Bull. Korean Math. Soc. **42** (2005), No. 4, pp. 713–724

ON C-BOCHNER CURVATURE TENSOR OF A CONTACT METRIC MANIFOLD

Jeong-Sik Kim, Mukut Mani Tripathi, and Jaedong Choi

Abstract. We prove that a (κ, μ)-manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold. Several interesting corollaries of this result are drawn. Non-Sasakian (κ, μ)-manifolds with C-Bochner curvature tensor B satisfying $B(\xi, X) \cdot S = 0$, where S is the Ricci tensor, are classified. $N(\kappa)$-contact metric manifolds M^{2n+1}, satisfying $B(\xi, X) \cdot R = 0$ or $B(\xi, X) \cdot B = 0$ are classified and studied.

1. Introduction

In [4], Blair, Koufogiorgos and Papantoniou introduced the class of contact metric manifolds M with contact metric structures (φ, ξ, η, g), in which the curvature tensor R satisfies the equation

$$R(X,Y)\xi = (\kappa I + \mu h) R_0(X,Y)\xi, \quad X, Y \in TM,$$

where $(\kappa, \mu) \in \mathbb{R}^2$, $2h$ is the Lie derivative of φ in the direction ξ and R_0 is given by

$$R_0(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \quad X, Y, Z \in TM.$$

A contact Riemannian manifold belonging to this class is called a (κ, μ)-*manifold.* Characteristic examples of non-Sasakian (κ, μ)-manifolds are the tangent sphere bundles of Riemannian manifolds of constant sectional curvature not equal to one.

On the other hand, S. Bochner introduced a Kähler analogue of the Weyl conformal curvature tensor by purely formal considerations, which is now well known as the Bochner curvature tensor [5]. A geometric

Received April 12, 2004. Revised March 29, 2005.
2000 Mathematics Subject Classification: 53C50, 53B30.
Key words and phrases: contact metric manifold, (κ, μ)-manifold, $N(\kappa)$-contact metric manifold, Sasakian manifold, C-Bochner curvature tensor, E-Bochner curvature tensor, η-Einstein manifold, Einstein manifold.
This work was supported by Korea Science & Engineering Foundation Grant (R05-2004-000-11588-0).

meaning of the Bochner curvature tensor is given by D. Blair in [1]. By using the Boothby-Wang's fibration [6], M. Matsumoto and G. Chūman constructed *C-Bochner curvature tensor* [10] from the Bochner curvature tensor. In [7], H. Endo defined E-Bochner curvature tensor as an extended C-Bochner curvature tensor and showed that a K-contact manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold.

A K-contact manifold is always a contact metric manifold, but the converse is not true in general. Thus, it is worthwhile to study C-Bochner curvature tensor and E-Bochner curvature tensor in contact metric manifolds. Extending the result of H. Endo[7] to a (κ, μ)-manifold we prove the following

THEOREM 1.1. *A (κ, μ)-manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold.*

Then, we draw several corollaries of this result to $N(\kappa)$-contact metric manifolds [15], the unit tangent sphere bundles [4], $N(\kappa)$-contact space forms [8] and (κ, μ)-space forms [9].

In [12] and [14], contact metric manifolds satisfying $R(X, \xi) \cdot S = 0$ are studied. Motivated by these studies, we classify non-Sasakian (κ, μ)-manifolds with C-Bochner curvature tensor B satisfying $B(\xi, X) \cdot S = 0$. In fact, we prove the following

THEOREM 1.2. *Let M^{2n+1} be a non-Sasakian (κ, μ)-manifold. If the C-Bochner curvature tensor B satisfies $B(\xi, X) \cdot S = 0$, then we have one of the following:*

(i) *M^{2n+1} is flat and 3-dimensional;*
(ii) *M^{2n+1} is locally isometric to $E^{n+1}(0) \times S^n(4)$;*
(iii) *M^{2n+1} is an η-Einstein manifold;*
(iv) *M^{2n+1} is a 3-dimensional Einstein manifold.*

In a recent paper [13], $N(\kappa)$-contact metric manifolds satisfying $R(X, Y) \cdot B = 0$ are studied. Hence, we classify $N(\kappa)$-contact metric manifolds satisfying $B(\xi, X) \cdot R = 0$. In fact, we prove the following

THEOREM 1.3. *Let M^{2n+1} $(n > 1)$ be an $N(\kappa)$-contact metric manifold. If the C-Bochner curvature tensor B satisfies $B(\xi, X) \cdot R = 0$, then either M^{2n+1} is a Sasakian manifold or M^{2n+1} is locally isometric to the sphere $S^{2n+1}(1)$.*

In the last, motivated by studies of contact metric manifolds satisfying $R(X, \xi) \cdot R = 0$ ([12, 14]), we consider $N(\kappa)$-contact metric manifolds satisfying $B(\xi, X) \cdot B = 0$. In fact, we prove the following

THEOREM 1.4. *Let M^{2n+1} be an $N(\kappa)$-contact metric manifold. Then the C-Bochner curvature tensor B satisfies $B(\xi, X) \cdot B = 0$ if and only if M^{2n+1} is a Sasakian manifold.*

2. Contact metric manifolds

A $(2n+1)$-dimensional differentiable manifold M is called an almost contact manifold if either its structural group can be reduced to $U(n) \times 1$ or equivalently, there is an almost contact structure (φ, ξ, η) consisting of a $(1,1)$ tensor field φ, a vector field ξ, and a 1-form η satisfying

(1) $\varphi^2 = -I + \eta \otimes \xi, \quad$ and (one of) $\quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$

An almost contact structure is said to be *normal* if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by

$$J\left(X, \lambda\frac{d}{dt}\right) = \left(\varphi X - \lambda\xi, \eta(X)\frac{d}{dt}\right) \tag{2}$$

is integrable, where X is tangent to M, t the coordinate of $\mathbb{R}$ and λ a smooth function on $M \times \mathbb{R}$. The condition for being normal is equivalent to vanishing of the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ. Let g be a compatible Riemannian metric with (φ, ξ, η), that is,

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y) \tag{3}$$

or equivalently,

$$g(X, \varphi Y) = -g(\varphi X, Y) \quad \text{and} \quad g(X, \xi) = \eta(X) \tag{4}$$

for all $X, Y \in TM$. Then, M becomes an *almost contact metric manifold* equipped with an almost contact metric structure (φ, ξ, η, g).

An almost contact metric structure becomes a *contact metric structure* if

$$g(X, \varphi Y) = d\eta(X, Y), \quad X, Y \in TM. \tag{5}$$

In a contact metric manifold, the $(1,1)$-tensor field h is symmetric and satisfies

(6) $h\xi = 0,\ h\varphi + \varphi h = 0,\ \nabla\xi = -\varphi - \varphi h,\ \mathrm{trace}(h) = \mathrm{trace}(\varphi h) = 0,$

where ∇ is the Levi-Civita connection.

A normal contact metric manifold is a *Sasakian manifold.* An almost contact metric manifold is Sasakian if and only if

$$\nabla_X \varphi = R_0(\xi, X), \quad X \in TM, \tag{7}$$

while a contact metric manifold M is Sasakian if and only if

(8) $$R(X,Y)\xi = R_0\,(X,Y)\,\xi, \quad X,Y \in TM.$$

A contact metric manifold is called a *K-contact manifold* if ξ is a Killing vector field. An almost contact metric manifold is K-contact if and only if $\nabla\xi = -\varphi$. A K-contact manifold is a contact metric manifold, while converse is true if $h = 0$. A Sasakian manifold is always a K-contact manifold. A 3-dimensional K-contact manifold is a Sasakian manifold. Thus, a 3-dimensional contact metric manifold is Sasakian if and only if $h = 0$.

The (κ,μ)-*nullity distribution* $N(\kappa,\mu)$ ([4, 12]) of a contact metric manifold M is defined by

$$\begin{aligned} N(\kappa,\mu) : p \to{} & N_p(\kappa,\mu) \\ & = \{U \in T_pM \mid R(X,Y)U = (\kappa I + \mu h)R_0(X,Y)U\} \end{aligned}$$

for all $X, Y \in TM$, where $(\kappa,\mu) \in \mathbb{R}^2$. A contact metric manifold with $\xi \in N(\kappa,\mu)$ is called a (κ,μ)-*manifold*. For a (κ,μ)-manifold, it follows that $h^2 = (\kappa - 1)\,\varphi^2$. This class contains Sasakian manifolds for $\kappa = 1$ and $h = 0$. In fact, for a (κ,μ)-manifold, the conditions of being Sasakian manifold, K-contact manifold, $\kappa = 1$ and $h = 0$ are all equivalent. If $\mu = 0$, the (κ,μ)-nullity distribution $N(\kappa,\mu)$ is reduced to the κ-*nullity distribution* $N(\kappa)$ ([15]). If $\xi \in N(\kappa)$, then we call a contact metric manifold M an $N(\kappa)$-*contact metric manifold*. For more details we refer to [3].

3. (κ,μ)-manifolds with vanishing E-Bochner curvature tensor

In [10], Matsumoto and Chūman defined the *C-Bochner curvature tensor* in an almost contact metric manifold as follows:

(9)
$$\begin{aligned} B\,(X,Y) ={}& R\,(X,Y) - \frac{m-4}{2n+4} R_0\,(Y,X) + \frac{1}{2n+4}\Big\{ R_0\,(QY,X) \\ & - R_0\,(QX,Y) + R_0\,(Q\varphi Y,\varphi X) - R_0\,(Q\varphi X,\varphi Y) \\ & + 2g\,(Q\varphi X,Y)\,\varphi + 2g\,(\varphi X,Y)\,Q\varphi + \eta\,(Y)\,R_0\,(QX,\xi) \\ & + \eta\,(X)\,R_0\,(\xi,QY)\Big\} - \frac{m+2n}{2n+4}\{R_0\,(\varphi Y,\varphi X) \\ & + 2g\,(\varphi X,Y)\,\varphi\} + \frac{m}{2n+4}\{\eta\,(Y)\,R_0\,(\xi,X) + \eta\,(X)\,R_0\,(Y,\xi)\} \end{aligned}$$

where Q is the Ricci operator, r is the scalar curvature and $m = \frac{2n+r}{2n+2}$.

For a (κ, μ)-manifold M^{2n+1}, we have

$$R(X,Y)\xi = (\kappa I + \mu h) R_0(X,Y)\xi, \tag{10}$$

which is equivalent to

$$R(\xi, X) = R_0(\xi, (\kappa I + \mu h) X) = -R(X,\xi). \tag{11}$$

In particular, we get

$$R(\xi, X)\xi = \kappa(\eta(X)\xi - X) - \mu hX = -R(X,\xi)\xi. \tag{12}$$

From (9), (10), and (11), it follows that

$$B(X,Y)\xi = \left(\frac{2(\kappa-1)}{n+2} I + \mu h\right) R_0(X,Y)\xi, \tag{13}$$

$$B(\xi, X) = R_0\left(\xi, \left(\frac{2(\kappa-1)}{n+2} I + \mu h\right) X\right) = -B(X,\xi). \tag{14}$$

Consequently, we have

$$B(\xi,X)\xi = \frac{2(\kappa-1)}{n+2}(\eta(X)\xi - X) - \mu hX = -B(X,\xi)\xi, \tag{15}$$

$$\eta(B(X,Y)\xi) = 0, \tag{16}$$

$$\eta(B(\xi,X)Y) = \frac{2(\kappa-1)}{n+2}(g(X,Y) - \eta(X)\eta(Y)) + \mu g(hX,Y). \tag{17}$$

In [7], H. Endo extended the concept of C-Bochner curvature tensor to E-Bochner curvature tensor as follows:

$$\begin{aligned} B^e(X,Y)Z = {} & B(X,Y)Z - \eta(X)B(\xi,Y)Z \\ & - \eta(Y)B(X,\xi)Z - \eta(Z)B(X,Y)\xi. \end{aligned} \tag{18}$$

Then, he showed that a K-contact manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold. Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Let M^{2n+1} be a (κ, μ)-manifold. If E-Bochner curvature tensor of M^{2n+1} vanishes, then from (15) and (18), we have

$$0 = B^e(X,\xi)\xi = -B(X,\xi)\xi = \frac{2(\kappa-1)}{n+2}(\eta(X)\xi - X) - \mu hX, \tag{19}$$

which, in view of $h^2 = (\kappa - 1)\varphi^2$, implies that

$$h^2 = \frac{(n+2)\mu}{2} h. \tag{20}$$

Taking the trace of (20), we obtain

$$0 = \operatorname{trace}\left(h^2\right) = 2n\,(1-\kappa)\,. \tag{21}$$

From (21), we have $\kappa = 1$. Thus, M^{2n+1} becomes Sasakian.

COROLLARY 3.1. *An $N(\kappa)$-contact metric manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold.*

The unit tangent sphere bundle T_1M equipped with the standard contact metric structure is a (κ,μ)-manifold if and only if the base manifold M is of constant curvature c with $\kappa = c(2-c)$ and $\mu = -2c$ ([4]). In case of $c \neq 1$, the unit tangent sphere bundle is non-Sasakian [16]. We denote by $T_1M(c)$ the unit tangent sphere bundle of a space of constant curvature c with standard contact metric structure. Then, applying Theorem 1.1 to $T_1M(c)$, we have

COROLLARY 3.2. *If $T_1M(c)$ is of vanishing E-Bochner curvature tensor, then $c = 1$.*

In an almost contact metric manifold, if X is a unit vector which is orthogonal to ξ, we say that X and φX span a φ-section. If the sectional curvature $c\,(X)$ of all φ-sections is independent of X, we say that M is of pointwise constant φ-sectional curvature. If an $N(\kappa)$-contact metric manifold M is of pointwise constant φ-sectional curvature c, then we say it an *$N(\kappa)$-contact space form* $M(c)$. The curvature tensor of $M(c)$ is given by [8].

$$\begin{aligned}
4R(X,Y)Z = {} & (c+3)\,\{g\,(Y,Z)\,X - g\,(X,Z)\,Y\} \\
& + (c-1)\,\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
& + \eta(Y)g\,(X,Z)\,\xi - \eta(X)g\,(Y,Z)\,\xi \\
& + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2(\varphi X,Y)\varphi Z\} \\
& + 4\,(\kappa-1)\,\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\
& + \eta(X)g\,(Y,Z)\,\xi - \eta(Y)g\,(X,Z)\,\xi\} \\
& + 4\,\{g\,(hY,Z)\,X - g\,(hX,Z)\,Y \\
& + g\,(Y,Z)\,hX - g\,(X,Z)\,hY \\
& + \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX \\
& + \eta(Y)g\,(hX,Z)\,\xi - \eta(X)g\,(hY,Z)\,\xi\} \\
& + 2\,\{g\,(hY,Z)\,hX - g\,(hX,Z)\,hY \\
& + g\,(\varphi hX,Z)\,\varphi hY - g\,(\varphi hY,Z)\,\varphi hX\}
\end{aligned} \tag{22}$$

for all $X,Y,Z \in TM$, where c is a constant on M if $\dim(M) > 3$.

Now, applying Theorem 1.1 to an $N(\kappa)$-contact space form, we are able to state the following

COROLLARY 3.3. *An $N(\kappa)$-contact space form with vanishing E-Bochner curvature tensor is a Sasakian space form.*

Let M be a $(2n+1)$-dimensional (κ,μ)-manifold $(n>1)$. If M has constant φ-sectional curvature c then it is called a (κ,μ)*-space form* and is denoted by $M(c)$. The curvature tensor of $M(c)$ is given by [9]

$$\begin{aligned}
&R(X,Y)Z\\
&= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\}\\
&+ \frac{c-1}{4}\{2(X,\varphi Y)\varphi Z + g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X\}\\
&+ \frac{c+3-4\kappa}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\\
&+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\\
&+ \frac{1}{2}\{g(hY,Z)hX - g(hX,Z)hY\\
&+ g(\varphi hX,Z)\varphi hY - g(\varphi hY,Z)\varphi hX\}\\
&+ g(\varphi Y,\varphi Z)hX - g(\varphi X,\varphi Z)hY\\
&+ g(hX,Z)\varphi^2 Y - g(hY,Z)\varphi^2 X\\
&+ \mu\{\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY\\
&+ g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi\}
\end{aligned} \tag{23}$$

for all $X, Y, Z \in TM$, where $c+2\kappa = -1 = \kappa - \mu$ if $\kappa < 1$.

Now, applying Theorem 1.1 to a (κ,μ)-contact space form, we are able to state the following

COROLLARY 3.4. *A (κ,μ)-contact space form with vanishing E-Bochner curvature tensor is a Sasakian space form.*

REMARK 3.5. Theorem 1.1, Corollary 3.1, Corollary 3.2, Corollary 3.3 and Corollary 3.4 are valid for vanishing of C-Bochner curvature tensor also.

4. (κ,μ)-manifolds satisfying $B(\xi,X)\cdot S = 0$

Before proving Theorem 1.2, we give some results and a brief introduction to η-Einstein (κ,μ)-manifold.

LEMMA 4.1. [4] *In a non-Sasakian (κ, μ)-manifold M^{2n+1}, the Ricci operator Q is given by*

$$Q=(2(n-1)-n\mu)I+(2(n-1)+\mu)h+(2(1-n)+n(2\kappa+\mu))\eta\otimes\xi. \tag{24}$$

Consequently, the Ricci tensor S is given by

$$\begin{aligned} S(X,Y) = {} & (2(n-1)-n\mu)\, g(X,Y) + (2(n-1)+\mu)\, g(hX,Y) \\ & + (2(1-n)+n(2\kappa+\mu))\,\eta(X)\,\eta(Y). \end{aligned} \tag{25}$$

We also recall the following theorem due to D. Blair.

THEOREM 4.2. [2, 3] *A contact metric manifold M^{2n+1} satisfying $R(X,Y)\xi = 0$ is locally isometric to $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

THEOREM 4.3. [15] *If an $N(k)$-contact metric manifold of dimension ≥ 5 is Einstein, then it is necessarily Sasakian.*

We also need the following definition.

DEFINITION 4.4. [11] A contact metric manifold M is said to be *η-Einstein* if the Ricci operator Q satisfies

$$Q = aI + b\eta \otimes \xi,$$

where a and b are smooth functions on the manifold. In particular, if $b = 0$, then M is an *Einstein manifold*.

Now, we prove Theorem 1.2.

Proof of Theorem 1.2. Let M^{2n+1} be a non-Sasakian (κ, μ)-manifold. If $\kappa = 0 = \mu$, then from (10), we have $R(X,Y)\xi = 0$, which in view of Theorem 4.2 implies that M^{2n+1} satisfies one of the statements (i) and (ii). Now, we assume that κ and μ are not simultaneously zero. From (25), we have

$$\begin{aligned} S(hX,Y) = {} & (2(n-1)-n\mu)\, g(hX,Y) \\ & -(\kappa-1)(2(n-1)+\mu)\, g(X,Y) \\ & +(\kappa-1)(2(n-1)+\mu)\,\eta(X)\,\eta(Y), \end{aligned} \tag{26}$$

where we have used $\eta \circ h = 0$, $h^2 = (\kappa - 1)\varphi^2$ and (3). The condition $B(\xi, X) \cdot S = 0$ gives

$$S(B(\xi,X)Y,\xi) + S(Y, B(\xi,X)\xi) = 0. \tag{27}$$

In view of $Q\xi = 2n\kappa\xi$, we get

$$S(X,\xi) = 2n\kappa\eta(X), \tag{28}$$

which implies that

$$S(B(\xi,X)Y,\xi) = 2n\kappa\eta(B(\xi,X)Y), \tag{29}$$

Using (17) in the above equation, we get

$$\begin{aligned} S(B(\xi,X)Y,\xi) &= 2n\kappa\mu g(hX,Y) \\ &\quad + \frac{4n\kappa(\kappa-1)}{n+2}(g(X,Y)-\eta(X)\eta(Y)). \end{aligned} \tag{30}$$

In view of (15) and (28), we have

$$\begin{aligned} S(B(\xi,X)\xi,Y) &= \frac{4n\kappa(\kappa-1)}{n+2}\eta(X)\eta(Y) \\ &\quad - \frac{2(\kappa-1)}{n+2}S(X,Y) - \mu S(hX,Y). \end{aligned} \tag{31}$$

From (27), (30), and (31), we have

$$\begin{aligned} \frac{2(\kappa-1)}{n+2}S(X,Y) &= \frac{4n\kappa(\kappa-1)}{n+2}g(X,Y) \\ &\quad + 2n\mu g(hX,Y) - \mu S(hX,Y). \end{aligned} \tag{32}$$

Finally, from (25), (26), and (32), we have

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{33}$$

where

$$a = \frac{\dfrac{(\kappa-1)(4n\kappa+(n+2)(2(n-1)+\mu)\mu)}{n+2} - \dfrac{\mu(2+n\mu)(2(n-1)-n\mu)}{2(n-1)+\mu}}{\dfrac{2(\kappa-1)}{n+2} - \dfrac{\mu(2+n\mu)}{2(n-1)+\mu}},$$

and

$$b = -\frac{\dfrac{\mu((\kappa-1)(2(n-1)+\mu)^2+(2+n\mu)(n(2\kappa+\mu)-2(n-1)))}{2(n-1)+\mu}}{\dfrac{2(\kappa-1)}{n+2} - \dfrac{\mu(2+n\mu)}{2(n-1)+\mu}} \tag{34}$$

Thus, if $\mu \neq 0$, then M^{2n+1} is an η-Einstein manifold, which is the statement (iii). If $\mu = 0$ and $\kappa \neq 0$, then in view of (34), we get $b = 0$ in (33); thus M^{2n+1} becomes an Einstein manifold. Moreover, if $n > 1$, in view of Theorem 4.3, we conclude that M^{2n+1} is Sasakian, which is a contradiction. Therefore we have the statement (iv). This completes the proof.

In view of Theorem 1.2, we are able to state the following two corollaries.

COROLLARY 4.5. *Let M^{2n+1} be a non-Sasakian $N(\kappa)$-contact metric manifold of dimension ≥ 5. If the C-Bochner curvature tensor B satisfies $B(\xi, X) \cdot S = 0$, then M^{2n+1} is locally isometric to $E^{n+1}(0) \times S^n(4)$.*

COROLLARY 4.6. *Let M^{2n+1} be a non-Sasakian $N(\kappa)$-contact metric manifold with $n > 1$ and $\kappa \neq 0$. Then, the C-Bochner curvature tensor B never satisfies $B(\xi, X) \cdot S = 0$.*

5. $N(\kappa)$-contact metric manifolds satisfying $B(\xi, X) \cdot R = 0$

In an $N(\kappa)$-contact metric manifold M^{2n+1}, we have

$$R(X,Y)\xi = \kappa R_0(X,Y)\xi, \tag{35}$$

$$R(\xi, X) = \kappa R_0(\xi, X) = -R(X, \xi). \tag{36}$$

Consequently,

$$B(X,Y)\xi = \frac{2(\kappa - 1)}{n+2} R_0(X,Y)\xi, \tag{37}$$

$$B(\xi, X) = \frac{2(\kappa - 1)}{n+2} R_0(\xi, X) = -B(X, \xi). \tag{38}$$

Proof of Theorem 1.3. The condition $B(\xi, X) \cdot R = 0$ gives

$$0 = [B(\xi, X), R(Y, Z)]\xi - R(B(\xi, X)Y, Z)\xi - R(Y, B(\xi, X)Z)\xi,$$

which in view of (38) provides

$$\begin{aligned} 0 = \frac{2(\kappa - 1)}{n+2} \{ & g(X, R(Y,Z)\xi)\xi - \eta(R(Y,Z)\xi)X - g(X,Y)R(\xi, Z)\xi \\ & + \eta(Y)R(X,Z)\xi - g(X,Z)R(Y,\xi)\xi + \eta(Z)R(Y,X)\xi \\ & - \eta(X)R(Y,Z)\xi + R(Y,Z)X \} \end{aligned}$$

Using (36), we get

$$\frac{2(\kappa - 1)}{n+2} (R(Y,Z) - \kappa R_0(Y,Z)) = 0.$$

Therefore, either $\kappa = 1$ or

$$R(Y,Z)X = \kappa(g(X,Z)Y - g(X,Y)Z). \tag{39}$$

It is well known that, except for the flat 3-dimensional case, a contact metric manifold of constant curvature is Sasakian and of constant curvature $+1$. Therefore, in view of (39), M is locally isometric to the sphere $S^{2n+1}(1)$. Thus, the proof is complete.

REMARK 5.1. If the $N(\kappa)$-contact metric manifold is assumed to be complete and simply connected, then in the Theorem 1.3, local isometry is replaced by global isometry.

6. $N(\kappa)$-contact metric manifolds satisfying $B(\xi, X) \cdot B = 0$

This section is devoted to the proof of Theorem 1.4.

Proof of Theorem 1.4. The condition $B(\xi, X) \cdot B = 0$ gives

$$0 = [B(\xi, X), B(Y, Z)]\,\xi - B(B(\xi, X)Y, Z)\,\xi - B(Y, B(\xi, X)Z)\,\xi,$$

which in view of (38) provides

$$\begin{aligned} 0 = \frac{2(\kappa-1)}{n+2} \{ & g(X, B(Y, Z)\xi)\xi - \eta(B(Y, Z)\xi)X - g(X, Y)B(\xi, Z)\xi \\ & + \eta(Y)B(X, Z)\xi - g(X, Z)B(Y, \xi)\xi + \eta(Z)B(Y, X)\xi \\ & - \eta(X)B(Y, Z)\xi + B(Y, Z)X\}. \end{aligned}$$

Using (37), we get

$$\frac{2(\kappa-1)}{n+2}\left(R\Big(Y, Z\Big) - \frac{2(\kappa-1)}{(n+2)}R_0(Y, Z)\right) = 0.$$

Therefore, either $\kappa = 1$ or

$$B(Y, Z)X = \frac{2(\kappa-1)}{(n+2)}\left(g(X, Z)Y - g(X, Y)Z\right).$$

Contracting Y in the above equation, we conclude that

$$0 = \frac{2(\kappa-1)}{(n+2)}\left(2ng(X, Z)\right),$$

which gives $\kappa = 1$. Thus in the both cases M^{2n+1} is a Sasakian manifold.

Conversely, if M^{2n+1} is a Sasakian manifold, then in view of (38) we have $B(\xi, X) \cdot B = 0$.

References

[1] D. E. Blair, *On the geometric meaning of the Bochner tensor*, Geom. Dedicata **4** (1975), 33–38.

[2] ———, *Two remarks on contact metric structures*, Tôhoku Math. J. **29** (1977), 319–324.

[3] ———, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, 203. Birkhauser Boston, Inc., Boston, MA, 2002.

[4] D. E. Blair, T. Koufogiorgos, and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. **91** (1995), no. 1–3, 189–214.

[5] S. Bochner, *Curvature and Betti numbers*, Ann. of Math. **50** (1949), no. 2, 77–93.

[6] W. M. Boothby and H. C. Wang, *On contact manifolds*, Ann. of Math. **68** (1958), 721–734.

[7] H. Endo, *On K-contact Riemannian manifolds with vanishing E-contact Bochner curvature tensor*, Colloq. Math. **62** (1991), no. 2, 293–297.

[8] ———, *On the curvature tensor fields of a type of contact metric manifolds and of its certain submanifolds*, Publ. Math. Debrecen **48** (1996), no. 3–4, 253–269.

[9] T. Koufogiorgos, *Contact Riemannian manifolds with constant φ-sectional curvature*, Tokyo J. Math. **20** (1997), no. 1, 13–22.

[10] M. Matsumoto and G. Chūman, *On the C-Bochner curvature tensor*, TRU Math. **5** (1969), 21–30.

[11] M. Okumura, *Some remarks on space with a certain contact structure*, Tôhoku Math. J. **14** (1962), 135–145.

[12] B. J. Papantoniou, *Contact Riemannian manifolds satisfying $R(\xi, X) \cdot R = 0$ and $\xi \in (k, \mu)$-nullity distribution*, Yokohama Math. J. **40** (1993), no. 2, 149–161.

[13] G. Pathak, U. C. De, and Y.-H. Kim, *Contact manifolds with C-Bochner curvature tensor*, Bull. Calcutta Math. Soc. **96** (2004), no. 1, 45–50.

[14] D. Perrone, *Contact Riemannian manifolds satisfying $R(X, \xi) \cdot R = 0$*, Yokohama Math. J. **39** (1992), no. 2, 141–149.

[15] S. Tanno, *Ricci curvatures of contact Riemannian manifolds*, Tôhoku Math. J. **40** (1988), 441–448.

[16] Y. Tashiro, *On contact structures of tangent sphere bundles*, Tôhoku Math. J. **21** (1969), 117–143.

Jeong-Sik Kim, Department of Mathematics and Mathematical Information Yosu National University, Yosu 550-749, Korea
E-mail: jskim315@hotmail.com

Mukut Mani Tripathi, Department of Mathematics and Astronomy, Lucknow University, India
E-mail: mmt66@satyam.net.in

Jaedong Choi, Department of Mathematics, Korea Air Force Academy, Cheongwon 363-849, Korea
E-mail: jdong@afa.ac.kr