the same section we introduce the distinguished subalgebras of interest, which are built out of $\mathcal{D}$ and the unitaries coming from the group of units S^*. We also discuss conditional expectations onto the diagonal subalgebras of $C^*(S)$ and $C_r^*(S)$.

Our first findings about injectivity of a $*$-homomorphism on $C^*(S)$ are the subject of Section 4. We show in Theorem 4.3 that injectivity can be phrased as a non-vanishing condition involving projections from $\mathcal{D}$, similar to [11, Theorem 3.7], when the semigroup S has at most an identity element as unit, or, in the presence of non-trivial units, satisfies a technical condition on the left action of S^* on the space $\mathcal{J}(S)$. In Section 5 we identify a number of conditions on a right LCM semigroup S which imply that $C^*(S)$ is purely infinite and simple. These conditions include a characterisation of the left action of S^* on $\mathcal{J}(S)$ which is a refined version of effective action; we call this strongly effective. In a short Section 6 we discuss injectivity of the canonical surjection from $C^*(S)$ onto $C_r^*(S)$ and illustrate this with semigroups of the form $G \rtimes_\theta P$. Section 7 initiates the study of injectivity of $*$-homomorphisms on $C^*(S)$ phrased in terms of a core subalgebra that is built from $\mathcal{D}$ and the unitaries corresponding to the group of units S^* in S. The final section, Section 8, is devoted to applications. Here we discuss the validity of the properties of right LCM semigroups introduced in Sections 4 and 5. The main class of examples is that of semidirect products of the form $G \rtimes_\theta P$, and via Theorem 8.12 we provide examples of purely infinite simple $C^*(S)$ from this class. We also take the opportunity to examine the Zappa-Szép product semigroups $X^* \bowtie G$ coming from self-similar actions (G, X) as considered in [2, 16, 18]; in particular, we examine some of the properties of semigroups introduced in the paper. While at this stage we cannot apply our C^*-algebraic results to this class of semigroups, we plan to examine these problems in further work.

2. Some results on semigroups

By a semigroup S we understand a non-empty set S with an associative operation. We refer to [4] and [14] for basic properties of semigroups. Semigroups with an identity element for the operation are known as monoids. Here we shall use the terminology semigroup and specify existence of an identity when this is the case. All semigroups considered in this work are discrete. A semigroup S is *left cancellative* if $pq = pr$ implies $q = r$ for all $p, q, r \in S$; *right cancellative* if $qp = rp$ implies $q = r$ for all $p, q, r \in S$; and *cancellative* if it is both left and right cancellative.

Given a semigroup S with identity 1_S, an element x in S is invertible if there is $y \in S$ such that $xy = yx = 1_S$. We denote by S^* the group of invertible elements of S (also called the group of units of S). We shall write $S^* \neq \emptyset$ in case the group of units is non-trivial (possibly consisting only of the identity element), and we write $S^* = \emptyset$ otherwise. If S is cancellative and $x \in S^*$, then x^{-1} will denote the inverse of x.

The Green relations on a semigroup are well-known; see for example [14, Chapter 2]. The left Green relation $\mathcal{L}$ is $a\mathcal{L}b$ if and only if $Sa = Sb$ for $a, b \in S$. Likewise, the right Green relation $\mathcal{R}$ is given by $a\mathcal{R}b$ if and only if $aS = bS$ for $a, b \in S$.

Suppose that S is a semigroup with $S^* \neq \emptyset$. Since $Sx = S$ whenever $x \in S^*$, we see that $a = xb$ for some $x \in S^*$ implies that $a\mathcal{L}b$. If S is right cancellative, the reverse implication holds and, moreover, the element x in S^* is unique. Indeed, let $Sa = Sb$. Then there are $c, d \in S$ such that $b = ca$ and $a = db$, so $b = cdb$ and $a = dca$. Thus right cancellation implies $cd = 1_S = dc$, showing that $c, d \in S^*$. If right cancellation is replaced with left cancellation in the previous considerations, then $a\mathcal{R}b$ is the same as $a = by$ for a unique $y \in S^*$.

If $S^* = \emptyset$, we will assume throughout this paper that S has the following property: if $a, b \in S$ satisfy $aS = bS$, then $a = b$. This is what happens in the case that $S^* = \{1_S\}$.

Given a semigroup S, a right ideal R is a non-empty subset of S such that $RS \subseteq R$. The *principal right ideals* of S are all the right ideals of the form $pS := \{ps \mid s \in S\}$ for $p \in S$. Given a principal right ideal pS, an element $r \in pS$ is called a *right multiple* of p. The right ideal generated by $p \in S$ is defined as $\{p\} \cup pS$; we shall denote it $\langle p \rangle$.

Remark 2.1. If S has an identity it is clear that $pS = \langle p \rangle$. For an arbitrary left cancellative semigroup S and $p \in S$, a sufficient condition to have $pS = \langle p \rangle$ is that there is an idempotent $t \in S$, i.e. $t = tt$, such that $p = pt$. Note that if p is a regular element of S, in the sense that there is $s \in S$ such that $p = psp$, then $t = sp$ is an idempotent such that $p = pt$. Thus $p \in pS$ whenever p is a regular element in a semigroup S.

Definition 2.2. A semigroup S is *right LCM* if it is left cancellative and every pair of elements p and q with a right common multiple has a right least common multiple r.

It is clear that a semigroup S is right LCM if it is left cancellative and for any p, q in S, the intersection of principal right ideals $pS \cap qS$ is either empty or of the form rS for some $r \in S$. This property of semigroups is called *Clifford's condition* in [17, §4.1] and [27]. In general, right least common multiples are not unique: if r is a right least common multiple of p and q, then so is rx for any $x \in S^*$.

The quasi-lattice ordered groups treated in [26] are examples of right LCM semigroups with unique right least common multiples. We discuss other examples in Section 8. The main class of examples of semigroups that is considered in the present work is that of semidirect product semigroups. We introduce next our conventions for a semidirect product of semigroups.

For a semigroup T we let $\operatorname{End} T$ denote the semigroup of all homomorphisms $T \to T$. The identity endomorphism is id_T. An action $P \overset{\theta}{\curvearrowright} T$ of a semigroup P on T is a homomorphism $\theta : P \to \operatorname{End} T$, i.e. $\theta_p\theta_q = \theta_{pq}$ for all $p, q \in P$. If T has an identity 1_T, we shall require that $\theta_p(1_T) = 1_T$ for all $p \in P$. In case P has an identity 1_P, we shall further require that θ_{1_P} is the identity endomorphism of T.

Definition 2.3. Let T, P be semigroups and $P \overset{\theta}{\curvearrowright} T$ an action. The *semidirect product* of T by P with respect to θ, denoted $T \rtimes_\theta P$, is the semigroup $T \times P$ with composition given by

$$(s, p)(t, q) = (s\theta_p(t), pq),$$

for $s, t \in T$ and $p, q \in P$.

Examples of semidirect products are $ax + b$-semigroups, where T comprises the additive structure and P the multiplicative structure in some ring or field. It is known that $T \rtimes_\theta P$ is right cancellative when T and P are both right cancellative, and $T \rtimes_\theta P$ is left cancellative when T and P are both left cancellative and, in addition, θ is an action by injective endomorphisms of T.

In the next result we describe S^* in the case of a semidirect product $S = G \rtimes_\theta P$ in which G is a group.

Lemma 2.4. *Let G be a group, P a semigroup and $P \overset{\theta}{\curvearrowright} G$ an action such that $G \rtimes_\theta P$ is left cancellative. If P has an identity, then $(G \rtimes_\theta P)^* = G \rtimes_\theta P^*$ holds; otherwise $G \rtimes_\theta P$ does not have an identity.*

Proof. If P has an identity element 1_P, the identity element of $G \rtimes_\theta P$ is given by $(1_G, 1_P)$. Now let $(g, x) \in (G \rtimes_\theta P)^*$. By definition, there is $(h, y) \in G \rtimes_\theta P$ such that $(g\theta_x(h), xy) = (g, x)(h, y) = (1_G, 1_P)$. Thus, $x \in P^*$. Conversely, if $x \in P^*$ and $g \in G$, the inverse of (g, x) is given by $(\theta_{x^{-1}}(g^{-1}), x^{-1})$. The second case is obvious. $\square$

Remark 2.5. Let G be a group, P a semigroup with $P^* = \{1_P\}$ and $P \overset{\theta}{\curvearrowright} G$ an action such that $G \rtimes_\theta P$ is left cancellative. Given $(g, p) \in G \rtimes_\theta P$, we have $(g, p)(h, 1_P) = (g\theta_p(h)g^{-1}, 1_P)(g, p)$ for any $h \in G$. By Lemma 2.4, $a(G \rtimes_\theta P)^* \subset (G \rtimes_\theta P)^* a$ for any a in $G \rtimes_\theta P$. This observation motivates the next considerations.

In [4, §10.3], a subset H of a semigroup S is called centric if $aH = Ha$ for every $a \in S$. For a semigroup S with $S^* \neq \emptyset$, we shall consider two one-sided versions of this condition.

Definition 2.6. Given a semigroup S with $S^* \neq \emptyset$, let (C1) and (C2) be the conditions:

(C1) $aS^* \subseteq S^* a$ for all $a \in S$.
(C2) $S^* a \subseteq aS^*$ for all $a \in S$.

Proposition 2.7. *Let S be a semigroup with $S^* \neq \emptyset$. Consider the equivalence relation on S given as follows: for $a, b \in S$,*

$$a \sim b \text{ if } a = xb \text{ for some } x \in S^*.$$

If S satisfies (C1)*, then $\sim$ is a congruence on S. Consequently, if $\mathcal{S} := S/_\sim$ denotes the collection of equivalence classes $[a] := \{b \in S \mid b \sim a\}$, then $\mathcal{S}$ is a semigroup with identity $[1_S]$. Moreover, $\mathcal{S}^* = \{[1_S]\}$.*

Proof. It is routine to check that $\sim$ is an equivalence relation. To show that it is a congruence on S, we must show that whenever $a \sim b$ then $cad \sim cbd$ for all c, d in S. Let $x \in S^*$ such that $a = xb$. By (C1), there is $x' \in S^*$ such that $cx = x'c$. Then $cad = cxbd = x'cbd$, giving the claim. Thus $[a_1] \cdot [a_2] := [a_1 a_2]$ for $a_1, a_2 \in S$ is a well-defined operation which turns $\mathcal{S}$ into a semigroup with identity $[1_S]$.

Suppose that $[a][b] = [1_S] = [b][a]$ for $a, b \in S$. Then $ab = x$ and $ba = y$ for $x, y \in S^*$, which shows that $bx^{-1} = y^{-1}b$ is an inverse for a. Similarly, $b \in S^*$, and thus $[a] = [b] = [1_S]$. $\square$

Remark 2.8. The relation $\sim$ from Proposition 2.7 is closely related to the left Green relation: since $Sx = S$ whenever $x \in S^*$, we see that $a \sim b$ implies $a\mathcal{L}b$. If S is right cancellative, then also $a\mathcal{L}b$ implies $a \sim b$.

Our interest is in semigroups S that are left cancellative and often cancellative. So we would like to know when the semigroup $\mathcal{S}$ from Proposition 2.7 inherits these properties. One sufficient condition for left cancellation to pass from S to $\mathcal{S}$ is spelled out in the next lemma, whose immediate proof we omit.

Lemma 2.9. *Let S be a semigroup with $S^* \neq \emptyset$ and satisfying* (C1)*. If S is right cancellative, then $\mathcal{S}$ is right cancellative. Further, $\mathcal{S}$ is left cancellative if S is left cancellative and has the following property:*

$$ab = xac \text{ for } a, b, c \in S, x \in S^* \Longrightarrow \exists y \in S^* \text{ with } xa = ay.$$

Proposition 2.10. *Let P be a semigroup with $P^* \neq \emptyset$, G a group and $P \overset{\theta}{\curvearrowright} G$ an action by injective group endomorphisms of G. Denote by $S = G \rtimes_\theta P$ the resulting semidirect product.*

(a) *If P satisfies* (C1)*, then so does S.*
(b) *If P is right cancellative and satisfies* (C1)*, then $\mathcal{S}$ is right cancellative.*
(c) *If P is left cancellative and P^* is centric, then $\mathcal{S}$ is left cancellative.*

Proof. For (a), let $(g, p) \in S$ and $(g', x) \in S^* = G \rtimes P^*$, according to Lemma 2.4. Choose by (C1) an element $y \in P^*$ such that $px = yp$. It follows that $(g, p)(g', x) = (g'', y)(g, p)$ for $g'' = g\theta_p(g')\theta_y(g^{-1})$. For assertion (b), note that S has (C1) by (a) and is right cancellative, so the claim follows by Lemma 2.9.

To prove (c), first note that $\mathcal{S}$ is well-defined since S has (C1). Suppose we have elements (g, p), (h, q), (k, r) in S and $(g_0, p_0) \in S^*$ such that $(g, p)(h, q) = (g_0, p_0)(g, p)(k, r)$. Therefore $(g\theta_p(h), pq) = (g_0\theta_{p_0}(g\theta_p(k)), p_0pr)$. Since P^* is centric, there is a unique $p_1 \in P^*$ such that $p_0p = pp_1$. Choosing $g_1 = h\theta_{p_1}(k^{-1})$ in G we have $(g_0, p_0)(g, p) = (g, p)(g_1, p_1)$. Hence Lemma 2.9 applies and shows that $\mathcal{S}$ is left cancellative. ☐

The next result shows that cancellative semigroups which are semidirect products of the form $G \rtimes_\theta P$, with $P^* = \{1_P\}$, can be characterised abstractly as cancellative semigroups S that satisfy (C1) and for which the quotient map of S onto $\mathcal{S}$ admits a homomorphism lift.

Proposition 2.11. *There is a bijective correspondence between the class of cancellative semigroups S with identity 1_S satisfying* (C1) *and such that the quotient map from S onto $\mathcal{S}$ admits a transversal homomorphism which embeds $\mathcal{S}$ into S and the class of semidirect product semigroups $G \rtimes_\theta P$ arising from a cancellative semigroup P with $P^* = \{1_P\}$, which acts by injective endomorphisms of a group G.*

Proof. Suppose S is cancellative with 1_S, satisfies (C1), and is such that there is an embedding of $\mathcal{S}$ as a subsemigroup of S which is a right inverse for the quotient map $S \to \mathcal{S}$. For ease of notation, we identify $\mathcal{S} \subseteq S$. Then for each $p \in \mathcal{S}$ we have a map $\theta_p : S^* \to S^*$, where $\theta_p(x)$ is the unique element of S^* satisfying $px = \theta_p(x)p$. Note that such an element exists because of (C1), and is unique because S is right cancellative. We claim that $\theta : p \mapsto \theta_p$ is an action of $\mathcal{S}$ by injective endomorphisms of S^*. For each $p \in \mathcal{S}$ and $x, y \in S^*$ we have $\theta_p(xy)p = pxy = \theta_p(x)py = \theta_p(x)\theta_p(y)p$, which by right cancellation means $\theta_p(xy) = \theta_p(x)\theta_p(y)$. Since we obviously have $\theta_p(1_S) = 1_S$, each θ_p is an endomorphism of S^*. For each $p, q \in \mathcal{S}$ and $x \in S^*$ we have $\theta_{pq}(x)pq = pqx = p\theta_q(x)q = \theta_p(\theta_q(x))pq$, which by right cancellation means $\theta_{pq}(x) = \theta_p(\theta_q(x))$, and so θ is an action. Hence we can form the semidirect product $S^* \rtimes_\theta \mathcal{S}$. We have each θ_p injective because $\theta_p(x) = \theta_p(y)$ implies $px = \theta_p(x)p = \theta_p(y)p = py$, resulting in $x = y$.

The map $\phi : S^* \rtimes_\theta \mathcal{S} \to S$ given by $\phi((x,p)) = xp$ is a homomorphism because

$$\phi((x,p))\phi((y,q)) = xpyq = x\theta_p(y)pq = \phi((x,p)(y,q)).$$

For each $r \in S$ we choose $p \in \mathcal{S}$ as the representative of r in $\mathcal{S}$. Then $r = xp$ for some $x \in S^*$, which means $r = \phi((x,p))$, and hence ϕ is surjective. For injectivity note that $\phi((x,p)) = \phi((y,q))$ means p and q differ by a unit. Hence as elements of $\mathcal{S}$ they must be equal. Then right cancellation gives $x = y$. So $\phi : S^* \rtimes_\theta \mathcal{S} \to S$ is an isomorphism. Moreover, $\mathcal{S}$ is cancellative because S is cancellative, and we have $\mathcal{S}^* = \{1_S\}$ because $[x] = [1_S]$ for all $x \in S^*$. Since $\mathcal{S}^* = \{1_S\}$, $\mathcal{S}$ trivially satisfies (C1).

Now suppose that P is cancellative with $P^* = \{1_P\}$ and acts by injective endomorphisms on a group G. Then we know from the discussion on semidirect products prior to Lemma 2.4 that $G \rtimes_\theta P$ is cancellative. We also know from Proposition 2.10 that $G \rtimes_\theta P$ satisfies (C1). Denote by $\mathcal{S}_{G,P}$ the semigroup obtained by applying Proposition 2.7 to $G \rtimes_\theta P$, and consider the map $\pi : \mathcal{S}_{G,P} \to G \rtimes_\theta P$ given by $\pi([(g,p)]) = (1_G, p)$. Since $(G \rtimes_\theta P)^* = G \times \{1_P\}$, the equality $[(g,p)] = [(h,q)]$ implies $p = q$, which means $\pi([(g,p)]) = \pi([(h,q)])$. So π is well defined. We have

$$\pi([(g,p)][(h,q)]) = \pi([(g\theta_p(h), pq)]) = pq = \pi([(g,p)])\pi([(h,q)])$$

for each $[(g,p)], [(h,q)] \in \mathcal{S}_{G,P}$, and so π is a homomorphism. Moreover, π is obviously unital. Finally, for each $[(g,p)], [(h,q)] \in \mathcal{S}_{G,P}$ we have

$$\pi([(g,p)]) = \pi([(h,q)]) \Longrightarrow p = q,$$

so $(g,p) = (gh^{-1}, 1_P)(h,q)$, resulting in $[(g,p)] = [(h,q)]$. Thus π is injective, and hence a semigroup embedding in $G \rtimes_\theta P$. □

3. Right LCM semigroup C^*-algebras

3.1. **Semigroup C^*-algebras.** In [19], Li constructed the reduced and the full C^*-algebras $C_r^*(S)$ and $C^*(S)$ associated to a left cancellative semigroup S with identity. In this work we shall allow semigroups that do not necessarily have an identity, so we start by investigating to what extent the construction of $C_r^*(S)$ and $C^*(S)$ from [19] still makes sense.

Let S be a left cancellative semigroup, and let $\{\varepsilon_t\}_{t\in S}$ denote the canonical orthonormal basis of $\ell^2(S)$ such that $(\varepsilon_s|\varepsilon_t) = \delta_{s,t}$ for $s,t \in S$. For each $p \in S$ let V_p be the operator in $\mathcal{L}(\ell^2(S))$ given by $V_p\varepsilon_t = \varepsilon_{pt}$ for all $t \in S$. We have $V_p^*V_p = I$ in $\mathcal{L}(\ell^2(S))$, so that V_p is an isometry for every $p \in S$. We define the *reduced* C^*-algebra $C_r^*(S)$ to be the unital C^*-subalgebra of $\mathcal{L}(\ell^2(S))$ generated by V_p for all $p \in S$.

Given $p \in S$, clearly $V_pV_p^*\varepsilon_s = 0$ when $s \notin pS$. Left cancellation implies that $V_pV_p^*\varepsilon_s = \varepsilon_s$ when $s \in pS$. Thus the range projection $V_pV_p^*$ of V_p is the orthogonal projection onto the subspace $\ell^2(pS)$ corresponding to the principal right ideal pS. We shall denote this projection by E_{pS}. With reference to Remark 2.1, note that p need not belong to pS. However, p is contained in pS if S has an identity or if p is a regular element of S. We summarise some properties of the elements V_p and E_{pS} in the next lemma, whose proof we omit.

Lemma 3.1. *Let S be a left cancellative semigroup that does not necessarily have an identity. Then for each p in S, the range projection of V_p is equal to the orthogonal projection E_{pS} onto the subspace $\ell^2(pS)$. Further, the isometries V_p and the projections E_{pS} satisfy the relations:*

(1) $V_pV_q = V_{pq}$;
(2) $V_pE_{qS}V_p^* = E_{pqS}$;
(3) $E_{pS}E_{qS} = E_{pS\cap qS}$

for all $p, q \in S$.

Recall from [19] that for each right ideal X and $p \in S$, the sets

$$pX = \{px \mid x \in X\} \quad \text{and} \quad p^{-1}X = \{y \in S \mid py \in X\}$$

are also right ideals. Li [19, §2.1] defines the set of *constructible right ideals* $\mathcal{J}(S)$ to be the smallest family of right ideals of S satisfying

(1) $\emptyset, S \in \mathcal{J}(S)$ and
(2) $X \in \mathcal{J}(S), p \in S \Longrightarrow pX, p^{-1}X \in \mathcal{J}(S)$.

An inductive argument as in the proof of [19, Lemma 3.3] shows that (1) and (2) imply

(3) $X, Y \in \mathcal{J}(S) \Longrightarrow X \cap Y \in \mathcal{J}(S)$.

The full C^*-algebra for a left cancellative semigroup S will be defined in terms of generators and relations similar to what is done in [19] for semigroups with identity.

Definition 3.2. Let S be a left cancellative semigroup. The *full semigroup C^*-algebra $C^*(S)$* is the universal unital C^*-algebra generated by isometries $(v_p)_{p\in S}$ and projections $(e_X)_{X\in\mathcal{J}(S)}$ satisfying

(L1) $v_pv_q = v_{pq}$;
(L2) $v_pe_Xv_p^* = e_{pX}$;
(L3) $e_\emptyset = 0$ and $e_S = 1$; and
(L4) $e_Xe_Y = e_{X\cap Y}$,

for all $p, q \in S$, $X, Y \in \mathcal{J}(S)$.

The *left regular representation* is the $*$-homomorphism $\lambda : C^*(S) \to C_r^*(S)$ given by $\lambda(v_p) = V_p$ for all $p \in S$.

In [19], the set of constructible right ideals $\mathcal{J}(S)$ is called *independent* if for every choice of $X, X_1, \ldots, X_n \in \mathcal{J}(S)$ we have

$$X_j \subsetneqq X \text{ for all } 1 \le j \le n \Longrightarrow \bigcup_{j=1}^{n} X_j \subsetneqq X.$$

Equivalently, $\mathcal{J}(S)$ is independent if $\bigcup_{j=1}^n X_j = X$ implies $X_j = X$ for some $1 \le j \le n$.

The next two lemmas explain why right LCM semigroups form a particularly tractable class of semigroups. The proof of the first of these lemmas is left to the reader.

Lemma 3.3. *If S is a right LCM semigroup, then $\mathcal{J}(S) = \{\emptyset, S\} \cup \{pS \mid p \in S\}$.*

Lemma 3.4. *Let S be a right LCM semigroup. Then $\bigcup_{X\in F} X \subsetneqq S$ holds for all finite subsets $F \subset \mathcal{J}(S) \setminus \{S\}$ if and only if $\mathcal{J}(S)$ is independent.*

Proof. Clearly, independence of $\mathcal{J}(S)$ implies $\bigcup_{X\in F} X \subsetneqq S$ for all finite $F \subset \mathcal{J}(S)\setminus\{S\}$. Conversely, let $X, X_1, \ldots, X_n \in \mathcal{J}(S)$ satisfy $X_i \subsetneqq X$. Since S is right LCM, Lemma 3.3 gives $p, p_1, \ldots, p_n \in S$ with $X = pS, X_i = p_iS$ for $i = 1, \ldots, n$. For each $i = 1, \ldots, n$, $X_i \subsetneqq X$ implies that $p_i = pp_i'$ for some $p_i' \in S$ with $p_i'S \subsetneqq S$. Thus

$$\bigcup_{1\leq i\leq n} X_i = p \bigcup_{1\leq i\leq n} p_i'S \text{ and } X = pS.$$

By left cancellation, $\bigcup_{1\leq i\leq n} X_i = X$ is equivalent to $\bigcup_{1\leq i\leq n} p_i'S = S$. However, the second statement is false by the choice of $p_i'S$. Hence $\bigcup_{1\leq i\leq n} X_i \subsetneqq X$ and $\mathcal{J}(S)$ is independent. $\square$

Remark 3.5. Let S be a left cancellative semigroup and $\mathcal{J}(S)$ the family of constructible right ideals. Let F be a finite subset of $\mathcal{J}(S)\setminus\{S\}$. Note that if S has an identity 1_S, then $\bigcup\limits_{X\in F} X \subsetneqq S$ holds. Indeed, if we had $\bigcup\limits_{X\in F} X = S$, then there would exist $X \in F$ such that $1_S \in X$, so $X = S$ since X is a right ideal, a contradiction.

Corollary 3.6. *If S is a right LCM semigroup with identity, then $\mathcal{J}(S)$ is independent.*

Proof. This follows from [27, Proposition 2.3.5]. Alternatively, apply Lemma 3.4 and Remark 3.5. $\square$

If S does not have an identity, we can always pass to its unitisation $\tilde{S} = S\cup\{1_S\}$, where we declare $1_Sp = p = p1_S$ for all $p \in \tilde{S}$.

Lemma 3.7. *If S is a right LCM semigroup with $S^* = \emptyset$, then for every $p, q \in S$ we have $pS \cap qS = \emptyset$ precisely when $p\tilde{S} \cap q\tilde{S} = \emptyset$, and*

$$pS \cap qS = rS \text{ if and only if } p\tilde{S} \cap q\tilde{S} = r\tilde{S}$$

for $r \in S$. In particular, $\tilde{S}$ is right LCM and $\mathcal{J}(\tilde{S})$ is independent.

Proof. Let $p, q \in S$. It is clear that $pS \cap qS$ is empty if and only if $p\tilde{S} \cap q\tilde{S}$ is. Suppose next that $pS\cap qS \neq \emptyset$. In case $pS = qS$, the standing assumption imposed on semigroups without identity element forces $p = q$, and so $p\tilde{S} = q\tilde{S}$. Assume therefore that $pS \neq qS$, and let $r \in S$ with $pS \cap qS = rS$. Then

$$pS \cap qS = rS \subset r\tilde{S} \subseteq p\tilde{S} \cap q\tilde{S}.$$

We claim that $p\tilde{S} \cap q\tilde{S} \subseteq r\tilde{S}$. Let $t \in p\tilde{S} \cap q\tilde{S}$. If $t \in pS \cap qS$, then clearly $t \in rS \subset r\tilde{S}$. Assume that $t = q = ps$ for some $s \in S$. Then $t = q1_S \in q\tilde{S}$ and $t \in pS \subset p\tilde{S}$, so $t \in r\tilde{S}$. The case that $t = p = qu$ for some $u \in S$ is similar, and the claim is established.

Since left cancellation in $\tilde{S}$ is inherited from S, this shows that $\tilde{S}$ is a right LCM semigroup. Thus $\mathcal{J}(\tilde{S})$ is independent according to Corollary 3.6. $\square$

The following example shows that independence of $\mathcal{J}(S)$ need not hold in general for semigroups without an identity:

Example 3.8. Let $S = 2\mathbb{N}^\times \cup 3\mathbb{N}^\times$ be endowed with composition given by multiplication. Then $\mathcal{J}(S)$ is not independent. Indeed, for $X_1 = 2\mathbb{N}^\times = 3^{-1}(2S)$ and $X_2 = 3\mathbb{N}^\times = 2^{-1}(3S)$, we have $X_i \subsetneqq S$ but $X_1 \cup X_2 = S$. We remark that S is not right LCM.

One can modify the previous example to get a right LCM semigroup with $S^* = \emptyset$ such that $\mathcal{J}(S)$ is independent.

Example 3.9. Consider the set $S = \mathbb{N}^\times \setminus \{1\}$ with composition given by multiplication. Then S is a right LCM semigroup with $S^* = \emptyset$. We claim that $\mathcal{J}(S)$ is independent. For this it suffices to show that $\bigcup_{X \in F} X \subsetneqq S$ holds for all finite $F \subset \mathcal{J}(S) \setminus \{S\}$. Assume that $\bigcup_{i=1}^n X_i = S$ for $X_1, \dots, X_n$ in $\mathcal{J}(S) \setminus \{S\}$. Since S contains $n+1$ relatively prime elements $p_1, \dots, p_{n+1}$, we can find $i_0 \in \{1, \dots, n\}$ and $j, k \in \{1, \dots, n+1\}$ with $j \neq k$ such that $p_j, p_k \in X_{i_0}$. But this implies that $X_{i_0} = S$, a contradiction. The underlying idea is that as long as there are infinitely many prime right ideals, $\mathcal{J}(S)$ is independent.

Remark 3.10. For a left cancellative semigroup S, the range projection $v_p v_p^*$ of the generating isometry v_p in $C^*(S)$ equals e_{pS}:

$$v_p v_p^* \overset{(L3)}{=} v_p e_S v_p^* \overset{(L2)}{=} e_{pS}.$$

Thus, if S has an identity, then v_x is a unitary in $C^*(S)$ if (and only if) $x \in S^*$. If S is right LCM, then Lemma 3.3 shows that $C^*(S)$ is generated already by $(v_p)_{p \in S}$.

3.2. **Spanning families and distinguished subalgebras.** When S is a right LCM semigroup we have a description of its C^*-algebra $C^*(S)$ in terms of a spanning set of monomials of the kind that span C^*-algebras associated to quasi-lattice ordered pairs; see [11]. This assertion could be deduced from [27, Proposition 3.2.15]; however we include a proof since here we do not assume that S necessarily has an identity.

Lemma 3.11. *Let S be a right LCM semigroup. If S has an identity, then $C^*(S) = \overline{\operatorname{span}}\{v_p v_q^* \mid p, q \in S\}$. If $S^* = \emptyset$, then $C^*(S) = \overline{\operatorname{span}}\{v_p v_q^* \mid p, q \in \tilde{S}\}$.*

Proof. In each case, the right-hand side is closed under taking adjoints and, due to Remark 3.10, contains the generators of $C^*(S)$. Hence, we only need to show that the right-hand side is multiplicatively closed. Using (L1), it suffices to show that the product of v_q^* and v_p for arbitrary p and q in S is 0 or has the form $v_{p'} v_{q'}^*$ for some $p', q' \in S$. By Remark 3.10, we have

$$v_q^* v_p = v_q^* e_{qS} e_{pS} v_p \overset{(L4)}{=} v_q^* e_{qS \cap pS} v_p.$$

Since S is right LCM, we know that $pS \cap qS$ is either empty, in which case $e_{qS \cap pS} = 0$ by (L3), or $pS \cap qS = rS$ for some $r \in pS \cap qS$. If we let $p', q' \in S$ be such that $pp' = qq' = r$ in S (which are uniquely determined since S is left cancellative), then

$$v_q^* v_p = v_q^* e_{rS} v_p = v_q^* v_{qq'} v_{pp'}^* v_p = v_{q'} v_{p'}^*$$

establishes the claim for the second case. □

Definition 3.12. Let S be a left cancellative semigroup. Define a subalgebra of $C^*(S)$ by

$$\mathcal{D} := C^*(\{e_X \mid X \in \mathcal{J}(S)\}).$$

If $S^* \neq \emptyset$, define further the subalgebras

$$\mathcal{C}_O := C^*(\{v_p v_x v_p^* \mid p \in S, x \in S^*\}) \text{ and } \mathcal{C}_I := C^*(\{e_{pS}, v_x \mid p \in S, x \in S^*\}).$$

These are, respectively, the *diagonal*, the *outer core* and the *inner core* of $C^*(S)$.

It is clear that $\mathcal{D} = \overline{\operatorname{span}}\{e_X \mid X \in \mathcal{J}(S)\}$. The other two subalgebras satisfy the following:

Lemma 3.13. *Let S be a right LCM semigroup with $S^* \neq \emptyset$. Then*

(i) $\mathcal{D} \subseteq \mathcal{C}_I \subseteq \mathcal{C}_O$;
(ii) $\mathcal{C}_I = \overline{\operatorname{span}}\{e_{pS}v_x \mid p \in S,\ x \in S^*\}$; *and*
(iii) *if* $S^* = \{1_S\}$, *then* $\mathcal{D} = \mathcal{C}_I = \mathcal{C}_O$.

Proof. Parts (i) and (iii) are immediate verifications. For assertion (ii) we use (L2) and (L4) to get

$$e_{pS}v_x e_{qS}v_y = e_{pS}v_x e_{qS}v_x^* v_x v_y = e_{pS \cap xqS}v_{xy},$$

for each $p, q \in S,\ x, y \in S^*$. Hence $\{e_{pS}v_x \mid p \in S,\ x \in S^*\}$ is closed under multiplication. Since $(e_{pS}v_x)^* = v_x^* e_{pS} = e_{x^{-1}pS}v_{x^{-1}}$, claim (ii) follows. □

3.3. **Conditional expectations onto canonical diagonals.** Let S be a left cancellative semigroup. The diagonal $\mathcal{D}_r$ in $C_r^*(S)$ is defined to be the subalgebra $\mathcal{D}_r = \overline{\operatorname{span}}\{E_X \mid X \in \mathcal{J}(S)\}$. We show next that when S is right LCM and also right cancellative, there is a canonical faithful conditional expectation from $C_r^*(S)$ onto its diagonal. The result was motivated by [19, Lemma 3.11] and is a generalisation to cancellative right LCM semigroups of a similar result proved for quasi-lattice ordered groups; see [26, Remark 3.6] and [29]. More precisely, it is a consequence of the normality of the coaction in [29, Proposition 6.5] and of [29, Lemma 6.7] that the Wiener-Hopf algebra $\mathcal{T}(G, S)$, i.e. the reduced C^*-algebra of a quasi-lattice ordered group (G, S), admits a faithful conditional expectation onto its canonical diagonal.

Proposition 3.14. *If S is a cancellative right LCM semigroup, then the canonical map $\Phi_{\mathcal{D},r} : C_r^*(S) \longrightarrow \mathcal{D}_r$ given by $\Phi_{\mathcal{D},r}(V_pV_q^*) = \delta_{p,q}V_pV_p^*$ for $p, q \in S$ is a faithful conditional expectation.*

Proof. It was proved in [19, Section 3.2] that there is a faithful conditional expectation $E : \mathcal{L}(\ell^2(S)) \longrightarrow \ell^\infty(S)$ characterised by $(E(T)\varepsilon_s|\varepsilon_s) = (T\varepsilon_s|\varepsilon_s)$ for all $s \in S$ and all $T \in \mathcal{L}(\ell^2(S))$. Clearly, $\mathcal{D}_r \subset \ell^\infty(S)$. We will show that the converse inclusion holds. Note that $C_r^*(S)$ is the closure of the span of elements $V_pV_q^*, p, q \in S$. Therefore it suffices to show that $E(V_pV_q^*) \in \mathcal{D}_r$ for any $p, q \in S$. Let $s \in S$. If $s \notin qS$, then $V_q^*\varepsilon_s = 0$, and for $s \in qS$ of the form $s = qs'$ we have $V_q^*\varepsilon_s = \varepsilon'_s$. Thus if $E(V_pV_q^*) \neq 0$, then there is $s' \in S$ such that $ps' = qs'$. Right cancellation then implies $p = q$, so $V_pV_q^* \in \mathcal{D}_r$. Since $\Phi_{\mathcal{D},r} = E$ in this case, the proposition follows. □

A successful strategy to prove injectivity of representations of $C^*(S)$ uses the classical idea of Cuntz from [5], which involves expecting onto a diagonal subalgebra and constructing a projection with good approximation properties. To pursue this path, we need a faithful conditional expectation from $C^*(S)$ onto $\mathcal{D}$. Such a map can be specified by its image on the spanning elements of $C^*(S)$ as follows:

$$\Phi_{\mathcal{D}}(v_pv_q^*) = \begin{cases} v_pv_p^*, & \text{if } p = q, \\ 0, & \text{if } p \neq q. \end{cases} \tag{3.1}$$

Thus in examples we need to ensure that (3.1) does extend to $C^*(S)$ and that it is faithful on positive elements. We now describe one such situation.

Let us recall the notion of a semigroup crossed product by endomorphisms; see e.g. [11]. Let S be a semigroup with identity and A a unital C^*-algebra with an action $S \overset{\alpha}{\curvearrowright} A$ by endomorphisms. A non-degenerate representation of (A,S,α) in a unital C^*-algebra B is given by a unital $*$-homomorphism $\pi_A : A \longrightarrow B$ and a semigroup homomorphism $\pi_S : S \longrightarrow \mathrm{Isom}(B)$, where $\mathrm{Isom}(B)$ denotes the semigroup of isometries in the C^*-algebra B. The pair (π_A, π_S) is said to be covariant if it satisfies the covariance condition

$$\pi_S(s)\pi_A(a)\pi_S(s)^* = \pi_A(\alpha_s(a)) \text{ for all } a \in A \text{ and } s \in S.$$

Assuming that there is a covariant pair, the semigroup crossed product $A \rtimes_\alpha S$ is the unital C^*-algebra generated by a pair (ι_A, ι_S) which is universal for non-degenerate covariant representations. This is to say that whenever (π_A, π_S) is a non-degenerate covariant representation of (A,S,α) in a C^*-algebra B, there is a homomorphism $\overline{\pi} : A \rtimes_\alpha S \longrightarrow B$ such that

$$\pi_A = \overline{\pi} \circ \iota_A \text{ and } \pi_S = \overline{\pi} \circ \iota_S.$$

The crossed product $A \rtimes_\alpha S$ is uniquely determined (up to canonical isomorphism) by this property. If the action α is by injective endomorphisms, then there is always a covariant pair and $A \rtimes_\alpha S$ is non-trivial; see [10].

It was observed in [19] that whenever S is a left cancellative semigroup with identity, then there is an action τ of S by endomorphisms of $\mathcal{D}$ given by $\tau_p(e_X) = v_p e_X v_p^* = e_{pX}$ for all $p \in S$ and $X \in \mathcal{J}(S)$. The semigroup crossed product $\mathcal{D} \rtimes_\tau S$ is the universal C^*-algebra generated by a pair $(\iota_{\mathcal{D}}, \iota_S)$ of homomorphisms of $\mathcal{D}$ and S, respectively, subject to the covariance condition $\iota_S(p)\iota_{\mathcal{D}}(e_X)\iota_S(p)^* = \iota_{\mathcal{D}}(e_{pX})$ for all $p \in S$ and $X \in \mathcal{J}(S)$. As shown in [19, Lemma 2.14], the C^*-algebras $C^*(S)$ and $\mathcal{D} \rtimes_\tau S$ are canonically isomorphic through the isomorphism that sends v_p to $\iota_S(p)$ and e_X to $\iota_{\mathcal{D}}(e_X)$. We have the following consequence of Lemma 3.11.

Corollary 3.15. *Given a right LCM semigroup S, let τ be the action of S on $\mathcal{D}$ given by conjugation with v_p for $p \in S$. If S has an identity, then $\mathcal{D} \rtimes_\tau S = \overline{\mathrm{span}}\{\iota_S(p)\iota_S(q)^* \mid p,q \in S\}$. If $S^* = \emptyset$, then $\mathcal{D} \rtimes_\tau S = \overline{\mathrm{span}}\{\iota_S(p)\iota_S(q)^* \mid p,q \in \tilde{S}\}$ holds.*

Recall that a semigroup S is said to be *right reversible* if $Sp \cap Sq$ is non-empty for all $p, q \in S$; see [4, §10.3]. If S embeds into a group, we refer to the subgroup generated by the image of S as the *enveloping group* of S. Note that this group is unique up to canonical isomorphism in case it exists.

Proposition 3.16. *Let S be a right LCM semigroup with identity such that S is right reversible and its enveloping group $\mathcal{G} = S^{-1}S$ is amenable. Then there is a faithful conditional expectation from $C^*(S)$ onto $\mathcal{D}$ characterised by* (3.1).

Proof. The first observation is that the action τ admits a left inverse, β, given by

$$\beta_p(e_X) = v_p^* e_X v_p = e_{p^{-1}X}$$

for $p \in S$ and $X \in \mathcal{J}(S)$. It was proved in [20, Corollary 2.9] that β_p defines an endomorphism of $\mathcal{D}$ for each $p \in S$, the reason for this being that $p^{-1}X \cap p^{-1}Y = p^{-1}(X \cap Y)$ holds for all $X, Y \in \mathcal{J}(S)$. It is clear that β is an action of S such that $\beta_p \circ \tau_p = \mathrm{id}$ for all $p \in S$. Moreover,

$$(\tau_p \circ \beta_p)(e_X) = v_p v_p^* e_X v_p v_p^* = e_{pS} e_X e_{pS} = e_X \tau_p(1)$$

for every $p \in S$ and $X \in \mathcal{J}(S)$. Thus $\tau_p \circ \beta_p$ is simply the cut-down to the corner associated to the projection $\tau_p(1)$.

One consequence of the existence of β is that τ_p is injective for every $p \in S$. Hence Theorems 2.1 and 2.4 of [10] show that $\mathcal{D}$ embeds in $\mathcal{D} \rtimes_\tau S$.

As a second consequence of the existence of β, note that [15, Proposition 3.1(1)] implies that there is a coaction of $\mathcal{G}$ whose fixed-point algebra is $\iota_\mathcal{D}(\mathcal{D})$. Thus there is a conditional expectation $\Phi_\mathcal{D}$ from $\mathcal{D} \rtimes_\tau S$ onto $\iota_\mathcal{D}(\mathcal{D})$ such that

$$\Phi_\mathcal{D}(\iota_S(p)\iota_S(q)^*) = \begin{cases} \iota_S(p)\iota_S(p)^*, & \text{if } p = q, \\ 0, & \text{if } p \neq q. \end{cases}$$

Identifying $\iota_S(p)$ with v_p and $\iota_\mathcal{D}(e_{pS})$ with e_{pS} gives existence of the claimed expectation. Under the assumption that the enveloping group $\mathcal{G}$ is amenable, the map $\Phi_\mathcal{D}$ is faithful on positive elements; cf. [28, Lemma 1.4]. Note that the last conclusion may also be reached for the semigroup dynamical system $(\mathcal{D}, S, \tau)$ by invoking [8, Lemma 8.2.5]. $\square$

3.4. **From quasi-lattice order groups to right LCM semigroups.** It turns out that a good part of the general strategy of Laca and Raeburn [11] for proving injectivity of representations of $C^*(S)$ in the case that S is part of a quasi-lattice order (G, S) can be extended to the class of right LCM semigroups, although the arguments become more delicate due to the presence of non-trivial units. The next several results make this claim precise.

Notation 3.17. In Lemma 3.1 we introduced isometries V_p for $p \in S$ and projections E_{pS} for $pS \in \mathcal{J}(S)$ in $C_r^*(S)$ that satisfy conditions (L1)–(L4). Later in the paper we shall mainly be interested in families of isometries and projections satisfying (L1)–(L4) inside an arbitrary C^*-algebra B. In order to avoid unnecessary notational adornment we shall still use V_p, E_{pS} in that case.

Given a family of commuting projections $(E_i)_{i \in I}$ in a unital C^*-algebra B and finite subsets $A \subset F$ of I, we denote

$$Q^E_{F,A} := \prod_{i \in A} E_i \prod_{j \in F \setminus A} (1 - E_j).$$

If the family is $(e_X)_{X \in \mathcal{J}(S)}$ in $C^*(S)$, we write $Q^e_{F,A}$ for the corresponding projections. In the case of a right LCM semigroup S, finite subsets of $\mathcal{J}(S)$ are determined by finite subsets of S; see Lemma 3.3.

If S is a left cancellative semigroup with identity such that $\mathcal{J}(S)$ is independent, then [19, Corollary 2.22] and [19, Proposition 2.24] show that the left regular representation λ from $C^*(S)$ to $C_r^*(S)$ restricts to an isomorphism from $\mathcal{D}$ onto the diagonal $\mathcal{D}_r$. This allows us to show:

Lemma 3.18. *Let S be a right LCM semigroup. Then the left regular representation λ restricts to an isomorphism from the diagonal $\mathcal{D}$ of $C^*(S)$ onto the diagonal $\mathcal{D}_r$ of $C_r^*(S)$.*

Proof. If S has an identity, then $\mathcal{J}(S)$ is independent by Corollary 3.6. Hence the lemma is simply an application of the mentioned results from [19]. Now suppose $S^* = \emptyset$ holds. Then $\mathcal{J}(\tilde{S})$ is independent according to Lemma 3.7 and Corollary 3.6. Moreover, by Lemma 3.7, we have

$$pS \cap qS = rS \text{ if and only if } p\tilde{S} \cap q\tilde{S} = r\tilde{S} \text{ for all } p, q, r \in S.$$

This fact and the standing hypothesis $S \neq \emptyset$ imply that the maps

$$\begin{array}{lcl} \mathcal{D} & \longrightarrow & \tilde{\mathcal{D}} \\ e_S & \mapsto & e_{\tilde{S}} \\ e_{pS} & \mapsto & e_{p\tilde{S}} \end{array} \quad \text{and} \quad \begin{array}{lcl} \mathcal{D}_r & \longrightarrow & \tilde{\mathcal{D}}_r \\ E_S & \mapsto & E_{\tilde{S}} \\ E_{pS} & \mapsto & E_{p\tilde{S}} \end{array}$$

are isomorphisms, where $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}_r$ denote the diagonal subalgebra of $C^*(\tilde{S})$ and $C_r^*(\tilde{S})$, respectively. Since $\mathcal{J}(\tilde{S})$ is independent, $\tilde{\lambda} : \tilde{\mathcal{D}} \longrightarrow \tilde{\mathcal{D}}_r$ is an isomorphism. Altogether, we get a commutative diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\lambda|_{\mathcal{D}}"] \arrow[d, "\cong"'] & \mathcal{D}_r \arrow[d, "\cong"] \\ \tilde{\mathcal{D}} \arrow[r, "\cong"'] & \tilde{\mathcal{D}}_r \end{tikzcd} \tag{3.2}$$

which proves that $\lambda|_{\mathcal{D}}$ is an isomorphism. ☐

Proposition 3.19. *Suppose S is a right LCM semigroup and π is a $*$-homomorphism of $C^*(S)$. Let $E_X := \pi(e_X)$ for $X \in \mathcal{J}(S)$ and $V_p := \pi(v_p)$ for $p \in S$. Then the following statements are equivalent:*

(I) *$\pi|_{\mathcal{D}} : \mathcal{D} \longrightarrow \pi(\mathcal{D})$ is an isomorphism.*

(II) *$Q^E_{F,A} \neq 0$ for all non-empty finite subsets F of $\mathcal{J}(S)$ and all non-empty subsets $A \subset F$ satisfying*

$$\bigcap_{X \in A} X \cap \bigcap_{Y \in F \setminus A} S \setminus Y \neq \emptyset.$$

(III) *$Q^E_{F,\emptyset} \neq 0$ for all non-empty subsets $F \subset \mathcal{J}(S) \setminus \{S\}$.*

Proof. Lemma 3.18 implies that the left regular representation λ restricts to an isomorphism from $\mathcal{D}$ onto $\mathcal{D}_r$. Thus assuming (I) and letting $A \subset F$ be finite non-empty subsets of $\mathcal{J}(S)$ satisfying the non-empty intersection condition of (II), it follows that $\lambda(Q^e_{F,A}) \neq 0$. Hence $Q^e_{F,A} \neq 0$, which by injectivity of $\pi|_{\mathcal{D}}$ gives that $Q^E_{F,A} \neq 0$. This shows that (I) implies (II). Conversely, it suffices to note that by [19, Lemma 2.20], condition (I) is equivalent to the implication $Q^E_{F,A} = 0 \Longrightarrow Q^e_{F,A} = 0$ for all non-empty finite subsets F of $\mathcal{J}(S)$ and all non-empty subsets $A \subset F$. Thus (I) and (II) are equivalent.

Consider next a non-empty finite subset $F \subset \mathcal{J}(S) \setminus \{S\}$. If S has an identity, then Lemma 3.4 provides independence of $\mathcal{J}(S)$. In particular, we have $\bigcup_{X \in F} X \subsetneqq S$. Hence $Q^e_{F,\emptyset} \neq 0$ because its image under λ is non-zero. In case $S^* = \emptyset$, Lemma 3.7 shows that $F \subset \mathcal{J}(S) \setminus \{S\}$ corresponds to a finite subset $\tilde{F} \subset \mathcal{J}(\tilde{S}) \setminus \{\tilde{S}\}$. As $\tilde{S}$ is a right LCM semigroup with identity, we get $Q^e_{\tilde{F},\emptyset} \neq 0$. According to Lemma 3.18, this is equivalent to $Q^e_{F,\emptyset} \neq 0$. Since π carries $Q^e_{F,\emptyset}$ to $Q^E_{F,\emptyset}$, it follows that (I) implies (III).

Thus it remains to prove that (III) yields (II). Assume (III) and let $F \subset \mathcal{J}(S)$ be a non-empty subset and $A \subset F$ non-empty satisfying the non-empty intersection condition of (II). Let $\sigma_A \in S$ such that $\sigma_A S = \bigcap_{X\in A} X$ and $\bigcup_{Y \in F\setminus A} Y \neq S$. Thus,

$$Q^E_{F,A} = Q^E_{A,A}\, Q^E_{F\setminus A,\emptyset}\, Q^E_{A,A} = V_{\sigma_A} \prod_{Y\in F\setminus A} (1 - V^*_{\sigma_A} E_Y V_{\sigma_A}) V^*_{\sigma_A}.$$

Each $Y \in F\setminus A$ has the form $Y = p_Y S$ for some $p_Y \in S$. Since S is right LCM, there exists $q_Y \in S$ such that $\sigma_A^{-1} Y = q_Y S$ and $\sigma_A q_Y S = \sigma_A S \cap p_Y S$. Thus $\sigma_A^{-1} Y$ is a proper right ideal of S if and only if $\sigma_A \notin Y$. The choice of F and A therefore guarantees that $\sigma_A^{-1} Y \neq S$ for all $Y \in F \setminus A$. Hence $Q^E_{\sigma_A^{-1}(F\setminus A),\emptyset} \neq 0$ by (III). From

$$Q^E_{\sigma_A^{-1}(F\setminus A),\emptyset} = \prod_{Y\in F\setminus A} (1 - E_{\sigma_A^{-1}(Y)})$$

and $V^*_{\sigma_A} E_Y V_{\sigma_A} = E_{\sigma_A^{-1}(Y)}$, we obtain that

$$Q^E_{F,A} = V_{\sigma_A} \prod_{Y\in F\setminus A} (1 - E_{\sigma_A^{-1}(Y)}) V^*_{\sigma_A} \neq 0$$

since V_{σ_A} is an isometry. This finishes the proof of the proposition. □

The following result is a variant of [11, Lemma 1.4].

Lemma 3.20. *If $(E_i)_I$ are commuting projections in a unital C^*-algebra B and $A \subset F$ are finite subsets of I, then each $Q^E_{F,A}$ is a projection, $\sum_{A\subset F} Q^E_{F,A} = 1$, and we have*

$$\sum_{i\in F} \lambda_i E_i = \sum_{A\subset F} \Big(\sum_{i\in A} \lambda_i\Big) Q^E_{F,A} \tag{3.3}$$

for any choice of complex numbers $\{\lambda_i \mid i \in I\}$ and, moreover,

$$\Big\| \sum_{i\in F} \lambda_i E_i \Big\| = \max_{\substack{A\subset F\\ Q^E_{F,A}\neq 0}} \Big| \sum_{i\in A} \lambda_i \Big|. \tag{3.4}$$

Proof. Since the projections E_i commute, $Q^E_{F,A}$ is a projection. The second assertion is obtained via

$$1 = \prod_{i\in F} (E_i + 1 - E_i) = \sum_{A\subset F} Q^E_{F,A}.$$

Equation (3.3) as well as equation (3.4) follow immediately from this. □

We now set up a conventional notation which will be used repeatedly in the sequel. Let S be a right LCM semigroup. We let

$$t_F := \sum_{p,q\in F} \lambda_{p,q} v_p v_q^* \quad \text{and} \quad t_{F,\mathcal{D}} := \sum_{p\in F} \lambda_{p,p} e_{pS} \tag{3.5}$$

denote an arbitrary, but fixed finite linear combination in $C^*(S)$ and its image in $\mathcal{D}$ under $\Phi_{\mathcal{D}}$, where F is a finite subset of S when S has an identity, or, in case $S^* = \emptyset$, F is a finite subset of $\tilde{S}$, and $\lambda_{p,q} \in \mathbb{C}$ for $p, q \in F$.

We will decompose $t_F - t_{F,\mathcal{D}}$ into further terms, based on a suitable subset $A \subset F$ depending on the choice of the $\lambda_{p,q}$'s. We are interested in combinations t_F with $t_{F,\mathcal{D}} \neq 0$, so we shall make this a standing assumption.

Lemma 3.21. *Let S be a right LCM semigroup and t_F, $t_{F,\mathcal{D}}$ be as in* (3.5)*. Then there exists a non-empty subset $A \subset F$ such that the projection $Q^e_{F,A}$ is non-zero and satisfies the following:*

(i) $Q^e_{F,A} v_p v_q^* Q^e_{F,A} = 0$ *for all $p, q \in F$ with $p \notin A$ or $q \notin A$.*
(ii) $\|Q^e_{F,A} t_{F,\mathcal{D}} Q^e_{F,A}\| = \|t_{F,\mathcal{D}}\|$.
(iii) *If $t_{F,\mathcal{D}}$ is positive, then we may take $Q^e_{F,A} t_{F,\mathcal{D}} Q^e_{F,A} = \|t_{F,\mathcal{D}}\| Q^e_{F,A}$.*

Proof. The projections $(e_{pS})_{p\in F}$ commute because of $e_{pS}e_{qS} = e_{pS\cap qS}$ for any $p, q \in S$. Applying Lemma 3.20 yields $A \subset F$ which satisfies $Q^e_{F,A} \neq 0$, and

$$\|Q^e_{F,A} t_{F,\mathcal{D}} Q^e_{F,A}\| = \|t_{F,\mathcal{D}}\|.$$

If t_F is positive, then we may choose $Q^e_{F,A} t_{F,\mathcal{D}} Q^e_{F,A}$ to be a multiple of $Q^e_{F,A}$. As $t_{F,\mathcal{D}} \neq 0$, we must have $A \neq \emptyset$. The fact that $Q^e_{F,A} \neq 0$ and the right LCM property of S imply that

$$Q^e_{F,A} = \prod_{q \in F\setminus A} (e_{\sigma_A S} - e_{\sigma_A S\cap qS}),$$

where $\sigma_A \in S$ is such that $\sigma_A S = \bigcap_{p\in A} pS$. We claim that $Q^e_{F,A} v_p v_q^* Q^e_{F,A} = 0$ for $p \in F \setminus A$. Indeed, if we have $p \notin A$, then $Q^e_{F,A} v_p v_q^* Q^e_{F,A}$ contains a factor of $(1-e_{pS})v_p = v_p - v_p = 0$, and hence $Q^e_{F,A} v_p v_q^* Q^e_{F,A} = 0$. Similarly, $v_q^*(1-e_{qS}) = 0$, so we get $Q^e_{F,A} v_p v_q^* Q^e_{F,A} = 0$ for $q \in F \setminus A$. $\square$

Before we state the next result we introduce some notation. Assume the hypotheses of Lemma 3.21 and let A be the finite subset of F satisfying (i)–(iii). Fix $\sigma_A \in S$ such that $\bigcap_{p\in A} pS = \sigma_A S$ (this element is not unique for the given A; in case $S^* \neq \emptyset$ then $\sigma_A x$ for any $x \in S^*$ will satisfy the same identity as σ_A). For each $p \in A$, let $p_A \in S$ denote the element satisfying $pp_A = \sigma_A$. By left cancellation, this element is unique. Define now

$$\begin{aligned} t_{F,1} &= \sum_{\substack{p,q\in F, p\neq q\\ p\notin A \text{ or } q\notin A}} \lambda_{p,q} v_p v_q^*,\\ t_{F,2} &= \sum_{\substack{p,q\in A, p\neq q\\ p_A S\neq q_A S}} \lambda_{p,q} v_p v_q^*, \text{ and}\\ t_{F,3} &= \sum_{\substack{p,q\in A, p\neq q\\ p_A S= q_A S}} \lambda_{p,q} v_p v_q^*. \end{aligned}$$

The sum $t_{F,3}$ will only be relevant here when $|S^*| > 1$. When $|S^*| \leq 1$, we distinguish two cases: if $S^* = \emptyset$, our standing assumption says that $sS = tS$ forces $s = t$ for $s, t \in S$. Hence a term in $t_{F,3}$ would correspond to $p_A = q_A$, which implies $pp_A = qp_A$. Thus, if the semigroup S is also right cancellative, we would get $p = q$, a contradiction. The same argument rules out $t_{F,3}$ when $S^* = \{1_S\}$.

Lemma 3.22. *Assume the hypotheses of Lemma* 3.21 *and let A be the finite subset of F satisfying* (i)–(iii)*. Fix σ_A as above. Define a subset of $A \times A$ by*

$$A_1 = \{(p,q) \mid p \neq q, \exists x, y \in S, x, y \text{ not both units} : p_A x = q_A y, p_A S \cap q_A S = p_A x S\}.$$

Then

$$e_{\sigma_A S} t_{F,2} e_{\sigma_A S} = \sum_{(p,q)\in A_1} \lambda_{p,q} v_{\sigma_A x} v^*_{\sigma_A y}. \tag{3.6}$$

If $|S^*| > 1$, then also

$$e_{\sigma_A S} t_{F,3} e_{\sigma_A S} = \sum_{\substack{p,q \in A, p \neq q, \\ p_A = q_A x, x \in S^*}} \lambda_{p,q} v_{\sigma_A} v_x v_{\sigma_A}^*. \tag{3.7}$$

Proof. Clearly, $t_F = t_{F,\mathcal{D}} + t_{F,1} + t_{F,2} + t_{F,3}$. Let us look more closely at the cut-downs of $t_{F,2}$ and $t_{F,3}$ by $Q^e_{F,A}$. For $p, q \in A, p \neq q$, we have

$$\begin{aligned} e_{\sigma_A S} v_p v_q^* e_{\sigma_A S} &= v_{\sigma_A} v_{p_A}^* v_{q_A} v_{\sigma_A}^* \\ &= v_{\sigma_A} v_{p_A}^* e_{p_A S} e_{q_A S} v_{q_A} v_{\sigma_A}^* \\ &= \begin{cases} 0, & \text{if } p_A S \cap q_A S = \emptyset, \\ v_{\sigma_A x} v_{\sigma_A y}^*, & \text{if } p_A S \cap q_A S \neq \emptyset, \end{cases} \end{aligned}$$

where $x, y \in S$ satisfy $p_A x = q_A y$ and $p_A S \cap q_A S = p_A x S$. The choice of the pair (x, y) is unique up to composition from the right by S^*. Hence, $v_{\sigma_A x} v_{\sigma_A y}^*$ is independent of the choice of (x, y). Therefore, with regard to $t_{F,2}$, we only have to deal with $p, q \in A, p \neq q$, such that $p_A S \cap q_A S \neq \emptyset$. These are exactly the pairs (p, q) in A_1, so (3.6) follows.

If $v_p v_q^*$ are terms in $t_{F,3}$, then $p_A S = q_A S$ means that $p_A \mathcal{R} q_A$, where $\mathcal{R}$ is the right Green relation. Thus there exists $x \in S^*$ such that $p_A = q_A x$, and (3.7) follows. □

Lemma 3.23. *If S is a right LCM semigroup, then there are finite subsets A, F_1 of S with $A \subset F \subset F_1$ and $Q^e_{F_1,A} \neq 0$ such that*

$$\begin{aligned} Q^e_{F_1,A} t_F Q^e_{F_1,A} &= Q^e_{F_1,A}(t_{F,\mathcal{D}} + t_{F,3}) Q^e_{F_1,A} \text{ and} \\ \|Q^e_{F_1,A} t_{F,\mathcal{D}} Q^e_{F_1,A}\| &= \|t_{F,\mathcal{D}}\|. \end{aligned}$$

If $t_{F,\mathcal{D}}$ is positive, then we may take $Q^e_{F_1,A} t_{F,\mathcal{D}} Q^e_{F_1,A} = \|t_{F,\mathcal{D}}\| Q^e_{F_1,A}$.

Proof. We invoke the notation of Lemma 3.22. For each $(p, q) \in A_1$, let $\alpha_{p,q} \in S$ be given by

$$\alpha_{p,q} := \begin{cases} x & \text{if } x \in S \setminus S^*, \\ y & \text{if } x \in S^*, \end{cases}$$

and set $F_1 := F \cup \{\sigma_A \alpha_{p,q} \mid (p, q) \in A_1\}$. First of all, let us show that $Q^e_{F_1,A} \neq 0$ holds. Due to $Q^e_{F,A} \neq 0$, we know that $\sigma_A S \cap rS$ is a proper and non-empty subset of $\sigma_A S$ for each $r \in F \setminus A$. Choose for each $r \in F \setminus A$ an element $r' \in S \setminus S^*$ such that $\sigma_A S \cap rS = \sigma_A r' S$. It follows that $r'S \subsetneqq S$ and $\alpha_{p,q} S \subsetneqq S$ for all $r \in F \setminus A$ and all $(p, q) \in A_1$.

If S has an identity, $\mathcal{J}(S)$ is independent by Corollary 3.6 and hence we get

$$\bigcup_{r \in F \setminus A} r'S \cup \bigcup_{(p,q) \in A_1} \alpha_{p,q} S \subsetneqq S$$

as both index sets are finite. By taking complements and using the implication (I) $\Rightarrow$ (II) from Proposition 3.19, this shows that

$$\prod_{r \in F \setminus A} (1 - e_{r'S}) \prod_{(p,q) \in A_1} (1 - e_{\alpha_{p,q} S}) \neq 0.$$

In the case where $S^* = \emptyset$, we get $r'\tilde{S} \subsetneqq \tilde{S}$ and $\alpha_{p,q}\tilde{S} \subsetneqq \tilde{S}$ for all $r \in F \setminus A$ and all $(p, q) \in A_1$. By Lemma 3.7, $\mathcal{J}(\tilde{S})$ is independent. If we combine this with Proposition 3.19 and the isomorphism $\tilde{\mathcal{D}} \cong \mathcal{D}$ from Lemma 3.18, we also get

$$\prod_{r \in F\setminus A} (1 - e_{r'S}) \prod_{(p,q)\in A_1} (1 - e_{\alpha_{p,q}S}) \neq 0$$

in the case $S^* = \emptyset$.

Since v_{σ_A} is an isometry and $Q^e_{F_1,A}$ has the form

$$Q^e_{F_1,A} = v_{\sigma_A}\Big(\prod_{r\in F\setminus A} (1 - e_{r'S}) \prod_{(p,q)\in A_1} (1 - e_{\alpha_{p,q}S})\Big)v^*_{\sigma_A}, \tag{3.8}$$

it follows that $Q^e_{F_1,A} \neq 0$. Then $Q^e_{F_1,A}$ is a non-trivial subprojection of $Q^e_{F,A}$, so

$$\|Q^e_{F_1,A} t_{F,\mathcal{D}} Q^e_{F_1,A}\| = \|t_{F,\mathcal{D}}\|.$$

If $t_{F,\mathcal{D}}$ is positive, then we have $Q^e_{F_1,A} t_{F,\mathcal{D}} Q^e_{F_1,A} = \|t_{F,\mathcal{D}}\| Q^e_{F_1,A}$. Note that Lemma 3.21 implies $Q^e_{F_1,A} t_{F,1} Q^e_{F_1,A} = 0$, and (3.6) gives

$$Q^e_{F_1,A} t_{F,2} Q^e_{F_1,A} = Q^e_{F_1\setminus A,\emptyset} \sum_{(p,q)\in A_1} \lambda_{p,q} v_{\sigma_A x} v^*_{\sigma_A y} Q^e_{F_1\setminus A,\emptyset}.$$

Now suppose $(p, q) \in A_1$ and $\alpha_{p,q} = x$. Then $Q^e_{F_1\setminus A,\emptyset} v_{\sigma_A x} v^*_{\sigma_A y} Q^e_{F_1\setminus A,\emptyset}$ contains a factor $(1-e_{\sigma_A xS})v_{\sigma_A x}v^*_{\sigma_A y} = 0$ and hence $Q^e_{F_1,A} v_p v^*_q Q^e_{F_1,A} = 0$. A similar argument gives $Q^e_{F_1,A} v_p v^*_q Q^e_{F_1,A} = 0$ for $(p, q) \in A_1$ and $\alpha_{p,q} = y$. Therefore, we have verified $Q^e_{F_1,A} t_{F,2} Q^e_{F_1,A} = 0$, or in other words

$$Q^e_{F_1,A} t_F Q^e_{F_1,A} = Q^e_{F_1,A}(t_{F,\mathcal{D}} + t_{F,3}) Q^e_{F_1,A}. \qquad \square$$

Lemma 3.24. *Let S be a* cancellative *right LCM semigroup with $S^* = \emptyset$. Then there are finite subsets A, F_1 of S such that $A \subset F \subset F_1$, $Q^e_{F_1,A} \neq 0$ and*

$$\|Q^e_{F_1,A} t_F Q^e_{F_1,A}\| = \|t_{F,\mathcal{D}}\|.$$

If t_F is positive, then we may take $Q^e_{F_1,A} t_F Q^e_{F_1,A} = \|t_{F,\mathcal{D}}\| Q^e_{F_1,A}$.

Proof. It suffices to note that the sum $t_{F,3}$ is empty by our remark prior to Lemma 3.22. Then the finite subsets A, F_1 of S given by Lemma 3.23 satisfy the claim. $\square$

4. Uniqueness theorem for right LCM semigroup algebras using $\mathcal{D}$

In this section we prove a uniqueness theorem which involves a non-vanishing condition on elements of the diagonal subalgebra $\mathcal{D} \subset C^*(S)$. Our theorem will apply to right LCM semigroups S satisfying additional properties, including that S must be cancellative. One of these conditions, the one we call (D2), is rather technical. Before we state it we introduce two other conditions, which besides being closely related to (D2) are also likely to have more transparent formulations in examples. Indeed, they are often satisfied, while condition (D2) may be harder to obtain for large classes of semigroups.

Let S be a right LCM semigroup with $S^* \neq \emptyset$ and consider the action $S^* \curvearrowright \mathcal{J}(S)$ given by left multiplication, that is, $x \cdot X = xX$ for $x \in S^*$ and $X \in J(S)$. It is standard terminology that the action $S^* \curvearrowright \mathcal{J}(S)$ is effective if, by definition, for every x in $S^* \setminus \{1_S\}$ there is $X \in \mathcal{J}(S)$ such that $xX \neq X$. We next introduce three other properties that a semigroup can have.

Definition 4.1. Let S be a right LCM semigroup with $S^* \neq \emptyset$. We say that the action $S^* \curvearrowright \mathcal{J}(S)$ given by left multiplication is *strongly effective* if for all $x \in S^* \setminus \{1_S\}$ and $p \in S$, there exists $q \in pS$ such that $xqS \neq qS$.

Consider further the following two conditions that S can satisfy:

(D1) For all $x \in S^*$ and $X \in \mathcal{J}(S)$, we have $xX \cap X \neq \emptyset \Longrightarrow xX = X$.

(D2) If $s_0 \in S, s_1 \in s_0 S$ and $F \subset S$ is a finite subset with $s_1 S \cap \left(S \setminus \bigcup_{q \in F} qS \right) \neq \emptyset$, then, for every $x \in S^* \setminus \{1_S\}$, there is $s_2 \in s_1 S$ satisfying

$$s_2 S \cap \left(S \setminus \bigcup_{q \in F} qS \right) \neq \emptyset \text{ and } s_0^{-1} s_2 S \cap x s_0^{-1} s_2 S = \emptyset.$$

Remark 4.2. Suppose that the action $S^* \curvearrowright \mathcal{J}(S)$ is strongly effective and (D1) is satisfied. It is immediate to see that for every $x \in S^* \setminus \{1_S\}$ and $p \in S$ there exists $q \in pS$ such that $xqS \cap qS = \emptyset$. In fact, more is true. Let $s_0 \in S$, $s_1 \in s_0 S$, and $x \in S^* \setminus \{1_S\}$. Write $s_1 = s_0 r$ for some $r \in S$. By the previous observation applied to x and $r \in S$ there is $r' \in rS$ such that $xr'S \cap r'S = \emptyset$. If we now let $s_2 = s_0 r' \in s_1 S$, we have established condition (D2) in case F is the empty set. Conversely, if condition (D2) is satisfied, then by applying it with F equal the empty set and $s_0 = 1_S$ it follows that $S^* \curvearrowright \mathcal{J}(S)$ is strongly effective.

For convenience, we will denote the elements of F by $q_1, \ldots, q_{|F|}$ whenever $F \neq \emptyset$.

Theorem 4.3. *Let S be a cancellative right LCM semigroup such that $\Phi_{\mathcal{D}}$: $C^*(S) \to \mathcal{D}$ is a faithful conditional expectation. Let $(V_p)_{p \in S}$ and $(E_{pS})_{p \in S}$ be families of isometries and projections in a C^*-algebra B satisfying* (L1)–(L4). *Let $\pi := \pi_{V,E}$ be the associated $*$-homomorphism from $C^*(S)$ to B. Assume that one of the following conditions holds:*

(1) $|S^*| \leq 1$.
(2) $|S^*| > 1$ *and S satisfies condition* (D2).

Then $\pi : C^*(S) \to B$ is injective if and only if

$$\prod_{p \in F} (1 - E_{pS}) \neq 0 \textit{ for every finite } F \subset S \setminus S^*. \tag{4.1}$$

Remark 4.4. (a) We observe that for a quasi-lattice ordered pair (G, S) in the sense of [26], the semigroup S is right LCM with $S^* = \{1_S\}$. Thus the case (1) with $S^* = \{1\}$ of the theorem recovers [11, Theorem 3.7].

(b) Note that Theorem 4.3 does not apply to the case where S is a non-trivial group ($S = \{1_S\}$ amounts to $C^*(S) \cong \mathbb{C}$). The reason is that $S^* = S$ directs us to part (2) of Theorem 4.3 and (D2) fails in the group case for $F = \emptyset$: indeed, there exists $x \in S^* \setminus \{1_S\}$, but for every $p \in S$ we get $xpS \cap pS = S \neq \emptyset$.

(c) The hypotheses of part (1) of the theorem are satisfied in the case of the semigroup from Example 3.9 because $\Phi_{\mathcal{D}}$ is a faithful expectation; to see this, note that $\mathbb{N} \setminus \{1\}$ embeds in $\mathbb{Q}_+^* \setminus \{1\}$, hence in $\mathbb{Q}_+^*$, and the latter admits a dual action on $C^*(S)$ by [11, Remark 3.7]. Semigroups satisfying condition (D2) will be described in Examples 8.8 and 8.9.

The proof of this theorem requires some preparation. Note that by Proposition 3.19, condition (4.1) is equivalent to injectivity of π on $\mathcal{D}$. Relying on this equivalence, the key step in proving Theorem 4.3 is the following intermediate result:

Proposition 4.5. *Let S be a cancellative right LCM semigroup, and let $(V_p)_{p\in S}$ and $(E_{pS})_{p\in S}$ be families of isometries and projections in a C^*-algebra B satisfying* (L1)–(L4). *Let π be the associated $*$-homomorphism from $C^*(S)$ to B. Assume that one of the following conditions holds:*

(1) $|S^*| \leq 1$.
(2) $|S^*| > 1$ *and S satisfies condition* (D2).

If π is injective on $\mathcal{D}$, then the map

$$\sum_{p,q\in F} \lambda_{p,q} V_p V_q^* \mapsto \sum_{p\in F} \lambda_{p,p} V_p V_p^*,$$

where $F \subset S$ is finite and $\lambda_{p,q} \in \mathbb{C}$, is contractive, and hence extends to a contraction Φ of $\pi(C^*(S))$ onto $\pi(\mathcal{D})$.

One consequence of this proposition is that when π is the identity homomorphism, we obtain a contractive map from $C^*(S)$ onto $\mathcal{D}$ which is nothing but the conditional expectation $\Phi_{\mathcal{D}}$ from Theorem 4.3.

The established strategy to proving Proposition 4.5 is to express Φ on finite linear combinations of the spanning family $(v_p v_q^*)_{p,q\in S}$ as a cut-down by a suitable projection that will depend on the given linear combination. Fix therefore finite combinations $t_F \in C^*(S)$ and $t_{F,\mathcal{D}} \in \mathcal{D}$ as in (3.5). In view of our aim we assume, without loss of generality, that $t_{F,\mathcal{D}} \neq 0$ holds (otherwise 0 is a suitable projection). Most of the preparation needed to construct Φ was done in Section 3.4. For case (1), Lemma 3.24 will suffice.

Condition (D2) is relevant when there are non-trivial elements in S^*. These units will appear in the sum from (3.7) due to right cancellation in S: indeed, for $p, q \in A, p \neq q$ satisfying $p_A S = q_A S$ it follows that $pp_A = qp_A x$ for some $x \in S^*$. By right cancellation, necessarily $x \neq 1_S$. Thus there are $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S^*\setminus\{1_S\}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$e_{\sigma_A S} t_{F,3} e_{\sigma_A S} \quad = \quad \sum_{i=1}^{n} \lambda_i v_{\sigma_A} v_{x_i} v_{\sigma_A}^*. \tag{4.2}$$

Lemma 4.6. *Let S be a cancellative right LCM semigroup such that $|S^*| > 1$ and S satisfies condition* (D2). *Let A, F_1 be as in Lemma* 3.23. *Then there exists $p_F \in \sigma_A S$ such that $e_F := e_{p_F S}$ satisfies*

(i) $e_F Q^e_{F_1\setminus A,\emptyset} \neq 0$.
(ii) $\|e_F Q^e_{F_1\setminus A,\emptyset} t_F e_F Q^e_{F_1\setminus A,\emptyset}\| = \|t_{F,\mathcal{D}}\|$.
(iii) *If $t_{F,\mathcal{D}}$ is positive, then* $e_F Q^e_{F_1\setminus A,\emptyset} t_F e_F Q^e_{F_1\setminus A,\emptyset} = \|t_{F,\mathcal{D}}\| e_F Q^e_{F_1\setminus A,\emptyset}$.

Proof. Since $\mathcal{J}(S)$ is independent, $Q^e_{F_1,A} \neq 0$ is equivalent to $\sigma_A S \cap \Big(S\setminus \bigcup_{q\in F_1\setminus A} qS\Big) \neq \emptyset$. Applying (D2) to σ_A in place of s_0, s_1, the unit $x_1 \in S^*\setminus\{1_S\}$ and the finite set $F_1 \setminus A$ gives an element $s_2 \in \sigma_A S$ such that $x_1\sigma_A^{-1} s_2 S \cap \sigma_A^{-1} s_2 S = \emptyset$ and $s_2 S$ has non-empty intersection with $S\setminus \bigcup_{q\in F_1\setminus A} qS$. Next, we apply (D2) to σ_A as s_0, s_2 in place of s_1, the unit x_2 and $F_1 \setminus A$ resulting in an element $s_3 \in s_2 S$ such that $x_2\sigma_A^{-1} s_3 S \cap \sigma_A^{-1} s_3 S = \emptyset$ and $s_3 S$ has non-empty intersection with $S\setminus \bigcup_{q\in F_1\setminus A} qS$.

Note that we have

$$x_1\sigma_A^{-1}s_3S \cap \sigma_A^{-1}s_3S \overset{s_3\in s_2S}{\subset} x_1\sigma_A^{-1}s_2S \cap \sigma_A^{-1}s_2S = \emptyset.$$

Thus, proceeding inductively, we get $s_n \in \sigma_A S$ such that s_nS has non-empty intersection with $S\setminus \bigcup\limits_{q\in F_1\setminus A} qS$ and $x_i\sigma_A^{-1}s_nS \cap \sigma_A^{-1}s_nS = \emptyset$ for all $i = 1,\dots,n$. This translates to $Q^e_{F_1,A} \geq e_{s_nS}Q^e_{F_1\setminus A,\emptyset} \neq 0$ and $e_{s_nS}t_{F,3}e_{s_nS} = 0$. Let $p_F := s_n$. Since

$$e_FQ^e_{F_1\setminus A,\emptyset}t_Fe_FQ^e_{F_1\setminus A,\emptyset} = e_FQ^e_{F_1\setminus A,\emptyset}t_{F,\mathcal{D}}e_FQ^e_{F_1\setminus A,\emptyset},$$

an application of Lemma 3.23 shows that p_F satisfies (i)$-$(iii). ☐

Proof of Proposition 4.5. For any finite linear combination $T_F \subset \mathrm{span}\{\, V_pV_q^* \mid p,q \in S \,\}$, consider the corresponding element $t_F \in C^*(S)$. For case (1), we use Lemma 3.24 to obtain a non-zero projection $Q^e_{F_1,A} \in \mathcal{D}$ that satisfies

$$\|Q^e_{F_1,A}t_FQ^e_{F_1,A}\| = \|\Phi_{\mathcal{D}}(t_F)\|.$$

Since $\pi_{\mathcal{D}}$ is injective and $(V_p)_{p\in S}, (E_{pS})_{p\in S}$ are families of isometries and projections, respectively, satisfying (L1)–(L4), we get

$$\|Q^E_{F_1,A}T_FQ^E_{F_1,A}\| = \|\Phi(T_F)\|.$$

As $Q^E_{F_1,A} \neq 0$ is a projection, we get

$$\|\Phi(T_F)\| = \|Q^E_{F_1,A}T_FQ^E_{F_1,A}\| \leq \|T_F\|,$$

so Φ is contractive on a dense subset of $\pi(C^*(S))$. By standard arguments, it extends to a contraction from $\pi(C^*(S))$ to $\pi(\mathcal{D})$.

For case (2), run the same argument with $e_FQ^e_{F_1,A}$ given by Lemma 4.6 as the suitable replacement for $Q^e_{F_1,A}$. ☐

Proof of Theorem 4.3. Since S is a right LCM semigroup, Proposition 3.19 implies that condition (4.1) is equivalent to injectivity of π on $\mathcal{D}$. Obviously, $\pi|_{\mathcal{D}}$ is injective whenever π is injective, showing the forward implication in the theorem. To prove the reverse implication, we apply Proposition 4.5 to obtain the following commutative diagram:

$$\begin{tikzcd} C^*(S) \arrow[r, "\pi"] \arrow[d, "\Phi_{\mathcal{D}}"'] & \pi(C^*(S)) \arrow[d, "\Phi"] \\ \mathcal{D} \arrow[r, "\pi|_{\mathcal{D}}"'] & \pi|_{\mathcal{D}}(\mathcal{D}) \end{tikzcd} \tag{4.3}$$

Now, if $a \in C^*(S)_+, a \neq 0$, then $\Phi\circ\pi(a) = \pi|_{\mathcal{D}}\circ\Phi_{\mathcal{D}}(a) \neq 0$ as $\Phi_{\mathcal{D}}$ is faithful and $\pi|_{\mathcal{D}}$ is injective. Thus, we have $\pi(a) \neq 0$. Since injectivity of *-homomorphisms can be detected on positive elements, π is seen to be injective. ☐

5. Purely infinite simple $C^*(S)$ arising from right LCM semigroups

Suppose that S is a right LCM semigroup. Consider the following refinement of condition (D2):

(D3) If $s \in S$ and F is a finite subset of S with $sS \cap \big(S \setminus \bigcup_{q\in F} qS\big) \neq \emptyset$, then there is $s' \in sS$ such that $s'S \cap qS = \emptyset$ for all $q \in F$.

Whenever $F \neq \emptyset$, we will denote its elements by $q_1, \ldots, q_n$. In Section 8 we will see examples of semigroups satisfying conditions (D3) and (D2). To clarify the relationship between (D2) and (D3), we make the following observation:

Lemma 5.1. *Let S be a right LCM semigroup with $S^* \neq \emptyset$. If the action $S^* \curvearrowright \mathcal{J}(S)$ is strongly effective and S satisfies* (D1) *and* (D3)*, then* (D2) *holds.*

Proof. We saw in Remark 4.2 that the condition (D2) where $F = \emptyset$ is satisfied when $S^* \curvearrowright \mathcal{J}(S)$ is strongly effective and S satisfies (D1). Thus it remains to prove (D2) in case $F \neq \emptyset$. Let therefore $s_0, q_1, \ldots, q_n \in S$ and $s_1 \in s_0S$ with $s_1S \cap \left(S \setminus \bigcup_{i=1}^{n} q_iS\right) \neq \emptyset$. Applying (D3) yields an element $s \in s_1S \subset s_0S$ such that $sS \cap q_iS = \emptyset$ for $i = 1, \ldots, n$. Note that every $s_2 \in sS$ inherits this property, and therefore the first equation in (D2) is satisfied for such elements. Let $x \in S^*\setminus\{1_S\}$ and write $s = s_0r$ for some $r \in S$. By strong effectiveness and (D1) applied to x and rS we get $s' \in rS$ with the property that $s'S \cap xs'S = \emptyset$. Now $s_2 = s_0s' \in sS \subset s_1S$ satisfies $s_0^{-1}s_2S \cap xs_0^{-1}s_2S = s'S \cap xs'S = \emptyset$, proving (D2). $\square$

We note that (D1), (D3) and strong effectiveness are properties of a semigroup that can be more readily verified than (D2). The latter condition is quite close to the operator algebraic application it is designed for. Therefore, in Theorem 5.3 we provide an independent proof for the last two sets of assumptions, even though (3) may be deduced from the proof in the case (2). First we need a lemma.

Lemma 5.2. *Let S be a cancellative right LCM semigroup such that $|S^*| > 1$, the action $S^* \curvearrowright \mathcal{J}(S)$ is strongly effective, and* (D1), (D3) *are satisfied. Suppose $t_F \in C^*(S)$ and $t_{F,\mathcal{D}} \in \mathcal{D}$ are linear combinations as in* (3.5)*, and assume that t_F is a positive element in $C^*(S)$. Then $t_{F,\mathcal{D}}$ is positive and there is $p_F \in S$ such that $e_F := e_{p_FS}$ satisfies*

$$e_F t_F e_F = \|t_{F,\mathcal{D}}\| e_F.$$

Proof. As $t_{F,\mathcal{D}}$ is the image of t_F under the natural conditional expectation $C^*(S) \to \mathcal{D}$, it is also positive. According to Lemma 3.23, there are finite subsets A, F_1 of S with $A \subset F \subset F_1$ and $Q^e_{F_1,A} \neq 0$ such that

$$Q^e_{F_1,A} t_F Q^e_{F_1,A} = Q^e_{F_1,A}(t_{F,\mathcal{D}} + t_{F,3})Q^e_{F_1,A} \text{ and } Q^e_{F_1,A} t_{F,\mathcal{D}} Q^e_{F_1,A} = \|t_{F,\mathcal{D}}\| Q^e_{F_1,A}.$$

Since $|S^*| > 1$, the collection $\mathcal{J}(S)$ is independent by Corollary 3.6. According to Proposition 3.19, we have $Q^e_{F_1,A} \neq 0$ if and only if $\sigma_A S \cap \left(S \setminus \bigcup_{q\in F_1\setminus A} qS\right) \neq \emptyset$. By (D3), there is $p_0 \in \sigma_A S$ such that $p_0S \cap qS = \emptyset$ for all $q \in F_1\setminus A$. Hence, we have

$$e_{p_0S} t_F e_{p_0S} = e_{p_0S}(t_{F,\mathcal{D}} + t_{F,3})e_{p_0S} \text{ and } e_{p_0S} t_{F,\mathcal{D}} e_{p_0S} = \|t_{F,\mathcal{D}}\| e_{p_0S}.$$

Let $x_1, \ldots, x_n \in S^* \backslash \{1_S\}$ be the invertible elements that appear in (4.2) and let p_0' denote the element satisfying $p_0 = \sigma_A p_0'$. Applying strong effectiveness to p_0' and x_1 yields an element $p_1' \in p_0'S$ satisfying $x_1 p_1' S \neq p_1' S$. By (D1), this amounts to $x_1 p_1' S \cap p_1' S = \emptyset$. Proceeding inductively, where $p_i \in p_{i-1}S$ is given as $p_i = \sigma_A p_i'$ with $p_i' \in p_{i-1}'S$ satisfying $x_i p_i' S \cap p_i' S = \emptyset$, we obtain $p_n \in \sigma_A S$ such that

$$\begin{aligned} e_{p_nS} v_{\sigma_A} v_{x_i} v_{\sigma_A}^* e_{p_nS} &= v_{p_n} v_{p_n'}^* v_{x_i} v_{p_n'} v_{p_n}^* \\ &= v_{p_n} v_{p_n'}^* e_{p_i'S} v_{x_i} e_{p_i'S} v_{p_n'} v_{p_n}^* \qquad (p_n'S \subset p_i'S) \\ &= v_{p_n} v_{p_n'}^* e_{p_i'S \cap x_i p_i' S} v_{x_i} v_{p_n'} v_{p_n}^* \\ &= 0 \end{aligned}$$

for all $i = 1, \ldots, n$. Thus, $p_F := p_n$ satisfies the claim of the lemma. $\square$

Theorem 5.3. *Let S be a cancellative right LCM semigroup such that $\Phi_{\mathcal{D}} : C^*(S) \to \mathcal{D}$ is a faithful conditional expectation. Assume that* (D3) *and one of the following conditions holds:*

(1) $|S^*| \leq 1$.
(2) $|S^*| > 1$ *and S satisfies condition* (D2).
(3) $|S^*| > 1$, *S satisfies condition* (D1), *and the action $S^* \curvearrowright \mathcal{J}(S)$ is strongly effective.*

Then $C^*(S)$ is purely infinite and simple.

Proof. Recall from Lemma 3.11 that the linear span of the elements $v_p v_q^*$ is dense in $C^*(S)$. Every element from this linear span has the form $t_F = \sum\limits_{p,q \in F} \lambda_{p,q} v_p v_q^*$ for some finite $F \subset S$ and suitable $\lambda_{p,q} \in \mathbb{C}$. Moreover, $\Phi_{\mathcal{D}}(t_F) = t_{F,\mathcal{D}} = \sum\limits_{p \in F} \lambda_{p,p} e_{pS}$.

Let $a \in C^*(S)$ be positive and non-zero, and let $\varepsilon > 0$. Choose a positive linear combination t_F that approximates a up to within ε. If ε is sufficiently small, we have $t_F \neq 0$, which we will assume from now on. For the three different cases in the hypothesis of the theorem, we will use different methods to obtain a suitable small projection $e_F' := e_{q_FS}$ that annihilates the off-diagonal terms of t_F while picking up the norm of the diagonal part: that is,

$$e_F' t_F e_F' = \|t_{F,\mathcal{D}}\| e_F' = \|\Phi_{\mathcal{D}}(t_F)\| e_F'.$$

For case (1), we use Lemma 3.24, and for case (2) Lemma 4.6 to get a finite subset $F_2 = F_1 \setminus A$ of S and an element $p_F \in S$ such that $e_F = e_{p_FS}$ satisfies

(i) $e_F Q^e_{F_2,\emptyset} \neq 0$, and
(ii) $e_F Q^e_{F_2,\emptyset} t_F e_F Q^e_{F_2,\emptyset} = \|t_{F,\mathcal{D}}\| e_F Q^e_{F_2,\emptyset}$.

Since S is right LCM, (i) translates to $p_F S \cap \left(S \setminus \bigcup\limits_{q \in F_2} qS \right) \neq \emptyset$ according to Lemma 3.18. So we can apply (D3) to get an element $q_F \in p_F S$ such that $q_F S \cap qS = \emptyset$ for all $q \in F_2$. By (L4) this gives $e_{q_FS} \leq e_F Q^e_{F_2,\emptyset}$. Now $e_F' := e_{q_FS}$ satisfies $e_F' t_F e_F' = \|t_{F,\mathcal{D}}\| e_F'$ by (ii). For case (3), the existence of such a projection e_F' follows directly from Lemma 5.2.

We have $\|\Phi_{\mathcal{D}}(t_F)\| > 0$ since $\Phi_{\mathcal{D}}$ is faithful. Thus, $e_F' t_F e_F' = \|\Phi_{\mathcal{D}}(t_F)\| e_F'$ is invertible in the corner $e_F' C^*(S) e_F'$. If $\|a - t_F\|$ is sufficiently small, this implies that $e_F' a e_F'$ is positive and invertible in $e_F' C^*(S) e_F'$ as well, because

$$\|\Phi_{\mathcal{D}}(t_F)\| \overset{\varepsilon \searrow 0}{\longrightarrow} \|\Phi_{\mathcal{D}}(a)\| > 0.$$

Hence, if we denote its positive inverse by b, we get

$$\left(b^{\frac{1}{2}} v_{q_F}\right)^* e'_F a e'_F \left(b^{\frac{1}{2}} v_{q_F}\right) = v_{q_F}^* e'_F v_{q_F} = 1.$$

This implies that $C^*(S)$ is purely infinite and simple. □

6. Injectivity of the left regular representation of $C^*(S)$

A major question of interest in [19] and [20] is to determine conditions under which the left regular representation λ is an isomorphism $C^*(S) \cong C_r^*(S)$. In the context of right LCM semigroups, we have identified some classes of semigroups for which this isomorphism holds.

Proposition 6.1. *Assume the hypotheses of Theorem* 5.3. *Then the left regular representation $\lambda : C^*(S) \to C_r^*(S)$ is an isomorphism.*

Proof. The conclusion follows because in this case $C^*(S)$ is simple. □

Theorem 6.2. *Assume that S is a cancellative right LCM semigroup such that the conditional expectation $\Phi_{\mathcal{D}}$ is faithful. Then the left regular representation λ is an isomorphism from $C^*(S)$ onto $C_r^*(S)$.*

Proof. By Lemma 3.18, λ restricts to an isomorphism $\mathcal{D} \cong \mathcal{D}_r$. Then for $a \in C^*(S)_+, a \neq 0$, we have $\lambda|_{\mathcal{D}} \circ \Phi_{\mathcal{D}}(a) \neq 0$. From $\lambda|_{\mathcal{D}} \circ \Phi_{\mathcal{D}} = \Phi_{\mathcal{D},r} \circ \lambda$ and faithfulness of $\Phi_{\mathcal{D},r}$ established in Proposition 3.14 it follows that $\lambda(a) \neq 0$. Hence, λ is an isomorphism. □

Example 6.3. By Theorem 6.2 and Proposition 3.16 it follows that λ is an isomorphism in the case of $S = G \rtimes_\theta P$ that is right LCM, right reversible, and satisfies that $S^{-1}S$ is an amenable group.

Remark 6.4. There is an alternative approach to injectivity of λ for certain subsemigroups of amenable groups; see [20]. We refer to [20, Section 4] for the definition of the Toeplitz condition. Namely, if S is a left cancellative semigroup satisfying the conditions

(i) $\mathcal{J}(S)$ is independent,
(ii) S embeds into an amenable group H such that S generates H and $S \subset H$ satisfies the Toeplitz condition,

then $\lambda : C^*(S) \longrightarrow C_r^*(S)$ is an isomorphism; cf. the equivalence of (iii) and (v) in [20, Theorem 6.1] applied to $A = \mathbb{C}$ (where (v) is valid because H is amenable). The proof of [20, Theorem 6.1] depends on a relatively involved machinery of various crossed product constructions. The conclusion in Theorem 6.2 for right LCM semigroups is obtained through an analysis solely of the semigroup C^*-algebra $C^*(S)$.

7. A uniqueness result using $\mathcal{C}_I$

In this section we consider left cancellative semigroups with identity that satisfy condition (C1). By following an idea from [22] we show that it is possible to construct a conditional expectation from $C^*(S)$ onto $\mathcal{C}_I$ which may be used to reduce the question of injectivity of representations.

The proof of the next result is similar to [3, Proposition 3.5]. For a discrete group Γ, we denote by i_Γ the canonical homomorphism sending γ in Γ to the generating unitary $i_\Gamma(\gamma)$ in $C^*(\Gamma)$, and we let δ_Γ be the homomorphism $C^*(\Gamma) \to C^*(\Gamma) \otimes C^*(\Gamma)$ induced by the map $\gamma \to i_\Gamma(\gamma) \otimes i_\Gamma(\gamma)$.

Proposition 7.1. *Let S be a right LCM semigroup with identity and assume that there exists a homomorphism $\sigma : S \to T$ onto a subsemigroup T of a group Γ such that T generates Γ. Then there is a coaction $\delta : C^*(S) \to C^*(S) \otimes C^*(\Gamma)$ such that*

$$\delta(v_p v_q^*) = v_p v_q^* \otimes i_\Gamma(\sigma(p)\sigma(q)^{-1})$$

for all $p, q \in S$.

Proof. Since $C^*(S) = \overline{\text{span}}\{v_p v_q^* : p, q \in S\}$, we define a map from S to $C^*(S) \otimes C^*(\Gamma)$ by $s \mapsto v_s \otimes i_\Gamma(\sigma(s))$ for $s \in S$. A routine calculation shows that $\{v_s \otimes i_\Gamma(\sigma(s))\}_{s\in S}$ and $\{v_p v_p^* \otimes i_\Gamma(1_\Gamma)\}$ satisfy the relations that characterise $C^*(S)$; hence by the universal property we obtain the required homomorphism δ such that $\delta(v_s) = v_s \otimes i_\Gamma(\sigma(s))$ for all $s \in S$. Since $C^*(S)$ is unital and $\delta(1) = 1 \otimes i_\Gamma(1_\Gamma)$, the map δ is non-degenerate. If we let $\varepsilon : C^*(\Gamma) \to \mathbb{C}$ be the homomorphism integrated from $\gamma \mapsto 1$, then the equality $(\mathrm{id}_{C^*(S)} \otimes \varepsilon) \circ \delta = \mathrm{id}_{C^*(S)}$ shows that δ is an injective map. The coaction identity

$$(\delta \otimes \mathrm{id}_{C^*(\Gamma)}) \circ \delta = (\mathrm{id}_{C^*(S)} \otimes \delta_\Gamma) \circ \delta$$

is immediately checked on the generators v_s of $C^*(S)$. $\square$

By standard theory of coactions (see for example [28]), the spectral subspaces for δ are defined as $C^*(S)^\delta_{\sigma(s)} = \{a \in C^*(S) \mid \delta(a) = a \otimes i_\Gamma(\sigma(s))\}$. The space

$$C^*(S)^\delta = \{a \in C^*(S) \mid \delta(a) = a \otimes i_\Gamma(1_\Gamma)\}$$

is a C^*-subalgebra of $C^*(S)$, called the fixed-point algebra. There is always a conditional expectation $\Phi^\delta : C^*(S) \to C^*(S)^\delta$ such that $\Phi^\delta(a) = 0$ if $a \in C^*(S)^\delta_{\sigma(s)}$ with $\sigma(s) \neq 1_\Gamma$. Moreover, it is known that Φ^δ is faithful on positive elements precisely when the coaction δ is *normal*: this is for instance the case when Γ is amenable. Since $C^*(S) = \overline{\text{span}}\{v_p v_q^* : p, q \in S\}$, we have $C^*(S)^\delta = \overline{\text{span}}\{v_p v_q^* : p, q \in S, \sigma(p) = \sigma(q)\}$ and

$$\Phi^\delta(v_p v_q^*) = \begin{cases} v_p v_q^*, & \text{if } \sigma(p) = \sigma(q), \\ 0, & \text{otherwise.} \end{cases} \tag{7.1}$$

We will prove in Corollary 7.3 that the above fixed-point algebra $C^*(S)^\delta$ coincides with $\mathcal{C}_I$ when the semigroup S satisfies condition (C1).

Remark 7.2. Recall that $\mathcal{C}_I$ was defined as $C^*(\{v_x, e_{pS} \mid p \in S, x \in S^*\})$, where we assume S^* is non-trivial, and that basic properties of this subalgebra were established in Lemma 3.13. We claim that if S^* is non-trivial and S satisfies (C1), then $\mathcal{C}_O = \mathcal{C}_I$. To see this, note that Lemma 3.13 implies that it suffices to show that $v_p v_x v_p^* \in \mathcal{C}_I$ for all $p \in S$ and $x \in S^*$. By (C1) there exists $y \in S^*$ such that $px = yp$. Hence,

$$v_p v_x v_p^* = v_y e_{pS} = e_{ypS} v_y \in \mathcal{C}_I.$$

Given a right LCM semigroup S with $S^* \neq \emptyset$ and satisfying (C1), suppose that the monoid $\mathcal{S}$ constructed in Proposition 2.7 embeds into a group Γ such that $\mathcal{S}$ generates Γ. Proposition 7.1 applied to the canonical homomorphism $\sigma : S \to \mathcal{S}$, $\sigma(p) = [p]$ for $p \in S$ gives a coaction of Γ with associated conditional expectation as described in (7.1). Note that, in this situation, $\sigma(p) = \sigma(q)$ means precisely that $p = xq$ for some $x \in S^*$. Hence $v_p v_q^* = v_{xq} v_q^* = v_x e_{qS}$, which is in $\mathcal{C}_I$. Thus $C^*(S)^\delta \subseteq \mathcal{C}_I$, and since the reverse inclusion is immediate, the two subalgebras of

$C^*(S)$ are equal. In case S is not right cancellative, it may happen that $p = xq = x'q$ for different x, x' in S^*. However, $v_x e_{qS} = v_{xq}v_q^* = v_{x'q}v_q^* = v_{x'}e_{qS}$. We summarise these considerations in the following result.

Corollary 7.3. *Let S be a right LCM semigroup such that $S^* \neq \emptyset$ and* (C1) *holds. Assume that $\mathcal{S}$ embeds into a group Γ which is generated by the image of $\mathcal{S}$. Then there is a well-defined conditional expectation $\Phi_{\mathcal{C}_I} : C^*(S) \to \mathcal{C}_I$ such that*

$$\Phi_{\mathcal{C}_I}(v_p v_q^*) = \begin{cases} v_x e_{qS}, & \text{if } p = xq \text{ for some } x \in S^*, \\ 0, & \text{if } p \not\sim q. \end{cases} \tag{7.2}$$

If Γ is amenable, then $\Phi_{\mathcal{C}_I}$ is faithful on positive elements.

Our main result about injectivity of representations in terms of their restriction to $\mathcal{C}_I$ is the following theorem.

Theorem 7.4. *Let S be a cancellative right LCM semigroup with identity 1_S such that S satisfies* (C1) *and the semigroup $\mathcal{S}$ constructed in Proposition* 2.7 *embeds into a group Γ in such a way that Γ is generated by $\mathcal{S}$. Assume that the conditional expectation $\Phi_{\mathcal{C}_I} : C^*(S) \longrightarrow \mathcal{C}_I$ constructed in Corollary* 7.3 *is faithful, and that there is a faithful conditional expectation Φ_0 from $\mathcal{C}_I$ onto $\mathcal{D}$ such that $\Phi_0(e_{qS}v_x) = \delta_{x,1_S}e_{qS}$ for all $q \in S$ and $x \in S^*$. Then a $*$-homomorphism $\pi : C^*(S) \longrightarrow B$ is injective if and only if $\pi|_{\mathcal{C}_I}$ is injective.*

Proof. One direction of the theorem is clear, so assume that $\pi|_{\mathcal{C}_I}$ is injective. We must prove that π is injective.

Let $\Phi := \Phi_0 \circ \Phi_{\mathcal{C}_I}$ be the faithful conditional expectation from $C^*(S)$ to $\mathcal{D}$ obtained by composing the two given expectations. The idea of the proof is to construct a contraction $\Phi^\pi : \pi(C^*(S)) \to \mathcal{D}$ such that $\Phi^\pi \circ \pi = \Phi$. Then the injectivity of π will follow from a standard argument: let $a \in C^*(S)_+$ with $a \neq 0$. From $\Phi^\pi(\pi(a)) = \Phi(a)$ and the fact that Φ is faithful on positive elements it follows that $\pi(a) \neq 0$.

Let $F \subset S$ be finite and $t_F \in C^*(S)$ a linear combination of $v_p v_q^*, p, q \in F$ with scalars $\lambda_{p,q}$ in $\mathbb{C}$ such that t_F is positive and non-zero. Then $\Phi(t_F) \neq 0$. We have

$$\begin{aligned}
\Phi(t_F) &= \sum_{p \sim q} \lambda_{p,q}(\Phi_0 \circ \Phi_{\mathcal{C}_I})(v_p v_q^*) \\
&= \sum_{p \sim q, p = xq} \lambda_{p,q}\Phi_0(v_x e_{qS}) \\
&= \sum_{\{p,q \in F | p = q\}} \lambda_{p,q} e_{qS},
\end{aligned}$$

which is $t_{F,\mathcal{D}}$ and is non-zero.

According to Lemma 3.23, there are finite subsets $A \subset F \subset F_1 \subset S$ such that $Q^e_{F_1,A} t_F Q^e_{F_1,A} \in \mathcal{C}_O$ and $Q^e_{F_1,A} t_{F,\mathcal{D}} = \|t_{F,\mathcal{D}}\| Q^e_{F_1,A} \neq 0$. Remark 7.2 shows that $\mathcal{C}_O = \mathcal{C}_I$, so $Q^e_{F_1,A} t_F Q^e_{F_1,A} \in \mathcal{C}_I$. Therefore, we get

$$\begin{aligned}
\Phi(Q^e_{F_1,A} t_F Q^e_{F_1,A}) &= Q^e_{F_1,A}\Phi(t_F)Q^e_{F_1,A} \\
&= Q^e_{F_1,A} t_{F,\mathcal{D}} Q^e_{F_1,A} \\
&= \|t_{F,\mathcal{D}}\| Q^e_{F_1,A} \neq 0.
\end{aligned}$$

Since $\pi|_{C_I}$ is injective, it follows that

$$\begin{array}{rcl} \|\pi(t_F)\| & \geq & \|\pi(Q^e_{F_1,A})\pi(t_F)\pi(Q^e_{F_1,A})\| \\ & = & \|\pi(Q^e_{F_1,A}t_F Q^e_{F_1,A})\| \\ & = & \|Q^e_{F_1,A}t_F Q^e_{F_1,A}\| \\ & \geq & \|\Phi(Q^e_{F_1,A}t_F Q^e_{F_1,A})\| \\ & = & \|t_{F,\mathcal{D}}\|. \end{array}$$

Thus $\Phi^\pi : \pi(C^*(S)) \to \mathcal{D}$ given by $\Phi^\pi(\pi(t_F)) := t_{F,\mathcal{D}}$ is a well-defined contraction with the desired properties. $\square$

For the moment the only examples of semigroups S we know for which the hypotheses of Theorem 7.4 are satisfied are semidirect products $G \rtimes_\theta P$ covered by Proposition 3.16. However, we expect that the theorem will apply in situations where G or the enveloping group of P are non-amenable, for example when they are free groups. The challenge is to generalise the arguments of [11] that show existence of a faithful conditional expectation onto $\mathcal{D}$ from the case $S^* = \{1_S\}$ to the case that there are non-trivial units.

Corollary 7.5. *Assume the notation and hypotheses of Theorem 7.4. Then the left regular representation $\lambda : C^*(S) \longrightarrow C^*_r(S)$ is an isomorphism.*

Proof. Let $\Phi = \Phi_0 \circ \Phi_{\mathcal{C}_I}$ be the faithful conditional expectation from Theorem 7.4. Since S has an identity, $\mathcal{J}(S)$ is independent, and therefore λ restricts to an isomorphism $\mathcal{D} \cong \mathcal{D}_r$, [19]. Thus the argument of Theorem 6.2 can be used. $\square$

Results with the flavour of a gauge-invariant uniqueness theorem have been proved for many classes of C^*-algebras; see [3] and the references therein. In our context, a straightforward version is as presented in the next proposition.

Proposition 7.6. *Let S be a cancellative right LCM semigroup with identity such that S satisfies (C1) and the conditional expectation $\Phi_{\mathcal{C}_I} : C^*(S) \longrightarrow \mathcal{C}_I$ constructed in Corollary 7.3 is faithful. Then a $*$-homomorphism $\pi : C^*(S) \to B$ is injective if and only if $\pi|_{\mathcal{C}_I}$ is injective and B admits a coaction ϵ of the enveloping group Γ of $\mathcal{S}$ such that π is $\delta - \epsilon$-equivariant, i.e. $(\pi \otimes \mathrm{id}_{C^*(\Gamma)}) \circ \delta = \epsilon \circ \pi$.*

Proof. If B admits a coaction ϵ as in the hypothesis, then there is a conditional expectation Φ^ϵ from B onto $\pi(\mathcal{C}_I)$ such that $\Phi^\epsilon \circ \pi = (\pi|_{\mathcal{C}_I}) \circ \Phi_{\mathcal{C}_I}$. Now the standard argument shows that injectivity of π on $\mathcal{C}_I$ can be lifted to $C^*(S)$. $\square$

8. Applications

8.1. **Semidirect products of groups by semigroups.** The new class of right LCM semigroups covered in this work is that of semidirect products of a group by the action of a semigroup. Throughout this subsection, let G be a group, P a cancellative right LCM semigroup with identity 1_P, and $P \overset{\theta}{\curvearrowright} G$ an action by injective group endomorphisms of G. The semidirect product $G \rtimes_\theta P$ is denoted by S for convenience of notation.

Definition 8.1. The action θ is said to *respect the order* on P if for all $p, q \in P$ with $pP \cap qP \neq \emptyset$, we have $\theta_p(G) \cap \theta_q(G) = \theta_r(G)$, where $r \in P$ is any element such that $pP \cap qP = rP$. This is well-defined since $r_1P = pP \cap qP = r_2P$ implies that $r_1 = r_2x$ for some $x \in P^*$, which means $\theta_{r_1}(G) = \theta_{r_2x}(G) = \theta_{r_2}(G)$ because θ_x is an automorphism of G.

Proposition 8.2. *If θ respects the order, then S is a right LCM semigroup.*

Proof. Since both G and P are left cancellative and θ acts by injective maps, S is left cancellative. Suppose $g_1, g_2 \in G$ and $p_1, p_2 \in P$ such that $(g_1, p_1)S \cap (g_2, p_2)S \neq \emptyset$. Then $p_1P \cap p_2P \neq \emptyset$, and since P is right LCM, there is $q \in P$ satisfying $p_1P \cap p_2P = qP$. Denote by $p_1', q_1' \in P$ the elements satisfying $q = p_1p_1' = p_2p_2'$. We must also have $h_1, h_2 \in G$ such that $g_1\theta_{p_1}(h_1) = g_2\theta_{p_2}(h_2)$. We claim that

$$(g_1, p_1)S \cap (g_2, p_2)S = (g_1\theta_{p_1}(h_1), q)S.$$

Since $(g_1\theta_{p_1}(h_1), q) = (g_1, p_1)(h_1, p_1') = (g_2, p_2)(h_2, p_2')$, the right ideal $(g_1\theta_{p_1}(h_1), q)S$ is contained in $(g_1, p_1)S \cap (g_2, p_2)S$.

For the reverse containment, suppose that $(g, s), (h, t) \in S$ with $(g_1, p_1)(g, s) = (g_2, p_2)(h, t)$. Then $g_1\theta_{p_1}(g) = g_2\theta_{p_2}(h)$ and $p_1s = p_2t$. We now immediately have $p_1s = p_2t = qq'$ for some $q' \in P$. The identities $g_1\theta_{p_1}(h_1) = g_2\theta_{p_2}(h_2)$ and $g_1\theta_{p_1}(g) = g_2\theta_{p_2}(h)$ yield $\theta_{p_1}(h_1^{-1}g) = \theta_{p_2}(h_2^{-1}h)$. Since θ respects the order on P, we have $\theta_{p_1}(G) \cap \theta_{p_2}(G) = \theta_q(G)$, and hence $\theta_{p_1}(h_1^{-1}g) = \theta_q(k)$ for some $k \in G$. Then

$$\begin{aligned}(g_1, p_1)(g, s) = (g_1\theta_{p_1}(g), p_1s) = (g_1\theta_{p_1}(h_1)\theta_{p_1}(h_1^{-1}g), p_1s) &= (g_1\theta_{p_1}(h_1), q)(k, q') \\ &\in (g_1\theta_{p_1}(h_1), q)S.\end{aligned}$$

So the reverse containment holds, and hence S is right LCM. $\square$

Since the focus of this paper is on right LCM semigroups we shall assume from now on that θ respects the order. The structure of $\mathcal{J}(S)$ is determined by the semigroup P and the collection of cosets $\{G/\theta_p(G)\}_{p \in P}$.

Lemma 8.3. *For any $g, h \in G$ and $p \in P$ we have*

$$(g, p)S \cap (h, p)S = \begin{cases} (g, p)S, & \text{if } g^{-1}h \in \theta_p(G), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proof. If the intersection is non-empty, we have $g\theta_p(g_1) = h\theta_p(h_1)$ for some $g_1, h_1 \in G$. Then $g^{-1}h = \theta_p(g_1h_1^{-1}) \in \theta_p(G)$, as needed. $\square$

Corollary 8.4. *Let P satisfy* (C2). *For any $(g, p) \in S$ and $(h, x) \in S^*$, the following are equivalent:*

(i) $(h, x)(g, p)S \neq (g, p)S$;
(ii) $(h\theta_x(g), p)S \cap (g, p)S = \emptyset$;
(iii) $g^{-1}h\theta_x(g) \notin \theta_p(G)$.

In particular, S satisfies condition (D1).

Proof. Take $(g, p) \in S$ and $(h, x) \in S^*$. By Lemma 2.4, we have that $x \in P^*$. Condition (C2) gives $y \in P^*$ with $xp = py$. Therefore,

$$(h, x)(g, p)S \cap (g, p)S = (h\theta_x(g), p)S \cap (g, p)S.$$

By Lemma 8.3, these intersections are non-empty if and only if $g^{-1}h\theta_x(g) \in \theta_p(G)$, in which case the ideals $(h, x)(g, p)S$ and $(g, p)S$ coincide. It follows immediately from this that S satisfies condition (D1). $\square$

Lemma 8.5. *Let P satisfy* (C2). *Then the action $S^* \curvearrowright \mathcal{J}(S)$ from Definition* 4.1 *is strongly effective if and only if it is effective.*

Proof. Strong effectiveness implies effectiveness. Assume therefore that $S^*\curvearrowright\mathcal{J}(S)$ is effective. Let $(g,p)\in S$ and $(h,x)\in(G\times P^*)\setminus\{(1_G,1_P)\}$, where $S^*=G\times P^*$ by Lemma 2.4. If $(h,x)(g,p)S\neq(g,p)S$ holds, then (g,p) itself does the job required for strong effectiveness.

Now let $(h,x)(g,p)S=(g,p)S$. We have to find an element $(g',p')\in S$ satisfying

$$(h,x)(g,p)(g',p')S\neq(g,p)(g',p')S. \tag{8.1}$$

It follows from Corollary 8.4 that $g^{-1}h\theta_x(g)=\theta_p(\tilde{g})$ for some $\tilde{g}\in G$. Using (C2) to find y in S^* such that $xpp'=pp'y$, the left-hand side of (8.1) rewrites as

$$(h,x)(g,p)(g',p')S=(h\theta_x(g)\theta_{xp}(g'),xpp')S=(h\theta_x(g)\theta_{py}(g'),pp')S.$$

Thus to prove (8.1) we need to ensure that

$$(g\theta_p(g'))^{-1}h\theta_x(g)\theta_{py}(g')=\theta_p((g')^{-1})g^{-1}h\theta_x(g)\theta_{py}(g')\notin\theta_{pp'}(G).$$

Since $g^{-1}h\theta_x(g)=\theta_p(\tilde{g})$ and θ_p is injective, this is equivalent to

$$(g')^{-1}\tilde{g}\theta_y(g')\notin\theta_{p'}(G).$$

Since $x\neq 1_P$ implies, by right cancellation in P, that $y\neq 1_P$, we see that the existence of (g',p') is guaranteed by effectiveness of the action applied to $(\tilde{g},y)\in S^*\setminus\{1_S\}$. Thus $S^*\curvearrowright\mathcal{J}(S)$ is strongly effective. ☐

Since an action of a group on a space is effective precisely when the intersection of all stabiliser subgroups is the trivial subgroup, Lemma 8.5 says that we can rephrase the property of $S^*\curvearrowright\mathcal{J}(S)$ being strongly effective in terms of stabilisers. We introduce first some notation. For each $(g,p)\in S$, let $S_{(g,p)}$ denote the subgroup of G equal to $g\theta_p(G)g^{-1}$. For the action $S^*\curvearrowright\mathcal{J}(S)$ from Definition 4.1, let $S^*_{(g,p)S}$ denote the stabiliser subgroup of $(g,p)S\in\mathcal{J}(S)$.

Lemma 8.6. *Let P satisfy* (C2) *and consider the action $S^*\curvearrowright\mathcal{J}(S)$ from Definition* 4.1. *Then the stabiliser subgroup of $(g,p)S\in\mathcal{J}(S)$ takes the form*

$$S^*_{(g,p)S}=\{(h,x)\in S^*\mid h\theta_x(g)\in g\theta_p(G)\}.$$

If $P^*=\{1_P\}$, then $S^*_{(g,p)S}=S_{(g,p)}\times\{1_P\}$.

Further, $S^*\curvearrowright\mathcal{J}(S)$ is strongly effective if and only if

$$\bigcap_{(g,p)\in S}S_{(g,p)}=\{1_G\}.$$

In particular, if $P^*=\{1_P\}$ and G is abelian, then $S^*\curvearrowright\mathcal{J}(S)$ is strongly effective if and only if $\bigcap_{p\in P}\theta_p(G)=\{1_G\}$.

Proof. The claimed description of $S^*_{(g,p)S}$ follows from Corollary 8.4, and the characterisation of strongly effective follows from Lemma 8.5. ☐

We shall be able to say more for semigroups S where P is a countably generated free abelian semigroup with identity. For the purposes of the next results, we therefore assume that $P\cong\mathbb{N}^k$ for some $k\in\mathbb{N}$ or $P\cong\bigoplus_{\mathbb{N}}\mathbb{N}$. In this case, (C2) is automatic for P; hence $S=G\rtimes_\theta P$ satisfies (D1) by Corollary 8.4.

As indicated in the comment following Lemma 5.1, condition (D2) is harder to establish in full generality. The next result describes an obstruction to having (D2) satisfied by $S=G\rtimes_\theta P$.

Lemma 8.7. *Assume $P \cong \mathbb{N}^k$ for some $k \in \mathbb{N}$ or $P \cong \bigoplus_{\mathbb{N}} \mathbb{N}$. If there are $q_1, \dots, q_m \in P \setminus \{1_P\}$ such that $[G : \theta_{q_i}(G)] < \infty$ and*

$$\bigcap_{\substack{(g,p) \in S \\ p \in P \setminus (\bigcup_{1 \le i \le m} q_i P)}} S_{(g,p)} \supsetneqq \{1_G\},$$

then S does not satisfy (D2).

Proof. Suppose there are $q_1, \dots, q_m$ as prescribed above and pick an element

$$h \in \bigcap_{\substack{(g,p) \in S \\ p \in P \setminus (\bigcup_{1 \le i \le m} q_i P)}} S_{(g,p)}$$

with $h \neq 1_G$. Denote $[G : \theta_{q_i}(G)] = N_i$ in $\mathbb{N}^\times$, and choose, for each $i = 1, \dots, m$, a complete set of representatives $(h_{i,j})_{1 \le j \le N_i}$ for $G/\theta_{q_i}(G)$.

We claim that (D2) fails for the choice of elements $(g_0, p_0) = (g_1, p_1) = (1_G, 1_P)$ in S, $(h, 1_P)$ in S^*, and the finite subset $\{(h_{i,j}, q_i) \mid j = 1, \dots, N_i,\ i = 1, \dots, m\}$ of S. Note that we have $(g_1, p_1) \notin (h_{i,j}, q_i)S$ for all $j = 1, \dots, N_i$ and $i = 1, \dots, m$. If (D2) were to hold, it would imply the existence of (g_2, p_2) such that both $(g_2, p_2) \notin (h_{i,j}, q_i)S$ for all i, j and, by Corollary 8.4, also $h \notin S_{(g_2,p_2)}$. Hence, by the choice of h, there is at least one i with $p_2 \in q_i P$. For this i there is a unique $j \in \{1, \dots, N_i\}$ with $g_2 \in h_{i,j}\theta_{q_i}(G)$. In other words, we would get $(g_2, p_2) \in (h_{i,j}, q_i)S$, which is a falsehood. □

In the next two examples we describe some situations where S satisfies condition (D2).

Example 8.8. Let $G = \bigoplus_{\mathbb{N}} \mathbb{Z}$ and P be the unital subsemigroup of $\mathbb{N}^\times$ generated by 2 and 3. We shall denote $P = |2, 3\rangle$. Define an action θ of P by injective endomorphisms of G as follows: for $g = (g_n)_{n \in \mathbb{N}} \in G$, let

$$\theta_2(g) = 2g,\ (\theta_3(g))_0 = 3g_0 \text{ and } (\theta_3(g))_n = g_n \text{ for all } n \ge 1.$$

It is immediate that θ preserves the order, so S is right LCM by Proposition 8.2. Further, $[G : \theta_2(G)] = \infty$ and $[G : \theta_3(G)] = 3$. Note that $\bigcap_{n \in \mathbb{N}} \theta_{2^n}(G) = \{1_G\}$. We claim that $S = G \rtimes_\theta P$ satisfies (D2). It will follow from Lemma 8.11 that S does not satisfy (D3).

Suppose that we have $s_0 := (g_0, p_0) \in S, s_1 := (g_1, p_1) \in (g_0, p_0)S$ as well as $(h_1, q_1), \dots, (h_m, q_m) \in S$ such that

$$(g_1, p_1)S \cap \Big(S \setminus \bigcup_{1 \le i \le m} (h_i, q_i)S\Big) \neq \emptyset.$$

In particular, this implies $(g_1, p_1) \notin (h_i, q_i)S$ for each $i = 1, \dots, m$. In case $p_1 \in q_i P$ for some i, then necessarily $g_1 \notin h_i\theta_{q_i}(G)$, and therefore Lemma 8.3 implies that $(g_1, p_1)S \cap (h_i, q_i)S = \emptyset$. Without loss of generality we may thus assume that $p_1 \notin q_i P$ for all $i = 1, \dots, m$.

Now let $x = (g, 1_P)$ with $g \neq 1_G$. An element $s_2 := (g_2, p_2)$ as required in (D2) will have to satisfy $xs_0^{-1}s_2S \cap s_0^{-1}s_2S = \emptyset$. If we denote $r := p_0^{-1}p_2$, this requirement takes the form $(g, 1_P)(h', r)S \cap (h', r)S = \emptyset$ for some $h' \in G$. Now, using $\bigcap_{n \in \mathbb{N}} \theta_{2^n}(G) = \{1_G\}$, we can choose $n \in \mathbb{N}$ large enough so that $p_2 := p_1 2^n$ satisfies $g \notin \theta_r(G)$. By Corollary 8.4, this means that $x(h', r)S \cap (h', r)S = \emptyset$ for any $h' \in G$. Thus we have freedom to choose the first entry in s_2, and this choice must be made so that it ensures the second requirement in (D2). The crucial ingredient here is the fact that $[G : \theta_{2^k}(G)] = \infty$ for all $k \geq 1$, which will allow us to choose $g_2 \in g_1\theta_{p_1}(G)$ such that $(g_2, p_2) \notin (h_i, q_i)S$ for all i. To achieve this goal requires a careful argument.

If $p_2 \notin q_iP$ for all $i = 1, \dots, m$, then any choice of $g_2 \in g_1\theta_{p_1}(G)$ will ensure that $(g_2, p_2) \notin (h_i, q_i)S$ for all i. Assume next that $q_1, \dots, q_m$ are labelled in such a way that there is $m' \in \{1, \dots, m\}$ with the property that $p_2 \in q_iP$ implies $i \leq m'$. Note that the elements corresponding to $i = m'+1, \dots, m$ pose no obstruction to the choice of g_2 because for these indices i we have $(g_2, p_2) \notin (h_i, q_i)S$ irrespective of the choice of g_2. Possibly changing enumeration once more, we can assume that 1 is minimal in $\{1, \dots, m'\}$ in the sense that

$$p_1P \cap q_1P \subset p_1P \cap q_iP \Longrightarrow p_1P \cap q_1P = p_1P \cap q_iP \text{ for all } 1 \leq i \leq m',$$

and that $2, \dots, m'$ are assigned in such a way that there is m_1 with

$$p_1P \cap q_1P = p_1P \cap q_iP \Longrightarrow i \leq m_1 \text{ for } i \in 2, \dots, m'.$$

Let n_1 be such that $p_1P \cap q_1P = p_12^{n_1}$, and note that $1 \leq n_1 \leq n$. Since $[G : \theta_{2^{n_1}}(G)] = \infty$, there are infinitely many distinct principal right ideals of the form $(g_1\theta_{p_1}(g'), p_12^{n_1})S \subset (g_1, p_1)S$ with $g' \in G$. Since S is right LCM, of these infinitely many ideals, at most m_1 of them are not admissible for a choice of g_2 (because they are possibly contained in $(h_1, q_1)S, \dots, (h_{m_1}, q_{m_1})S$). Thus there is $g_{2,1} \in g_1\theta_{p_1}(G)$ with

$$(g_{2,1}, p_12^{n_1}) \notin (h_i, q_i)S \text{ for all } i = 1, \dots, m_1.$$

Replacing g_1 by $g_{2,1}$, p_1 by $p_12^{n_1}$, n by $n - n_1$, and $\{1, \dots, m'\}$ by $\{m_1+1, \dots, m'\}$, we can iterate this process. Thus at the second step we obtain an element $(g_{2,2}, p_12^{n_1}2^{n_2}) \in (g_{2,1}, p_12^{n_1})S$, for appropriate $1 \leq n_2 \leq n$ and $g_{2,2} \in G$, which also avoids additional ideals $(h_i, q_i)S$, where $i \in \{1, \dots, m_2\}$ is a subset of $\{m_1+1, \dots, m'\}$ for appropriate m_2. This process stops after finitely many steps (equal to $m \geq 1$ if $n_1 + \cdots + n_m = n$) because n is finite. Hence the final pair $(g_{2,m}, p_2)$ has the required properties.

The second example shows that we can also have (D2) in the absence of endomorphisms with infinite index:

Example 8.9. Let $G = \mathbb{Z}$, $P = \mathbb{N}^\times$ and θ be given by multiplication, i.e. $\theta_p(g) = pg$. Clearly, we have $[G : \theta_p(G)] = p < \infty$ for all $p \in P$. Also, note that for all $q_1, \dots, q_m \in P \setminus \{1_P\}$, we have

$$\bigcap_{p \in P \setminus (\bigcup_{1 \leq i \leq m} q_iP)} \theta_p(G) = \{1_G\}$$

since $P \setminus (\bigcup_{1 \leq i \leq m} q_iP)$ contains arbitrarily large positive integers. So there is an abundance of subsemigroups $Q \subset P$ for which the restricted action $\theta|_Q$ separates the points in G. We claim that $S = G \rtimes_\theta P$ satisfies (D2).

Let $(g_0, p_0) \in S$, $(g_1, p_1) \in (g_0, p_0)S$, $(h_1, q_1), \dots, (h_m, q_m) \in S \setminus \{1_S\}$ with $(g_1, p_1) \notin (h_i, q_i)S$ for $i = 1, \dots, m$, and $x = (g, 1_P) \in S$ with $g \neq 1_G$. For similar reasons as in Example 8.8, we can assume that $p_1 \notin q_iP$ holds for all i.

Now choose a prime $p \in P$ that does not divide any of the $q_1, \dots, q_m$. Take $n \geq 1$ such that $g \notin \theta_{p^n}(G) = p^n\mathbb{Z}$. If we let $p_2 := p_1p^n$ and $g_2 := g_1$, then $p_2 \notin q_iP$

and hence $(g_2,p_2) \notin (h_i,q_i)S$ for all i. Moreover, $p_0^{-1}p_2 \in p^nP$ as $p_1 \in p_0P$. Therefore $g \notin \theta_{p_0^{-1}p_2}(G)$. Hence Corollary 8.4 implies that $(g,1_P)(g_0,p_0)^{-1}(g_2,p_2)S \cap (g_0,p_0)^{-1}(g_2,p_2)S = \emptyset$, showing (D2).

Remark 8.10. One can relax the assumptions and consider semidirect products of suitable semigroups by semigroups instead, for instance positive cones G_+ in a group G on which we already have an action θ of a semigroup P. A natural assumption in this setting would be $\theta_p(G_+) \subset G_+$. Natural examples of this kind arise for $\mathbb{N}^k \subset \mathbb{Z}^k$ where θ takes values in $\mathrm{M}_k(\mathbb{N}) \cap \mathrm{GL}_k(\mathbb{Q})$.

8.2. **Examples of purely infinite simple semigroup C^*-algebras from semidirect products.** As before, we consider S of the form $G \rtimes_\theta P$, where P is a countably generated, free abelian semigroup with identity and $P \overset{\theta}{\curvearrowright} G$ is an action by injective group endomorphisms of G that respects the order. In this subsection, we show that Theorem 5.3 (3) applies to S if $[G:\theta_p(G)]$ is infinite for every $p \neq 1_P$. We illustrate the theorem with several concrete examples of semigroups whose semigroup C^*-algebra is purely infinite and simple.

Lemma 8.11. *Assume that $P \cong \mathbb{N}^k$ for some $k \geq 1$ or $P \cong \bigoplus_{\mathbb{N}} \mathbb{N}$. Then S satisfies* (D3) *if and only if the index $[G:\theta_p(G)]$ is infinite for every $p \neq 1_P$.*

Proof. To begin with, note that $S^* = G \times \{1_P\}$. Suppose that there exists $q \neq 1_P$ such that $[G:\theta_q(G)] = n < \infty$ and let $h_1,\dots,h_n \in G$ be a complete set of representatives for $G/\theta_q(G)$. We claim that (D3) fails for $(g,p) = (1_G,1_P) = 1_S$ and $(h_1,q),\dots,(h_n,q)$. To see this, note first that $(g,p) \notin (h_k,q)S$ for all $1 \leq k \leq n$. If $(g',p') \in (g,p)S$ is arbitrary, then there is a unique k such that $g' \in h_k\theta_q(G)$, i.e. $g' = h_k\theta_q(\tilde{g})$ for some $\tilde{g} \in G$. Since P is commutative, we have $(g',p')S \cap (h_k,q)S \supset (g',p'q)S \cap (h_k\theta_q(\tilde{g}),p'q)S$, which equals $(g',p'q)S$, so $(g',p')S \cap (h_k,q)S$ is non-empty.

Now suppose $[G:\theta_p(G)]$ is infinite for every $p \in P\backslash\{1_P\}$. Let $(g,p) \in S$ and $F \subset S$ be finite such that

$$(g,p)S \cap \Big(S \setminus \bigcup_{(h,q)\in F} (h,q)S\Big) \neq \emptyset.$$

Without loss of generality, we may assume $(g,p)S \cap (h,q)S \neq \emptyset$ and $p \neq q$ hold for all $(h,q) \in F$. Consider

$$F_P := \{r \mid pP \cap qP = rP \text{ for some } (h,q) \in F\}.$$

Pick $p_1 \in F_P$ which is minimal in the sense that for any other $r \in F_P$, $p_1 \in rP$ implies $r = p_1$. Let $(h_1,q_1),\dots,(h_n,q_n) \in F$ denote the elements satisfying $pP \cap q_iP = p_1P$. According to Proposition 8.2, the fact that $(g,p)S \cap (h_i,q_i)S \neq \emptyset$ for all $i = 1,\dots,n$ shows that we have

$$(g,p)S \cap (h_i,q_i)S = (g\theta_p(g_i'),p_1)S = (g,p)(g_i',p^{-1}p_1)S$$

for suitable $g_i' \in G$ and each $i = 1,\dots,n$. Since $p \neq q_i$, we have $p^{-1}p_1 \neq 1_P$ and hence the index $[G:\theta_{p^{-1}p_1}(G)]$ is infinite. In particular, there is $g_1 \in g\theta_p(G)$ such that

$$(g_1,p_1) \in (g,p)S \text{ and } (g_1,p_1)S \cap (h_i,q_i)S = \emptyset \text{ for all } i = 1,\dots,n.$$

Setting

$$F_1 := \{(h,q) \in F \mid (h,q)S \cap (g_1,p_1)S \neq \emptyset\},$$

we observe that $F_1 \subset F \setminus \{(h_1,q_1),\ldots,(h_n,q_n)\}$ so $F_1 \subsetneq F$. If F_1 is empty, then we are done, so let us assume that $F_1 \neq \emptyset$. Note that the minimal way in which p_1 was chosen implies $p_1 \notin pP \cap qP$ for all $(h,q) \in F_1$. This will allow us to conclude that

$$(g_1,p_1)S \cap \left(S \setminus \bigcup_{(h,q)\in F_1}, (h,q)S\right) \neq \emptyset$$

by invoking the choice of (g,p) and F. Indeed, if the intersection were empty, then there would be $(h,q) \in F_1$ with $(g_1,p_1)S \subset (h,q)S$; see Proposition 8.2. This would force $p_1 \in qP$ and therefore $p_1 \in p_1P \cap qP \subset pP \cap qP$, contradicting $(h,q) \in F_1$. Thus, we can iterate this process and, after finitely many steps, arrive at an element $(g',p') \in (g,p)S$ with the property $(g',p')S \cap (h,q)S = \emptyset$ for all $(h,q) \in F$. This completes the proof of the lemma. $\square$

Theorem 8.12. *Suppose G is a group, $P \cong \mathbb{N}^k$ for some $k \geq 1$ or $P \cong \bigoplus_{\mathbb{N}} \mathbb{N}$, and $P \overset{\theta}{\curvearrowright} G$ is an action by injective group endomorphisms of G respecting the order. Denote $S = G \rtimes_\theta P$. Assume that $\bigcap_{p\in P} \theta_p(G) = \{1_G\}$, $[G : \theta_p(G)]$ is infinite for every $p \neq 1_P$ and the conditional expectation $C^*(S) \xrightarrow{\Phi_{\mathcal{D}}} \mathcal{D}$ is faithful. Then $C^*(S)$ is purely infinite and simple.*

Proof. We intend to apply Theorem 5.3 (3). First, note that (D1) holds by Corollary 8.4 since (C2) is trivially satisfied for P. By Lemma 8.6, $\bigcap_{p\in P} \theta_p(G) = \{1_G\}$ corresponds to strong effectiveness of $S^* \curvearrowright \mathcal{J}(S)$. The fact that S satisfies (D3) follows from Lemma 8.11. Since $\Phi_{\mathcal{D}}$ is faithful, Theorem 5.3 (3) implies that $C^*(S)$ is purely infinite and simple. $\square$

Let us now look at some concrete examples. We start with a shift space:

Example 8.13. Let $P \cong \mathbb{N}^k$ for some $k \geq 1$ or $P \cong \bigoplus_{\mathbb{N}} \mathbb{N}$ and suppose G_0 is a countable amenable group. To avoid pathologies, let us assume that G_0 has at least two distinct elements. Then P admits a shift action θ on $G := \bigoplus_{P} G_0$ given by

$$(\theta_p((g_q)_{q\in P}))_r = \chi_{pP}(r)\ g_{p^{-1}r} \text{ for all } p,r \in P.$$

It is apparent that θ is an action by injective group endomorphism that respects the order and $\bigcap_{p\in P} \theta_p(G) = \{1_G\}$ holds. We note that S is a right reversible semigroup whose enveloping group $S^{-1}S$ is amenable because G and $P^{-1}P$ are amenable. Using Proposition 3.16 we conclude that $\Phi_{\mathcal{D}}$ is faithful. Finally, $[G : \theta_p(G)] < \infty$ holds for $p \neq 1_P$ only if G_0 is finite and $P \cong \mathbb{N}$. Indeed, if $p \neq 1_P$, then each element of $\bigoplus_{q\in P\setminus pP} G_0$ yields a distinct left-coset in $G/\theta_p(G)$. Clearly, this group is finite if and only if G_0 is finite and $P \cong \mathbb{N}$. So if P is not singly generated or G_0 is a countably infinite group, then $C^*(S)$ is purely infinite and simple by Theorem 8.12.

A variant of the next example with singly generated P and finite field $\mathcal{K}$ has been considered in [9, Example 2.1.4].

Example 8.14. Let $\mathcal{K}$ be a countably infinite field and let $G = \mathcal{K}[T]$ denote the polynomial ring in a single variable T over $\mathcal{K}$. We choose non-constant polynomials $p_i \in \mathcal{K}[T], i \in I$, for some index set I. Multiplying by p_i defines an endomorphism

θ_{p_i} of G with $[G : \theta_{p_i}(G)] = |\mathcal{K}|^{\deg(p_i)}$, where $\deg(p_i)$ denotes the degree of $p_i \in \mathcal{K}[T]$. Thus, if we let P be the semigroup generated by all the p_i's, in notation

$$P := |(p_i)_{i\in I}\rangle,$$

then the index of $\theta_p(G)$ in G is infinite for all $p \neq 1_P$. It is not hard to show that θ respects the order if and only if $(p_i) \cap (p_j) = (p_i p_j)$ holds for the principal ideals whenever $i \neq j$. Since every element in G has finite degree, $\bigcap_{p\in P} \theta_p(G) = \{1_G\}$ is automatic because the p_i are non-constant. The expectation $\Phi_{\mathcal{D}}$ is faithful for the same reason as in Example 8.13. Thus, provided the family $(p_i)_{i\in I}$ has been chosen accordingly, $C^*(S)$ is purely infinite and simple.

We next discuss a class of semigroups S based on a non-commutative group G.

Example 8.15. Let $G = \mathbb{F}_2$ be the free group in a and b. We define injective group endomorphisms θ_1, θ_2 of G by $\theta_1(a) = a^2, \theta_1(b) = b$, $\theta_2(a) = a, \theta_2(b) = b^2$ and set $P = |\theta_1, \theta_2\rangle \cong \mathbb{N}^2$. It is clear that the induced action θ of P on G respects the order. Additionally, $\bigcap_{p\in P} \theta_p(G) = \{1_G\}$ is easily checked using the word length coming from $\{a, a^{-1}, b, b^{-1}\}$. To see that $[G : \theta_1(G)] = \infty$ holds, note that the family $\big((ab)^j\big)_{j\geq 1}$ yields mutually distinct left-cosets in $G/\theta_1(G)$. The same argument with ba instead of ab shows that $\theta_2(G)$ has infinite index in G. Thus, $C^*(S)$ is purely infinite and simple provided that $\Phi_{\mathcal{D}}$ is faithful. One can show that this amounts to amenability of the action $\mathbb{F}_2 \overset{\tau}{\curvearrowright} \mathcal{D}$.

This example can be viewed as belonging to a larger class, as described next. The inspiration for these examples was [30, Example 2.3.9], where the single endomorphism θ_2 from Example 8.15 on $\mathbb{F}_2$ is considered.

Example 8.16. For $2 \leq n \leq \infty$, let $\mathbb{F}_n$ be the free group in n generators $(a_k)_{1\leq k\leq n}$. Fix $1 \leq d \leq n$ and choose for each $1 \leq i \leq d$ an n-tuple $(m_{i,k})_{1\leq k\leq n} \subset \mathbb{N}^\times$ such that

(1) for each $1 \leq i \leq d$, there exists k such that $m_{i,k} > 1$, and
(2) for all $1 \leq i, j \leq d, i \neq j$ and $1 \leq k \leq n$, $m_{i,k}$ and $m_{j,k}$ are relatively prime.

Then $\theta_i(a_k) = a_k^{m_{i,k}}$ defines an injective group endomorphism of $\mathbb{F}_n$ for each $1 \leq i \leq d$. We set $P = |(\theta_i)_{1\leq i\leq d}\rangle$. Using (2), one can show that the induced action θ of P on G respects the order. As in Example 8.15, $[G : \theta_p(G)]$ is infinite for every $p \neq 1_P$. The requirement $\bigcap_{p\in P} \theta_p(G) = \{1_G\}$ reduces to

(3) For each $1 \leq k \leq n$, there exists $1 \leq i \leq d$ satisfying $m_{i,k} > 1$.

So $C^*(S)$ is purely infinite and simple if conditions (1)−(3) above are satisfied and $\Phi_{\mathcal{D}}$ is faithful. As in Example 8.15, the latter corresponds to amenability of the action $\mathbb{F}_n \overset{\tau}{\curvearrowright} \mathcal{D}$.

8.3. **Semigroups from self-similar actions.** Another large class of right LCM semigroups arises from self-similar actions; cf. [2, 16, 18]. We won't be able to say much here, since conditions (D2) and (D3) are not likely to hold. However, these semigroups will satisfy condition (D1), and they will satisfy strong effectiveness in the presence of right cancellation. We include these observations here, as well as a description of those semigroups that satisfy (C1).

Let X be a finite alphabet. We write X^n for the set of all words of length n, and X^* for the set of all finite words. We let $\varnothing$ denote the empty word. Under concatenation of words, X^* is a semigroup (and is nothing more than $\mathbb{F}^+_{|X|}$). A *self-similar action* is a pair (G, X), where G is a group acting faithfully on X^* and such that for every $g \in G$ and $x \in X$, there exists a unique $g|_x \in G$ such that

$$g \cdot (xw) = (g \cdot x)(g|_x \cdot w). \tag{8.2}$$

The group element $g|_x$ is called the *restriction* of g to x. The restriction map can be extended iteratively to all finite words and satisfies

$$g|_{vw} = (g|_v)|_w, \quad gh|_v = g|_{h\cdot v}h|_v, \quad \text{and} \quad g|_v^{-1} = g^{-1}|_{g\cdot v},$$

for all $g, h \in G$ and $v, w \in X^*$. Moreover, the map $g : X^n \to X^n$ for $n \geq 1$ given by $w \mapsto g \cdot w$ is bijective. The proof of these properties and much more can be found in [24]. The Cuntz-Pimsner algebra $\mathcal{O}(G, X)$ of a self-similar group has been studied in [23, 25], and the Toeplitz algebra $\mathcal{T}(G, X)$ has been studied in [13].

To each self-similar action (G, X) there exists a semigroup $X^* \bowtie G$, which is the set $X^* \times G$ with multiplication given by

$$(x, g)(y, h) = (x(g \cdot y), g|_y h).$$

The semigroup $X^* \bowtie G$ was introduced in [16] and is an example of a Zappa-Szép product. The C^*-algebra $C^*(X^* \bowtie G)$ was studied in [2] and was shown to be isomorphic to $\mathcal{T}(G, X)$.

Denote $S = X^* \bowtie G$. Then S is right LCM and the principal right ideals are determined by the element of X^*, in the sense that $(w, g)X^* \bowtie G = (z, h)X^* \bowtie G$ if and only if $w = z$. The identity in $X^* \bowtie G$ is $(\varnothing, 1_G)$, and we have $(X^* \bowtie G)^* = \{\varnothing\} \times G$. For $x \in X$ let G_x denote the stabiliser subgroup of x in G. The map $\phi_x : G_x \to G$ given by $\phi_x(g) = g|_x$ is a homomorphism; see for example [18, Lemma 3.1].

We now show that S satisfies (D1), and we determine the precise conditions under which the action $S^* \curvearrowright \mathcal{J}(S)$ given by left multiplication on constructible ideals is strongly effective.

Lemma 8.17. *Let* (G, X) *be a self-similar action. Then* $X^* \bowtie G$ *satisfies* (D1) *from Definition* 4.1.

Proof. Let $(\varnothing, h) \in (X^* \bowtie G)^*$ and $(w, g)X^* \bowtie G \in \mathcal{J}(X^* \bowtie G)$ with

$$(\varnothing, h)(w, g)X^* \bowtie G \cap (w, g)X^* \bowtie G \neq \emptyset.$$

Then there are $(w', g'), (w'', g'')$ such that $(\varnothing, h)(w, g)(w', g') = (w, g)(w'', g'')$, and since w and $h \cdot w$ have the same length, this means $w = h \cdot w$. Then

$$(\varnothing, h)(w, g)X^* \bowtie G = (h \cdot w, h|_w g)X^* \bowtie G = (w, h|_w g)X^* \bowtie G = (w, g)X^* \bowtie G.$$

□

We know from [18, Proposition 3.11] that $X^* \bowtie G$ is right cancellative if and only if $\{w \in X^* : \exists g \in G \setminus \{1_G\}, g \cdot w = w \text{ and } g|_w = 1_G\} = \emptyset$. This condition also appears in the following result:

Lemma 8.18. *Let* (G, X) *be a self-similar action. Then the action* $S^* \curvearrowright \mathcal{J}(S)$ *given by left multiplication is strongly effective in the sense of Definition* 4.1 *if and only if*

$$\{w \in X^* : \exists g \in G \setminus \{1_G\}, g \cdot w = w \text{ and } g|_w = 1_G\} = \emptyset.$$

Proof. We prove the contrapositive of the forward implication. Suppose $w \in X^*$ and $g \in G$ with $g \cdot w = w$ and $g|_w = 1_G$. Then $(\varnothing, g) \in (X^* \bowtie G)^*$ and $(w, h) \in S$ satisfy

$$(\varnothing, g)(w, h)(z, k)X^* \bowtie G = (g \cdot w, g|_w h)(z, k)X^* \bowtie G = (w, h)(z, k)X^* \bowtie G,$$

for all $(z, k) \in X^* \bowtie G$. So the action is not strongly effective.

For the reverse implication, suppose $(\varnothing, g) \in (X^* \bowtie G)^*$ and $(w, g) \in X^* \bowtie G$. If $g \cdot w \neq w$, then $(\varnothing, g)(w, h)X^* \bowtie G \neq (w, h)X^* \bowtie G$. If $g \cdot w = w$, then $g|_w = 1_G$ by assumption. Choose $z \in X^*$ such that $g|_w \cdot (h \cdot z) \neq h \cdot z$. Then

$$\begin{aligned}(\varnothing, g)(w, h)(z, 1_G)X^* \bowtie G &= \big(w(g|_w \cdot (h \cdot z)), (g|_w h)|_z\big)X^* \bowtie G \\ &\neq (w(h \cdot z), (g|_w h)|_z)X^* \bowtie G \\ &= (w(h \cdot z), h|_z)X^* \\ &= (w, g)(z, 1_G)X^* \bowtie G.\end{aligned}$$

So the action is strongly effective. □

We can describe those semigroups $X^* \bowtie G$ that satisfy (C1). Recall from [18, page 22] (or [23]) that a self-similar action of G on X is *recurrent* if the action of G on X is transitive and the homomorphism ϕ_x is surjective for any $x \in X$. By [18, Lemma 1.3(8)], the last condition is equivalent to ϕ_w being surjective for all $w \in X^*$.

Lemma 8.19. *Let G be a recurrent self-similar action on X. Then $S := X^* \bowtie G$ satisfies* (C1). *In the converse direction, if S satisfies* (C1)*, then all maps ϕ_x for $x \in X$ are surjective.*

Proof. Let $(w, g) \in S$ and $(\varnothing, h) \in S^*$. We will show that there is $(\varnothing, k) \in S^*$ such that $(w, g)(\varnothing, h) = (\varnothing, k)(w, g)$. Since ϕ_w is surjective, there is $k \in G_w$ such that $\phi(k) = ghg^{-1}$. In other words, there is $k \in G$ with $k \cdot w = w$ and $k|_w = ghg^{-1}$. Then

$$(\varnothing, k)(w, g) = (k \cdot w, k|_w g) = (w, ghg^{-1}g) = (w, g)(\varnothing, h),$$

showing (C1).

In the other direction, let $x \in X$ and $g \in G$. From (C1) applied to $(x, g^{-1}) \in S$ and $(\varnothing, g) \in S^*$, there is $(\varnothing, k) \in S^*$ such that $(x, g^{-1})(\varnothing, g) = (x, e_G) = (\varnothing, k)(x, g^{-1})$, which is $(k \cdot x, k|_x g^{-1})$. This says that $k \in G_x$ and $\phi_x(k) = g$, showing surjectivity for all $x \in X$ (hence for all $x \in X^*$ by [18, Lemma 1.3(8)].) □

It would be interesting to know if for the latter class one can prove a uniqueness result using the expectation onto the C^*-subalgebra $\mathcal{C}_I$.

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