ON CAPILLARY FREE SURFACES IN THE ABSENCE OF GRAVITY

BY

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We study in this and in a related paper [5] the equation

$$\operatorname{div}\left(\frac{1}{W}\nabla u\right) \equiv \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{1}{W}\frac{\partial u}{\partial x_i}\right) = n\mathcal{H}(\mathbf{x}; u), \qquad W = (1+|\nabla u|^2)^{\frac{1}{2}}, \tag{1}$$

for a scalar function $u(\mathbf{x})$ over a region Ω in n-dimensional Euclidean space, $n \geqslant 2$. We assume the boundary Σ of Ω to satisfy smoothness hypotheses, which vary with the context. For some purposes it suffices that $\Sigma \in C^{(1)}$ in local parameters. However, at times, we shall have to refer, at least locally, to a mean curvature at points on Σ. Although our results could be stated in terms of generalized boundary curvatures in such cases, it is preferable for the purpose of these papers to assume that $\Sigma \in C^{(2)}$ at these points. Finally, certain results deal with boundaries on which regularity fails at a set of points, small enough to permit a restricted form of the divergence theorem to hold. The type of singular set that is admissible in this sense will be clarified later.

Our principal concern is the case of (1) of special physical interest,

$$\operatorname{div}\left(\frac{1}{W}\nabla u\right) = \varkappa u + nH(\mathbf{x}), \tag{2}$$

where $\varkappa$ is a constant. If $\varkappa = 0$ and $H \equiv$ const., then $u(\mathbf{x})$ defines a (non-parametric) surface of constant mean curvature H. If in addition $H = 0$, then (2) becomes the minimal surface equation. If $\varkappa \neq 0$ and $H \equiv$ const., then (2) becomes the equation for a surface whose mean curvature is proportional to its distance from a fixed reference plane. In this case H can be eliminated by adding a constant to $u(\mathbf{x})$.

Both cases are encountered physically as the equation for the height of a capillary surface in a cylindrical domain with base Ω and bounding walls Z generated by rays

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through Σ and parallel to the u axis. The case $\varkappa=0$ occurs in the absence of gravitational forces; if $\varkappa>0$, a gravitational field is directed parallel to the cylinder and towards the base, while if $\varkappa<0$ a gravity force points away from the base, as in an upside down capillary tube. One finds $\varkappa=((\varrho-\varrho_0)g)/\sigma$, where σ is the surface tension of the liquid in the tube, ϱ and ϱ_0 are the densities of the liquid and of the gas outside the liquid, and g is the gravitational acceleration (see, e.g., [1]). The solution surface $\mathcal{S}$ is to be determined by the equation and by the (physical) boundary condition that the angle γ (measured within the fluid) between $\mathcal{S}$ and Z is prescribed on the manifold $\mathcal{C}$ of contact. If $\boldsymbol{\nu}$ denotes the outer-directed unit normal to Σ, then this condition is

$$\mathrm{T}u\cdot\boldsymbol{\nu}\equiv W^{-1}\nabla u\cdot\boldsymbol{\nu}=\cos\gamma \quad \text{on } \Sigma. \tag{3}$$

For functions $u(\mathbf{x})$ in Ω, which need not be defined up to Σ, (3) is to be understood in the following sense: *The vector* $\boldsymbol{\nu}$ *is extended continuously into* Ω, *and* $\mathrm{T}u\cdot\boldsymbol{\nu}$ *is required to exist almost everywhere as a limit, as* Σ *is approached from points of* Ω. No further hypothesis about boundary behavior need be made, and we shall interpret (3) in this way throughout the text.

In practice, γ is determined experimentally and depends on the materials in the three-phase interface at $\mathcal{C}$. The physical situation of constant γ is of central interest in our work, although we are able often to discuss without essential change the more general case in which γ is allowed to vary along $\mathcal{C}$.

There are important differences of behavior between the cases $\varkappa=0$ and $\varkappa\neq0$, and different techniques are required to study them. For this reason we have divided our work into two parts. The first part discusses (1) for the case in which $\mathcal{H}$ is independent of u (corresponding to $\varkappa=0$ in (2)) and is covered in this paper. The second part, covered in a subsequent paper [5], corresponds to $\varkappa\neq0$ in (2); that is, $\mathcal{H}$ depends on u explicitly, as is the case for a capillary surface in a gravitational field.

Part of the work reported here and in [5] was done while the former author was at the University of Reading under a Science Research Council grant and at the National Physical Laboratory in Teddington, during 1970–71. Part was done while the latter author was at the University of Sussex in the Spring of 1970. Some of the results contained here and in [5] were announced in [3].

§ 1

In this section we give some geometric results that are of use in subsequent sections, but are of general interest in themselves. Consider an n dimensional domain Ω with boundary surface Σ that has mean curvature H^{Σ}. (We choose the sign of H^{Σ} so that $H^{\Sigma}>0$ when the curvature vector is directed along the interior normal.) Let $\mathcal{V}$ denote the volume of Ω, $\mathcal{A}$ the area of Σ, and $\boldsymbol{\nu}$ the outer-directed unit normal to Σ. Let r denote the distance from the origin to a general point of Σ.

We consider first the case in which Ω is star-shaped with respect to the origin. Then H^Σ can be extended to all of Ω as a function constant along rays from the origin, equal on each ray to the value at the intersection point with Σ. We denote by $\bar{H}^\Sigma$ the volume average of the extended H^Σ,

$$\bar{H}^\Sigma = \frac{1}{\mathcal{V}} \int_\Omega H^\Sigma(\mathrm{x})\, d\mathrm{x}.$$

LEMMA 1. *There holds*

$$n\bar{H}^\Sigma = \frac{\mathcal{A}}{\mathcal{V}}. \tag{4}$$

Proof: In a spherical coordinate system (r, ω), we may describe Σ by an equation of the form $r = f(\omega)$. We consider the function $F \equiv r/f(\omega)$, so that Σ is described, as well, by $F = 1$. We then have

$$H^\Sigma = \frac{1}{n-1} \operatorname{div} \frac{\nabla F}{|\nabla F|}\bigg|_{F=1} \tag{5}$$

and

$$\int_\Omega \operatorname{div} \frac{\nabla F}{|\nabla F|}\, d\mathrm{x} = \oint_\Sigma \frac{\nabla F}{|\nabla F|} \cdot \boldsymbol{\nu}\, d\sigma = \mathcal{A} \tag{6}$$

since ∇F is orthogonal to Σ on that surface and coincides in direction with $\boldsymbol{\nu}$.

If $0 < r < f(\omega)$, then

$$\frac{1}{n-1} \operatorname{div} \frac{\nabla F}{|\nabla F|} = \frac{f}{r} H^\Sigma. \tag{7}$$

This last result is evident geometrically, since the left side of (7) is the mean curvature of the similar surface obtained by contracting Σ with respect to the origin in the ratio r/f.

Placing (7) into (6) and integrating with respect to r yields

$$\oint H^\Sigma r^n d\omega = \mathcal{A}. \tag{8}$$

The quantity $r^n d\omega$ is, however, n times the volume element subtended at the origin by the solid angle $d\omega$. Thus, dividing (8) by $\mathcal{V}$, one obtains the desired result (4)(1).

From Lemma 1, we obtain the result,

(1) This result could have been obtained alternatively from a general integral formula, due originally to Minkowski [12], and given recently in a general formulation by Hsiung [10]. The proof we present seems particularly simple and adapted to the case considered here.

LEMMA 2. *If Ω is star shaped and if $\bar{H}^{\Sigma}$ equals or exceeds the mean curvature of a ball B^n of radius R, then there holds*

$$\frac{\mathcal{A}}{\mathcal{V}} \geqslant \frac{n}{R},$$

The result of Lemma 2 can also be obtained under a different hypothesis, which does not require that Ω be star shaped.

LEMMA 3. *If Ω lies interior to a ball B^n_R of radius R, then*

$$\frac{\mathcal{A}}{\mathcal{V}} \geqslant \frac{n}{R},$$

equality holding if and only if Ω coincides with B^n_R.

Lemmas 1 and 3 together yield:

COROLLARY: *If Ω is star shaped and lies interior to B^n_R, then*

$$\bar{H}^{\Sigma} \geqslant \frac{1}{R}.$$

Proof of Lemma 3. The isoperimetric inequality implies the existence of a ball $B^n_{R_0}$ of volume equal to that of Ω, such that $B^n_{R_0} \subset B^n_R$ and for which

$$\left.\frac{\mathcal{A}^n}{\mathcal{V}^{n-1}}\right]_{\Omega} \geqslant \left.\frac{\mathcal{A}^n}{\mathcal{V}^{n-1}}\right]_{B^n_{R_0}} = \gamma_n.$$

Thus,
$$\left.\frac{\mathcal{A}}{\mathcal{V}}\right]_{\Omega} \geqslant \left.\frac{\mathcal{A}}{\mathcal{V}}\right]_{B^n_{R_0}} = \left(\frac{\gamma_n}{\mathcal{V}}\right)^{1/n} = \frac{n}{R_0} \geqslant \frac{n}{R}.$$

Clearly, equality can hold only if Ω coincides with B^n_R.

We now consider any twice differentiable surface $u(\mathbf{x})$ defined in Ω and having continuous first derivatives up to Σ. Let $H(\mathbf{x})$ be its mean curvature, $H(\mathbf{x}) = (1/n)\operatorname{div}((1/W)\nabla u)$, $W = (1 + |\nabla u|^2)^{\frac{1}{2}}$. Let $\gamma(\mathbf{x})$ be the angle between the surface $u(\mathbf{x})$ and the (hyper-)cylinder with base Σ and generators parallel to the u-axis; then, denoting the surface average over Σ by $\hat{}$ and, as before, the volume average over Ω by $\bar{}$, we obtain for star-shaped Ω,

THEOREM 1. *There holds*

$$\bar{H}(\mathbf{x}) = \widehat{\cos\gamma\,\bar{H}^{\Sigma}}(\mathbf{x}). \tag{10}$$

Proof. From the above expression for $H(\mathbf{x})$, we have

$$n\int_{\Omega} H(\mathbf{x})\,d\mathbf{x} = \oint_{\Sigma} \frac{\nabla u}{W}\cdot \boldsymbol{\nu}\,d\sigma = \oint_{\Sigma} \cos\gamma\,d\sigma,$$

which, using (4), yields the result.

§ 2

We consider now solutions of

$$\operatorname{div}\left(\frac{1}{W}\nabla u\right) = nH(\mathbf{x}) \tag{11}$$

in Ω. Denote by $\{\Omega^j\}$ a sequence of domains exhausting Ω, whose boundaries Σ^j have uniformly bounded surface area, $\mathcal{A}^j = \mathcal{A}[\Sigma^j] < \mathcal{A} < \infty$. We find immediately the result:

LEMMA 4. *A necessary condition for the existence of a solution of* (11) *in* Ω *is that* $n\left|\int_{\Omega^j} H(\mathbf{x})d\mathbf{x}\right| < \mathcal{A}$ *for all* j.

The assertion follows from the relation

$$\mathfrak{I}_j = n\int_{\Omega^j} H(\mathbf{x})\,dx = \oint_{\Sigma^j} \mathbf{T}u\cdot\boldsymbol{\nu}\,d\sigma,$$

since $|\mathbf{T}u| < 1$ on Σ^j.

In particular, $\int_{\Omega} H(\mathbf{x})d\mathbf{x}$ can be defined for any solution as the limit of a suitable subsequence of the $\{\mathfrak{I}_j\}$. We remark, however, that $\lim \mathfrak{I}_j$ need not exist for every choice $\{\Omega^j\}$, as can be seen from simple examples.

LEMMA 5. *Suppose* $u(\mathbf{x})$ *satisfies* (11) *in* Ω, *with* $\mathbf{T}u\cdot\boldsymbol{\nu} = \cos\gamma$ *on* Σ. *Then there exists*

$$n\int_{\Omega} H(\mathbf{x})\,d\mathbf{x} = \lim_{j\to\infty} n\int_{\Omega^j} H(\mathbf{x})\,d\mathbf{x} = \oint_{\Sigma} \cos\gamma\,d\sigma \tag{12}$$

for any sequence $\{\Omega^j\}$ *exhausting* Ω.

Even under these hypotheses, the integral need not exist absolutely.

We note that in the volume and surface average notation introduced in section 1, (12) can be written

$$n\overline{H}(\mathbf{x}) = \frac{\mathcal{A}}{\mathcal{V}}\widehat{\cos\gamma} \tag{13}$$

and for the frequently encountered physical situation in which γ and $H(\mathbf{x})$ are constants (13) reduces to

$$nH = \frac{\mathcal{A}}{\mathcal{V}}\cos\gamma. \tag{14}$$

From Lemma 5 we conclude that solutions of (11) satisfying the boundary condition (3) can exist only for those γ (if any) that satisfy (12). We remark also that solutions of (11) in star-shaped domains must, of course, satisfy (10).

Combination of the geometrical results of section 1 with (13) yields

THEOREM 2. *If Σ satisfies the hypotheses of Lemma 2 or of Lemma 3, then there holds*

$$|\bar{H}(\mathbf{x})| \geqslant \frac{1}{R} |\widehat{\cos \gamma}|. \tag{15}$$

For the case in which γ and $H(\mathbf{x})$ are constants, (15) reduces to

$$|H| \geqslant \frac{1}{R} |\cos \gamma|. \tag{16}$$

§ 3

Here and in what follows we shall use symbols interchangeably to denote a set and its area, or volume; thus, Σ denotes a bounding surface and also its area, Ω denotes an open set and also its volume.

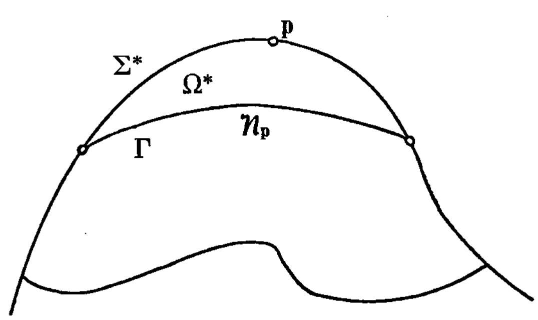


Fig. 1. One-sided neighborhood of p.

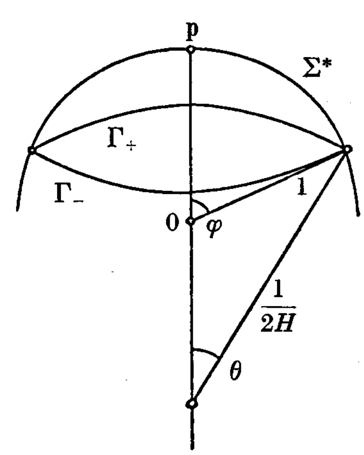


Fig. 2. Segment of unit disk.

We consider the local behavior of a solution $u(\mathbf{x})$ near a boundary point $\mathbf{p}$, and we suppose $u(\mathbf{x})$ to be defined in a one-sided n-dimensional neighborhood $\mathcal{N}_\mathbf{p}$ of $\mathbf{p}$, bounded by a piece $\Sigma_\mathbf{p} \in C^{(2)}$ of $(n-1)$-dimensional surface. Let Γ be an $(n-1)$-surface lying in $\mathcal{N}_\mathbf{p}$, and meeting $\Sigma_\mathbf{p}$ in an $(n-2)$-manifold that surrounds $\mathbf{p}$ on $\Sigma_\mathbf{p}$ (see Fig. 1). Let Ω^* be the part of $\mathcal{N}_\mathbf{p}$ bounded by Γ and $\Sigma_\mathbf{p}$, let Σ^* be the part of Σ contacting Ω^*. We integrate (11) over Ω^*, setting $\bar{H}^* = (1/\Omega^*) \int_{\Omega^*} H(\mathbf{x}) d\mathbf{x}$, $\widehat{\cos \gamma}^* = (1/\Sigma^*) \int_{\Sigma^*} \cos \gamma \, d\sigma$; noting $|Tu| < 1$ on Γ, we find:

LEMMA 6. *There holds*

$$\frac{n\bar{H}^*\Omega^*-\Gamma}{\Sigma^*} \leqslant \widehat{\cos\gamma^*} \leqslant \frac{n\bar{H}^*\Omega^*+\Gamma}{\Sigma^*}. \tag{17}$$

This simple result yields information of fairly precise character on the manner in which the curvature of the boundary near a point controls the permissible behavior of a solution at the point. We remark first that for prescribed intersection manifold of $\Sigma_{\mathbf{p}}$ with Γ, the best estimate that (17) will yield for $\cos\gamma^*$ will be obtained by minimizing the numerator on the right and maximizing that on the left. In the case $H(\mathbf{x}) \equiv \text{const.}$ in $\mathcal{N}_{\mathbf{p}}$, we see immediately that *an extremizing surface Γ passing through the intersection manifold, if it exists, will be a surface of constant mean curvature $H^{\Gamma}=(n/(n-1))H$ for the upper bound and $H^{\Gamma}=-(n/(n-1))H$ for the lower bound of $\widehat{\cos\gamma^*}$ in* (17)(1). This remark governs the considerations that follow.

3.1. As an example to illustrate the use of (17) for $n=2$, consider a solution $u(\mathbf{x})$ of (11) that is defined in a segment of a unit disk symmetric with respect to a boundary point $\mathbf{p}$, has constant mean curvature $0<H<\frac{1}{2}$, and constant boundary data $\cos\gamma$. Following the above remark, we choose for the upper bound the curve $\Gamma=\Gamma_+$, which has constant curvature $2H<1$, and for the lower bound the curve $\Gamma=\Gamma_-$ with curvature $-2H$ (see Fig. 2). We find

$$H-P_+(\varphi) \leqslant \cos\gamma \leqslant H+P_-(\varphi), \tag{18}$$

where
$$P_{\pm}(\varphi)=\frac{\theta+2H\sin(\varphi\pm\theta)}{4H\varphi}.$$

We have $\varphi^2P'_{\pm}]_{\varphi=0}=0$, also

$$(\varphi^2P'_{\pm})'=\frac{\varphi}{4H}\{[1\pm 2H\cos(\varphi\pm\theta)]\theta''-2H(1\pm\theta')^2\sin(\varphi\pm\theta)\}.$$

One computes
$$|\theta'(\varphi)|=2H\left|\frac{\cos\varphi}{\sqrt{1-4H^2\sin^2\varphi}}\right|<2H,$$

and $\theta''(\varphi)=-(1-\theta'^2)\tan\theta<0$ for $0<\varphi<\pi$. Hence $(\varphi^2P'_{\pm})'<0$, which implies $P'_{\pm}(\varphi)<0$ for $0<\varphi<\pi$. Thus $P_{\pm}(\varphi)$ decreases monotonically from $P_{\pm}(0)=1\pm H$ to $P_{\pm}(\pi)=0$. There follows from (18) a successively stronger (non-trivial) estimate for $\cos\gamma$ as the segment in which the solution is defined increases in size, until finally, for a solution defined in the entire disk, we obtain $H\leqslant\cos\gamma\leqslant H$. In fact, since in this case Γ degenerates to a point,

(1) The sign H is positive when the curvature vector is directed along the exterior normal to Ω^*.

we are led again to the consequence of (14), that for a solution of (11) defined in the entire disk, $\cos\gamma = H$.

If $H \geqslant \frac{1}{2}$ the method continues to yield a non-trivial lower bound for $\cos\gamma$. We obtain in this situation only a single stationary curve, with $H^{\Gamma} = -2H$, which is the only solution (interior to the disk) of the variational problem for this case. With increasing size of the segment Ω^* the bound becomes at first stronger; however, for sufficiently large Ω^* the bound weakens and eventually provides no information. The curve with $H^{\Gamma} = +2H$ can be used to provide a lower bound on $\cos\gamma$ for a solution defined in an exterior neighborhood of a boundary arc on the unit disk, that is, for a solution defined interior to a neighborhood of a boundary segment of a boundary Σ along which $H^{\Sigma} = -1$. This bound also becomes at first stronger, but eventually weakens and provides no information as the size of Ω^* increases.

The results of this section apply equally to the situation $H < 0$; this case reduces to that of positive H under the transformation $\tilde{u} = -u$, $\tilde{\gamma} = \pi - \gamma$, $\tilde{H} = -H$.

We note that the method yields information only in the case for which "extremal surfaces" Γ passing through the given boundary continuum (in this case two points) can be found. The same limitation applies to the following general considerations.

3.2. Consider again a one-sided neighborhood $\mathcal{N}_{\mathbf{p}}$ adjacent to $\Sigma_{\mathbf{p}} \subset \Sigma$.

THEOREM 3. *If either*

(i) $$H^{\Sigma} > \frac{n}{n-1} H_M \geqslant 0 \quad on\ \Sigma_{\mathbf{p}}$$

or

(ii) $$|H^{\Sigma}| < -\frac{n}{n-1} H_M \quad on\ \Sigma_{\mathbf{p}},$$

then

(a) *there exists* $\gamma_0 > 0$ *such that there is no solution* $u(\mathbf{x})$ *of* (11) *for which* $H(\mathbf{x}) \leqslant H_M$ *throughout* $\mathcal{N}_{\mathbf{p}}$ *and* $0 \leqslant \gamma < \gamma_0$ *on* $\Sigma_{\mathbf{p}}$;

(b) *there exists* $\gamma_1 < \pi$ *such that there is no solution* $u(\mathbf{x})$ *of* (11) *for which* $H(\mathbf{x}) \geqslant -H_M$ *throughout* $\mathcal{N}_{\mathbf{p}}$ *and* $\gamma_1 < \gamma \leqslant \pi$ *on* $\Sigma_{\mathbf{p}}$.

We examine the case (i, a). The proof we give is based on the right inequality of (17), which under the given hypotheses becomes, for a supposed solution,

$$\cos\gamma_0 < \widehat{\cos\gamma^*} \leqslant \frac{nH_M\Omega^* + \Gamma}{\Sigma^*} \equiv Q[\Omega^*]. \tag{19}$$

If $n = 2$, we can obtain "extremal" surfaces Γ explicitly as in 3.1; this procedure is not feasible if $n>2$, but since the essential requirements are local, it suffices to use as Γ a surface represented by appropriately chosen terms of a Taylor expansion.

We assume that Σ^* has, near $\mathbf{p}$, the representation

$$z^* = \frac{1}{2}\sum_1^{n-1} a_i^* x_i^2 + \dots$$

so that $\Sigma_1^{n-1} a_i^* = (n-1) H_{\mathbf{p}}^{\Sigma}$, where $H_{\mathbf{p}}^{\Sigma}$ is the mean curvature of Σ at the point $\mathbf{p}$. By hypothesis, $H_{\mathbf{p}}^{\Sigma} > (n/(n-1)) H_M$. As surface Γ, we introduce

$$z = \frac{1}{2}\sum_1^{n-1} a_i x_i^2 + \varepsilon$$

for suitable small $\varepsilon > 0$. Clearly we may choose $a_i < a_i^*$, $i = 1, \dots, n-1$, and such that $\Sigma_1^{n-1} a_i = nH_M$. The intersection set of the two surfaces projects asymptotically onto the ellipsoid $\Sigma_1^{n-1} (a_i^* - a_i) x_i^2 = 2\varepsilon$. Thus, for sufficiently small ε, $\Sigma_{\mathbf{p}}$ and Γ bound a simple region Ω^*. The calculation of the ratio $Q[\Omega^*]$ in (19) is a formal, if tedious, exercise. We find

$$Q[\Omega^*] = 1 + \frac{\varepsilon}{n+1}\left\{2nH_M - \sum_1^{n-1} (a_i^* + a_i)\right\} + o(\varepsilon).$$

Since $\Sigma_1^{n-1} (a_i^* + a_i) > 2\,\Sigma_1^{n-1} a_i = 2nH_M$, the result follows for case (i, a). The other cases in the theorem can be proved in the same way.

For the physical case in which H is constant and γ is prescribed continuously over all of Σ (and hence, from (13), H is determined), we obtain, using (13) and the notation introduced in § 3,

COROLLARY 3.1. *Suppose there is a point* $\mathbf{p} \in \Sigma$ *at which* $H_{\mathbf{p}}^{\Sigma} > \Sigma/((n-1)\Omega)$. *Then for any* $\Sigma_{\mathbf{p}} \ni \mathbf{p}$ *there exists* η, $0 < \eta < 1$, *such that there is no solution* $u(x)$ *of* (11) *with constant* H *in* Ω, *for which* $|\cos\gamma| > \eta$ on $\Sigma_{\mathbf{p}}$.

The value for η depends on the geometry of Ω and on the size of the neighborhood of $\mathbf{p}$ in Σ in which $H^{\Sigma} > \Sigma/((n-1)\Omega)$. If H and γ are both constant, the result can be put into a simple explicit form. We obtain then from (17):

COROLLARY 3.2. *If* $u(\mathbf{x})$ *is a solution of* (11) *with* $H(\mathbf{x}) \equiv H = \text{const.}$ *in* Ω, *and if* (3) *holds with* $\gamma = \text{const.}$ *on* Σ, *then*

$$|\cos\gamma| < \frac{\Gamma/\Sigma}{(\Sigma^*/\Sigma) - (\Omega^*/\Omega)} \tag{20}$$

for any choice of Γ *that makes the denominator positive.*

This estimate is (in particular) nontrivial whenever the condition of Corollary 3.1 holds; a consequence is that *the boundary value problem* (3; 11) *admits no solution for data that do not satisfy* (20).

The following reformulation of this result shows that solutions of the physical capillary problem in the absence of gravity are always unstable with respect to boundary perturbations.

COROLLARY 3.3. *Let Ω be an arbitrary bounded open set, and let γ be a prescribed constant, with $0<|(\pi/2)-\gamma|\leqslant\pi/2$. Then there exists a sequence $\{\Omega_j\}$ exhausting Ω, with analytic boundaries $\{\Sigma_j\}$, such that there is no solution of the problem* (3, 11) *with $H\equiv\text{const}$ in Ω_j, for any j.*

Thus if, in particular, the problem (3, 11) has a solution interior to Σ, then there is an arbitrarily small perturbation of Σ into an analytic Σ_j, such that there is no solution interior to Σ_j. If the requirement of analyticity is relaxed to infinite differentiability, then it suffices to perturb Σ in an arbitrary neighborhood of one of its points.

We note further the consequence of Corollary 3.1, that *in the case $|(\pi/2)-\gamma|=\pi/2$, the surfaces Σ_j can be chosen to approximate Σ not only in position but also in normal direction.*

When Ω is star shaped, we can use (4) to obtain a restatement of Corollary 3.1:

COROLLARY 3.4. *Suppose there is a point $\mathbf{p}\in\Sigma$ at which $H^{\Sigma}_{\mathbf{p}}>(n/(n-1))\bar{H}^{\Sigma}$. Then for any $\Sigma_{\mathbf{p}}\ni\mathbf{p}$ there exists η, $0<\eta<1$, such that there is no solution $u(\mathbf{x})$ of* (11) *with constant H in Ω, for which $|\cos\gamma|>\eta$ on $\Sigma_{\mathbf{p}}$.*

3.3. Although the estimate obtained by the above method of proof for Theorem 3 is certainly not precise, the theorem—and its corollaries—are qualitatively sharp in the sense that *for any prescribed constants H^{Σ}, H, γ, with $\pi>\gamma>0$, there exist solutions $u(\mathbf{x})$ of* (11) *near boundary surfaces Σ of mean curvature H^{Σ}, such that* $Tu\cdot\boldsymbol{\nu}=\cos\gamma$ on Σ. A convenient example is obtained by considering the surfaces of constant H possessing rotational symmetry about an axis.(1) In spherical coordinates, (11) becomes

$$\frac{d}{d\varrho}\varrho^{n-1}\frac{u_\varrho}{\sqrt{1+u_\varrho^2}}=nH\varrho^{n-1}$$

from which

$$\frac{u_\varrho}{\sqrt{1+u_\varrho^2}}=H\varrho+B\varrho^{1-n}. \tag{21}$$

(1) In the case $n=2$ these surfaces have been studied in a striking way by Delaunay [6], who obtained them as the rotation surfaces of the roulades of the conics.

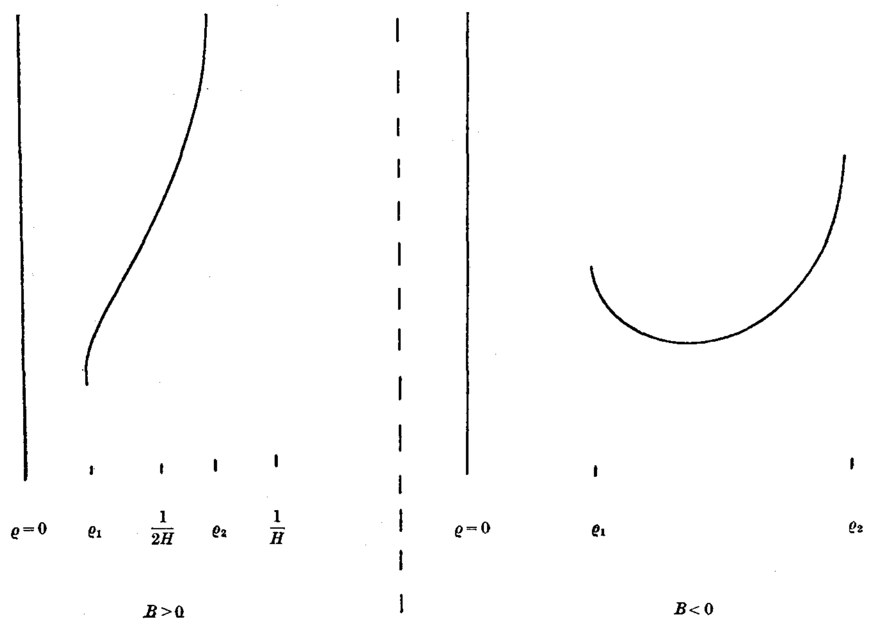


Fig. 3. Sections of rotationally symmetric surfaces with constant H, $n=2$.

A solution of (21) can exist only in an interval for which $|H\varrho + B\varrho^{1-n}| < 1$. We examine the case $H>0$, and distinguish the three possibilities:

(a) $B>0$. Solutions exist only if

$$B<\frac{1}{n}\left(\frac{n-1}{n}\right)^{n-1} H^{1-n}.$$

If this condition is satisfied, there is a solution in an annular region

$$0<\varrho_1<\frac{n-1}{nH}<\varrho_2<\frac{1}{H},$$

where ϱ_1, ϱ_2 assume all values in the indicated ranges as B varies from zero to its upper bound, and $u_\varrho \nearrow \infty$ as $\varrho \searrow \varrho_1$ and as $\varrho \nearrow \varrho_2$. A vertical section of a solution surface is shown in Fig. 3.

(b) $B=0$. In this case the unique solution of (21), up to a vertical displacement, is a sphere of radius H^{-1}.

(c) $B<0$. A solution exists in an interval (ϱ_1, ϱ_2), $0<\varrho_1<\infty$, $1/H<\varrho_2<\infty$, where ϱ_1 and ϱ_2 increase through all values of their ranges as B decreases from 0 to $-\infty$. In this case $u_\varrho \searrow -\infty$ as $\varrho \searrow \varrho_1$, $u_\varrho \nearrow +\infty$ as $\varrho \nearrow \varrho_2$ (Fig. 3).

The examples indicated above can now be constructed, for any prescribed constants H^{Σ}, H, γ, by appropriate choice, first of B, then of a concentric $(n-1)$ sphere Σ of suitable radius. The corresponding surfaces $u(\varrho)$ provide counterexamples in every configuration not covered by Theorem 3.

We do have, however:

COROLLARY 3.5. *If*(1) $H^{\Sigma}_{\mathbf{p}} > (n/(n-1))\, H_M \geqslant 0$ *or if* $|H^{\Sigma}_{\mathbf{p}}| < -(n/(n-1))\, H_M$, *then,*

(a) *there is no solution of* (11) *in any* $\mathcal{N}_{\mathbf{p}}$, *with* $H(\mathbf{x}) \leqslant H_M$ *in* $\mathcal{N}_{\mathbf{p}}$, *and such that* $\cos\gamma = 1$ *on* $\Sigma_{\mathbf{p}}$;

(b) *there is no solution of* (11) *in any* $\mathcal{N}_{\mathbf{p}}$, *with* $H(\mathbf{x}) \geqslant -H_M$ *in* $\mathcal{N}_{\mathbf{p}}$, *and such that* $\cos\gamma = -1$ *on* $\Sigma_{\mathbf{p}}$.

The result follows immediately from the method of proof of Theorem 3.

The surfaces obtained from (21) yield situations in which $\cos\gamma \equiv 1$ on Σ occurs with any H^{Σ} in the ranges $0 < H^{\Sigma} < (n/(n-1))\, H$ or $|H^{\Sigma}| > -(n/(n-1))\, H$ and in which $\cos\gamma \equiv -1$ on Σ occurs with any H^{Σ} in the ranges $0 < H^{\Sigma} < -(n/(n-1))\, H$ or $|H^{\Sigma}| < (n/(n-1))\, H$. In these situations, of course, the surfaces cannot be extended as solutions into the entire interior (or exterior) of Σ. In fact, an easy reasoning shows that *given any* ϱ_0 *in the closed interval* $\varrho_1 \leqslant \varrho_0 \leqslant \varrho_2$, *the rotationally symmetric surfaces constructed above are the unique ones of the given* H, *meeting the cylinder* Z_{ϱ_0} *over* Σ_{ϱ_0} *at the given angle and defined throughout either of the annuli* $\varrho_1 < \varrho < \varrho_0$ *or* $\varrho_0 < \varrho < \varrho_2$.

We point out finally that Theorem 3 is (at least qualitatively) sharp in still another sense. The conclusion that there is no solution for which $0 \leqslant \gamma < \gamma_0$ (respectively $\gamma_1 < \gamma \leqslant \pi$) on $\Sigma_{\mathbf{p}}$ cannot be strengthened to exclude these inequalities at an isolated point on $\Sigma_{\mathbf{p}}$, even for continuous data. For example, a lower hemisphere defines a solution of (11) with $H \equiv 1$ in $|\mathbf{x}| < 1$. The inequality (i) will then be satisfied on the arc Σ: $|\mathbf{x} - \frac{3}{4}| = \frac{1}{4}$. The hemisphere, considered as solution interior to Σ, defines a continuous $\gamma(\mathbf{x})$ on Σ, and $\gamma(\mathbf{x}) = 0$ at the point of contact of Σ with $|\mathbf{x}| = 1$. An analogous discussion applies to the inequality (ii).

3.4. The case $H^{\Sigma} = (n/(n-1))H$ has a special interest, and is not completely covered by Theorem 3. This situation is discussed in the following note [8], using other methods; it is shown there (in particular) that if $H \equiv H_M > 0$, there is no bounded solution in any $\mathcal{N}_{\mathbf{p}}$ for which $\cos\gamma = 1$ on Σ, while if $H \equiv H_M < 0$, there is no bounded solution for which $\cos\gamma = -1$ on Σ. J. Spruck has shown [13] that both these situations can occur for unbounded solutions. We show here, for the special case $n = 2$, that a solution can exist under these conditions when the value $H^{\Sigma} = (n/(n-1))\, H = 2H$ is achieved as discontinuous limit from

(1) For bounded solutions, the inequalities need not be strict, cf. § 3.4 and the note [8].

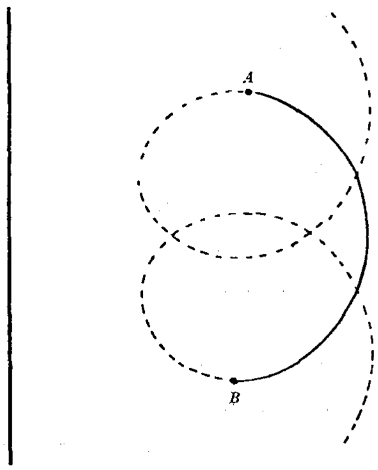
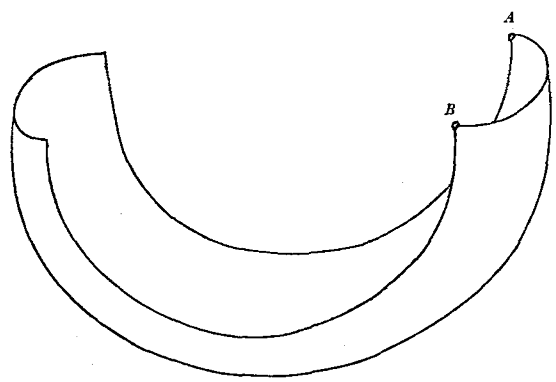
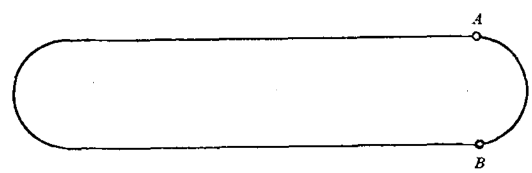


Fig. 4. Roulade of hyperbola.



a) Free surface of constant H



b) Section of bathtub

Fig. 5. Astronaut's bathtub for $\gamma = 0$.

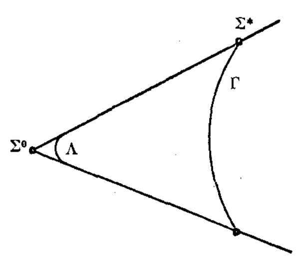


Fig. 6. Neighborhood of discontinuity set Σ°.

one side. Our example corresponds to Case (c) of § 3.3 but is best viewed in the sense of the Delaunay construction (see footnote p. 186) as surface of revolution of the roulade of a hyperbola. The roulade is shown in Fig. 4. We utilize only the portion indicated with solid line, which we rotate π radians about the axis, and next rotate the resulting surface till it is oriented as in Fig. 5a. We obtain a surface $u(\mathbf{x})$ of constant H, defined in the domain Ω whose boundary Σ is formed by the roulade, its reflected image, and the two straight lines joining them, and meeting the cylinder wall under Σ in the constant angle $\gamma = 0$. At the four points where the straight lines meet the roulades, Σ has curvature $2H$ as limit from within the roulades; from within the straight lines, however, the curvature at these points is zero.

If Σ is extended vertically downward to form a cylinder with base as in Fig. 5b, we obtain what we have called an "astronaut's bathtub". In a gravity-free situation, water covering the base and meeting the boundary walls in the angle $\gamma = 0$ would form the free surface $u(\mathbf{x})$, whose stability would presumably be ensured by the pressure of the adjacent atmosphere. We are of course not in position to recommend the actual use of such an apparatus for bathing purposes.

3.5. We apply here the inequality (17) to the case in which the tangent plane to the boundary Σ may have discontinuities on a small set $\Sigma^0 \subset \Sigma$. The size of Σ^0 will be limited by the hypothesis *that Σ^0 can be covered, from within Ω, by a sequence of smooth surfaces $\{\Lambda\}$, each of which meets Σ in a set of zero $(n-1)$-dimensional measure, and such that $\Lambda \to \Sigma^0$ and the area $\mathcal{A}^\Lambda$ of Λ tends to zero.*

We study solutions $u(\mathbf{x})$ of (11) in Ω; we assume, as before, that $\mathbf{T}u \cdot \boldsymbol{\nu}$ is defined on $\Sigma - \Sigma^0$ as a limit from within Ω. The essential interest in the material to follow lies in the fact that *no assumption is made on the behavior of $u(\mathbf{x})$ as points of Σ^0 are approached from within Ω.* In particular the growth of $u(\mathbf{x})$ near Σ^0 is in no way restricted by any hypothesis. We shall show that the geometry can nevertheless impose severe restrictions on the kinds of solutions that can exist in Ω near Σ^0.

Let $\mathbf{p} \in \Sigma^0$, let $\Gamma \subset \Omega$ be a (smooth) surface surrounding $\mathbf{p}$, which, together with a set $\Sigma^* \subset \Sigma$, bounds a domain $\Omega^* \subset \Omega$ (see Fig. 6). Let $\Lambda \subset \{\Lambda\}$. We integrate (11) over the part of Ω^* between Λ and Γ. Passing to the limit as $\Lambda \to \Sigma^0$, we find (since $|\mathbf{T}u| < 1$ and $\mathcal{A}^\Lambda \to 0$) that Lemma 6 holds in this configuration. That is, in the notation of that lemma, we have again

$$\frac{n\bar{H}^*\Omega^* - \Gamma}{\Sigma^*} \leqslant \widehat{\cos\gamma^*} \leqslant \frac{n\bar{H}^*\Omega^* + \Gamma}{\Sigma^*} \tag{22}$$

the average on Σ^* being taken now over points not in Σ^0.

THEOREM 4. *Suppose a sequence $\{\Gamma\}$ tending to $\mathbf{p}$ can be found, for which $|\bar{H}^*| < H < \infty$, and such that $\eta = \liminf (\Gamma/\Sigma^*) < 1$. Then there is no neighborhood of $\mathbf{p}$ in Σ^* throughout which $|\cos\gamma| > \eta_0 > \eta$.*

The proof is immediate from (22) and from the isoperimetric inequality, which yields

$$\limsup \frac{\Omega^*}{\Sigma^*} = \limsup \frac{2\Omega^*}{\Sigma^* + \Gamma(\Sigma^*/\Gamma)} \leqslant \limsup \frac{2\Omega^*}{\Sigma^* + \Gamma} = 0$$

for any sequence along which $\Gamma/\Sigma^* < 1$.

Theorem 4 yields a best possible result in all cases we have verified independently. In cases in which a geometric invariance is present, e.g., a cone generated by straight lines, it may even be unnecessary to apply the limiting process. As an example, we apply (22) to an $(n-1)$-dimensional cone K generated by rays from the origin in n-dimensional space.

Here Σ^0 is the (single) vertex point $\mathbf{p}$, and for $\{\Lambda\}$ one may choose spherical caps centered at $\mathbf{p}$.

We define the "half-angle" α of K by the relation $\sin\alpha = \min(\Gamma/\Sigma^*)$ among all planes Π that cut a section Γ from the solid cone $\mathcal{K}$ and intercept a closed surface $\Sigma^* \ni \mathbf{p}$ on K; if no such plane exists we set $\sin\alpha = 1$. Note that always $0 < \alpha \leqslant \pi/2$. We obtain:

THEOREM 5. *If $\alpha < \alpha_0 \leqslant |(\pi/2) - \gamma|$, then there is no solution $u(\mathbf{x})$ of (11) in the region $\mathcal{K}_r$ bounded by K and by any sphere Σ_r surrounding $\mathbf{p}$, for which $|H(\mathbf{x})| < H < \infty$ and for which (3) holds on the conical surface.*

If W is a wedge generated by K, the same criterion applies. We emphasize that no growth condition is imposed on the solution, as the vertex is approached from within the region or on K.

This result appears in [3] for the case $n = 2$, in which K consists of two rays from $\mathbf{p}$. The proof given in [3] extends without essential change to a cone (or wedge) symmetric about an axis (or hyperplane); however, it does not yield the more general statement given above.

Theorem 5 is sharp, at least in the symmetric case, in the sense that *solutions always exist if $(\pi/2) - \alpha \leqslant \gamma \leqslant (\pi/2) + \alpha$.* In fact, one verifies readily that a hemisphere making the prescribed angle γ with K solves the problem explicitly in this case.

3.6. We complement Theorem 5 and the above remark by showing that *if $u(\mathbf{x})$ satisfies (11) in a symmetric cone $\mathcal{K}_r$, with $H(\mathbf{x}) \geqslant H_0 > 0$ and $\alpha + \gamma \geqslant \pi/2$, then $u(\mathbf{x})$ is bounded above at $\mathbf{p}$. Precisely, $u(\mathbf{x}) \leqslant H_0^{-1} + \max_{x\in\Sigma_{r'}} u(\mathbf{x})$ for all $\mathbf{x}$ near $\mathbf{p}$, with any $r' < r$.*

We present a formal analytical proof of this statement in § 3.9. Here we give a direct geometrical proof, which requires however the additional hypothesis that $u(\mathbf{x})$ is (locally) of class $C^{(1)}$ up to the walls K.

Choose H_1, $H_0 > H_1 > 0$, and consider a lower hemisphere S: $v(\mathbf{x})$ of radius H_1^{-1}, with center on the axis of K at distance H_1^{-1} from $\mathbf{p}$. S cuts the cylindrical walls in an angle $\gamma_0 = (\pi/2) - \alpha$, and meets K at $\mathbf{p}$ in an undefined angle. If the center is displaced slightly away from the vertex then either the surfaces will no longer contact, or the angle of contact will decrease, so that we will then have $\gamma_0 < \gamma$, and $\mathbf{p}$ will be exterior to the domain Δ of definition of $v(\mathbf{x})$. The boundary of Δ (in $\mathcal{K}$) will consist of inner and outer spherical caps Σ_a and Σ_b, and of portions Σ^* of the conical walls.

We may suppose Σ_r lies interior to the domain of definition of $u(\mathbf{x})$, so that $u(\mathbf{x})$ is continuous on Σ_r. Let C be the smallest constant such that $v(\mathbf{x}) + C \geqslant u(\mathbf{x})$ in $\mathcal{K}_r \cap \Delta$. Then there is a point $\mathbf{q}$ in the closure of this set, at which equality is attained. Clearly $\mathbf{q}$ is not an interior point of $\mathcal{K}_r \cap \Delta$, as the mean curvature of S_C: $v(\mathbf{x}) + C$ is less than that of the given surface, so that the surfaces would have to cross at any inner point of tangency. Also, $\mathbf{q} \notin \Sigma_a$ or Σ_b, as $\partial v/\partial n = \infty$ at all such points(¹). Similarly, $\mathbf{q} \notin \Sigma^*$, since at such a point the tangential derivatives of v and of u would be equal, hence one would have again $(\partial v/\partial n)|_{\mathbf{q}} > (\partial u/\partial n)|_{\mathbf{q}}$ as a consequence of $\gamma_0 < \gamma$. Thus, $\mathbf{q} \in \Sigma_r$. We conclude that for all $\mathbf{x} \in \mathcal{K}_r \cap \Delta$, $u(\mathbf{x}) \leq H_1^{-1} + \max_{\Sigma_r} u(\mathbf{x})$. Letting the center of S slide back to its original position, and then letting $H_1 \to H_0$, the stated result follows.

Similarly, *if* $u(\mathbf{x})$ *satisfies* (11) *in* $\mathcal{K}_r$, *with* $H(\mathbf{x}) \leq H_0 < 0$ *and* $\alpha - \gamma \geq -(\pi/2)$, *then* $u(\mathbf{x})$ *is bounded below at* $\mathbf{p}$, *and* $u(\mathbf{x}) \geq -H_0^{-1} + \min_{\Sigma_r} u(\mathbf{x})$ *near* $\mathbf{p}$.

3.7. We may note that in the above construction, the sphere S must project onto at least one point of Σ_r. For otherwise the procedure would yield $v(\mathbf{x}) + C > u(\mathbf{x})$ in $\mathcal{K}_r \cap \Delta$ for every C. Thus, the procedure of § 3.6 yields as corollary that *if* $u(x)$ *satisfies* (11) *in* $\mathcal{K}_r$, *if* $H(\mathbf{x}) \geq H_0 > 0$ *and* $\alpha + \gamma \geq \pi/2$, *or if* $H(\mathbf{x}) \leq H_0 < 0$ *and* $\alpha - \gamma \geq -(\pi/2)$, *then* $r \leq 2H_0^{-1}$. That is, there is a bound, depending only on H_0, of the size of the domain in which the solution can be defined.

3.8. *The hypotheses of* § 3.6 *do not imply a bound below for the solution.* To see this, consider a spherical cap C in $\mathcal{K}_r$ centered at $\mathbf{p}$, and the lower half cylinder Z lying below C. A slight rotation of Z about an axis through two diametrical points on the sphere of intersection of C with K yields a surface $z(\mathbf{x})$ of constant mean curvature $H > 0$, defined in $\mathcal{K}_r$, for which $\alpha + \gamma > \pi/2$. Letting the angle of rotation tend back to zero yields a family of such surfaces with the same fixed H, whose ordinates achieve arbitrarily large negative values in $\mathcal{K}_r$.

The same example shows that the bounds of § 3.6 could not have been made to depend on the value of $u(\mathbf{x})$ at a single point of Σ_r.

3.9. It remains to prove the assertion of § 3.6 without the hypothesis of boundedness for $u(\mathbf{x})$ on K. To do so we use a general comparison principle satisfied by the solutions of (11), which is motivated by the procedure in § 3.6. Set $\mathrm{N}f \equiv \operatorname{div}((1/W)\nabla f)$, for any function $f(\mathbf{x})$.

THEOREM 6. *Let* $\Sigma = \Sigma^0 + \Sigma^\alpha + \Sigma^\beta$ *be a decomposition of* Σ, *such that* Σ^β *is of class* $C^{(1)}$ *and* Σ^0 *is small in the sense introduced in* § 3.5. *Let* $u(\mathbf{x})$, $v(\mathbf{x})$ *be of class* $C^{(2)}$ *in* Ω, *and suppose*

(i) $Nu \geq Nv$ *at all* $\mathbf{x} \in \Omega$

(ii) *for any approach to* Σ^α *from within* Ω, $\limsup\,[u - v] \leq 0$

(iii) *on* Σ^β, $(\mathbf{T}u - \mathbf{T}v) \cdot \boldsymbol{\nu} \leq 0$ *almost everywhere as a limit from points of* Ω.

(¹) The use of comparison surfaces with this property as a device for estimating solutions can be traced to S. Bernstein [2]; the procedure was further developed and stated as a formal lemma by Leray [11]. It was later rediscovered and applied in a different context (in spirit close to that of the present work) by one of the authors [7].

Then if Σ^α *can be so chosen that* $\Sigma^\alpha \subset \Sigma^0$, *there follows* $u(\mathbf{x}) \equiv v(\mathbf{x}) + \text{const}$. *Otherwise,* $u(\mathbf{x}) \leqslant v(\mathbf{x})$ *in* Ω; *if equality holds at a single point of* Ω, *then* $u(\mathbf{x}) \equiv v(\mathbf{x})$.

We note there is no hypothesis on smoothness or even of bounds for $u(\mathbf{x})$ or $v(\mathbf{x})$ or of their derivatives near Σ. The result clearly depends on the particular nonlinearity of the operator N.

Proof of Theorem 6. Suppose $u(\mathbf{x})$, $v(\mathbf{x})$ satisfy (i), (ii), (iii) and for some $\mathbf{x}_0 \in \Omega$ there holds $u(\mathbf{x}_0) - v(\mathbf{x}_0) = \frac{1}{2}M > 0$. Let $w(\mathbf{x}) = u(\mathbf{x}) - v(\mathbf{x})$ and let Ω_M be that subset of Ω, in which $0 < \varepsilon \leqslant w \leqslant M$, for some $\varepsilon > 0$. If $\varepsilon < \frac{1}{2}M$, then Ω_M is non-null and is bounded in part by a portion $\Sigma^* \subset \Sigma^\beta$, by a part (or all) of Σ^0, and by sets Γ_ε, Γ_M in Ω, on which $w = \varepsilon$ or M. Let Ω_M^Λ be the part of Ω_M lying exterior to one of the covering sets $\Lambda \subset \{\Lambda\}$.

We consider the formal relation

$$\begin{aligned}\int_{\Omega_M^\Lambda} [w(\mathbf{x}) - \varepsilon]\,(Nu - Nv)\,d\mathbf{x} \\ &= \oint_{[\Sigma^* + \Gamma_\varepsilon + \Gamma_M]^\Lambda} [w(\mathbf{x}) - \varepsilon]\,(\mathbf{T}u - \mathbf{T}v)\cdot \boldsymbol{\nu}\, d\sigma \\ &\quad - \int_{\Omega_M^\Lambda} \nabla w \cdot [\mathbf{T}u - \mathbf{T}v]\,dx + \oint_{\Lambda^*} [w(\mathbf{x}) - \varepsilon]\,(\mathbf{T}u - \mathbf{T}v)\cdot \boldsymbol{\nu}\,d\sigma. \end{aligned} \tag{23}$$

Here $\Lambda^* = \Lambda \cap \Omega_M$, and the superscript Λ denotes the part of the set lying exterior to Λ (see Figure 7).

Under the hypotheses, (23) makes sense as written if the first integral on the right is defined by a limiting procedure in terms of the other quantities that appear. In order to complete the proof, however, we need to define separately the contributions from Σ^* and from Γ_M. The information at our disposal does not yet permit a unique definition of these quantities, but we can attach a meaning to them that suffices for what is needed.

We approximate Σ^β from within Ω by a sequence $\{\tilde{\Sigma}^\beta\}$ of surfaces of class $C^{(1)}$ that converge to Σ^β pointwise and in normal direction, and we observe first that when applied to the restricted domain $\tilde{\Omega}_M^\Lambda$, all terms in (23) can be given an unambiguous meaning. This is so since $\mathbf{T}u$, $\mathbf{T}v$ are defined and smooth on $\tilde{\Sigma}^\beta$, and since Γ_ε, Γ_M are level sets of $w(\mathbf{x})$; thus, we can either choose ε and M so that Γ_ε, Γ_M are smooth (Sard's theorem), or we can use the method indicated in footnote (2) of [4] to show that the integration over Γ_ε, Γ_M can be defined regardless of possible irregularities in the sets.

When (23) is applied to $\tilde{\Omega}_M^\Lambda$, the integration over Σ^* becomes an integration over a set $\tilde{\Sigma}^* = \tilde{\Sigma}^\beta \cap \Omega_M^\Lambda$. Thus, $|w(\mathbf{x})| < M$ on $\tilde{\Sigma}^*$. We note further that $|\mathbf{T}f| < 1$ for any $f(\mathbf{x})$;

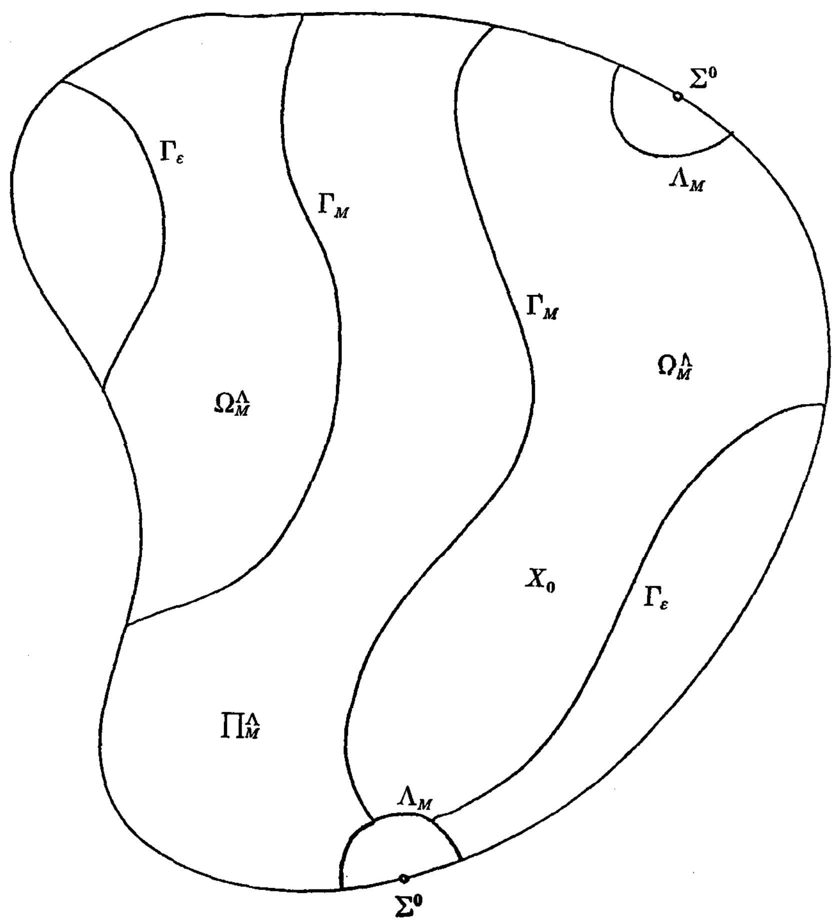


Fig. 7.

thus in particular $|\mathbf{T}u - \mathbf{T}v| < 2$ on $\tilde{\Sigma}^*$. Since the $\{\tilde{\Sigma}^\beta\}$ converge in area to Σ^β, we conclude there is a subsequence $\{\tilde{\Sigma}_j^\beta\}$, such that the corresponding integrals in (23) over $\tilde{\Sigma}_j^*$ converge, as $j \to \infty$, to a limit $\mathcal{L}^*$.

On $\tilde{\Sigma}_j^*$, $w(\mathbf{x}) > 0$. By hypothesis (iii), the functions $\varphi_j = (\mathbf{T}u - \mathbf{T}v) \cdot \boldsymbol{\nu}$ on $\tilde{\Sigma}_j^\beta$, considered as functions defined over Σ^β by the approximation procedure, satisfy $\limsup_{j\to\infty} \varphi_j(\mathbf{x}) \leqslant 0$ for each $\mathbf{x} \in \Sigma^\beta$. By the theorem of Egorov, for any $\delta, \eta > 0$, there exists $j_0(\delta, \eta)$ such that for all $j \geqslant j_0$ the set for which $\varphi_j(\mathbf{x}) > \delta$ has measure less than η. Using again the relations $|w| < M$, $|\mathbf{T}u - \mathbf{T}v| < 2$, we conclude immediately that $\mathcal{L}^* \leqslant 0$.

In (23) the contribution from Γ_ε vanishes. We proceed to evaluate the integral over Γ_M^Λ. To do so, we observe that Γ_M^Λ bounds, together with a set $\Sigma_M^\Lambda \subset \Sigma^\beta$ and portions $\Lambda_M \subset \Lambda$, the set Π_M^Λ: $w(\mathbf{x}) > M$, $\mathbf{x} \in \Omega^\Lambda$ (see Fig. 7).

We then have, formally,

$$\oint_{\Sigma_M^\Lambda}(\mathbf{T}u-\mathbf{T}v)\cdot\boldsymbol{\nu}\,d\sigma-\oint_{\Gamma_M^\Lambda}(\mathbf{T}u-\mathbf{T}v)\cdot\boldsymbol{\nu}\,d\sigma=\oint_{\Pi_M^\Lambda}(Nu-Nv)\,d\mathbf{x}-\oint_{\Lambda_M}(\mathbf{T}u-\mathbf{T}v)\cdot\boldsymbol{\nu}\,d\sigma, \quad (24)$$

the negative sign appearing in the second term because the orientation of $\boldsymbol{\nu}$ is taken here to coincide with that of the corresponding term in (23).

The terms in (24) must be interpreted by a limiting procedure consistent with the one used for (23). We introduce the same sequence $\{\tilde{\Sigma}_j^\beta\}$ and consider the portion $\tilde{\Sigma}_{M_j}=\tilde{\Sigma}_j^\beta\cap\Pi_M^\Lambda$. As before, there is a subsequence of these sets such that the integrals corresponding to the first term on the left in (24) converge to a limit $\mathcal{L}_M$, and using again the hypothesis (iii), the same reasoning as above yields $\mathcal{L}_M\leqslant 0$, for any Λ.

In terms of the given approximation procedure, (24) defines the integral over Γ_M^Λ in terms of other quantities whose sign is known and an integral over Λ whose sign is not known. We are however now in position to pass to the limit as $\Lambda\to\Sigma^0$. In (24), $w(\mathbf{x})$ does not appear explicitly in the integrals over Λ, while in (23), $|w(\mathbf{x})|<M$ in these integrations. Since $|\mathbf{T}u|<1$, $|\mathbf{T}v|<1$, and since by hypothesis the area $\mathcal{A}^\Lambda\to 0$, we find that (23) and (24) both hold with the superscript Λ deleted. We then have from (24)

$$\oint_{\Gamma_M}(\mathbf{T}u-\mathbf{T}v)\cdot\boldsymbol{\nu}\,d\sigma=\oint_{\Sigma_M}(\mathbf{T}u-\mathbf{T}v)\cdot\boldsymbol{\nu}\,d\sigma-\int_{\Pi_M}(Nu-Nv)\,d\mathbf{x}\leqslant 0 \quad (25)$$

by (i) and (iii); from (23) we obtain

$$\begin{aligned}\int_{\Omega_M}\nabla w\cdot[\mathbf{T}u-\mathbf{T}v]\,dx= &-\int_{\Omega_M}[w(\mathbf{x})-\varepsilon]\,(Nu-Nv)\,d\mathbf{x}\\ &+(M-\varepsilon)\oint_{\Gamma_M}(\mathbf{T}u-\mathbf{T}v)\cdot\boldsymbol{\nu}\,d\sigma\\ &+\oint_{\Sigma^*}[w(\mathbf{x})-\varepsilon]\,(\mathbf{T}u-\mathbf{T}v)\cdot\boldsymbol{\nu}\,d\sigma\leqslant 0 \end{aligned} \quad (26)$$

by (25), (i) and (iii). The integrand on the left side of (26) is however a positive definite form in the components of ∇w (this follows from the convexity of the area functional). We conclude that either $u(\mathbf{x})\leqslant v(\mathbf{x})$ in Ω, or else $\Omega_M\supset\Omega$ and $u(\mathbf{x})\equiv v(\mathbf{x})+\text{const.}$ in Ω.

If $\Sigma^\alpha\subset\Sigma^0$ we observe that the hypotheses of the theorem do not change if a constant is added to $u(\mathbf{x})$. For some $\mathbf{x}_0\in\Omega$, choose C_0 so that $u(\mathbf{x}_0)+C_0-v(\mathbf{x}_0)>0$. The above reasoning then yields $u(\mathbf{x})+C_0-v(\mathbf{x})\equiv\text{const.}$ in Ω, which was to be shown.

If $\Sigma^\alpha\not\subset\Sigma^0$ then in particular $\Sigma^\alpha\neq\phi$ and we conclude from (ii) that in either event $u(\mathbf{x})\leqslant v(\mathbf{x})$ in Ω. The maximum principle of *E*. Hopf [9] then yields the result that equality holds throughout Ω if it holds at a single interior point.

Another form of this comparison principle, suited to the situation encountered in a gravitational field, will be given in [5, § 3.6].

3.10. We now apply Theorem 6 to obtain a strengthened version of the result of § 3.6. We suppose again $u(\mathbf{x})$ satisfies (11) in a symmetric $\mathcal{K}_r$, with $H(\mathbf{x}) \geqslant H_0 > 0$. However no assumption is now made on smoothness of $u(\mathbf{x})$ up to the walls K_r; $u(\mathbf{x})$ is not required to be defined on K_r, and it is supposed only that $\alpha + \gamma \geqslant \pi/2$ as a limit (almost everywhere) from within $\mathcal{K}_r$.

In the physical case $n=2$, we show that $u(\mathbf{x}) < M + 2H_0^{-1}$ near Σ^0 whenever $u(\mathbf{x}) < M$ on a certain one dimensional interior subset, depending only on the geometry and not on the solution considered. If $n \geqslant 2$, there holds $u(\mathbf{x}) < M + H_0^{-1}$ in $\mathcal{K}_r$ whenever $u(\mathbf{x}) < M$ on some outer bounding surface Σ^α (figure 8).

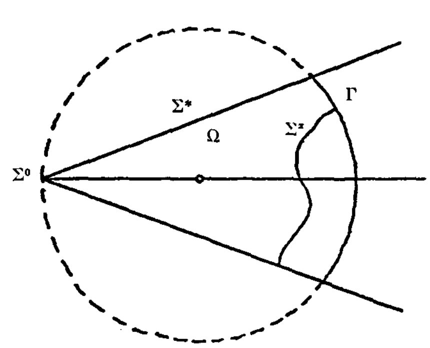


Fig. 8*a*

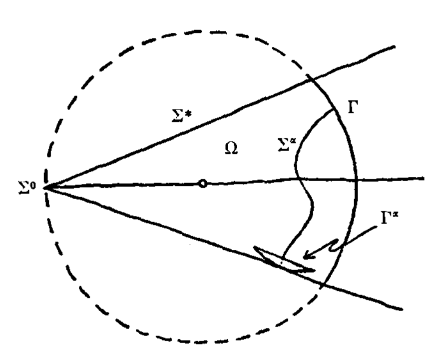


Fig. 8*b*

We choose for $v(\mathbf{x})$ a lower hemisphere S of radius H_0^{-1}, with projection (fig. 8 a) passing through the vertex Σ^0 and meeting $\mathcal{K}_r$ in a region Δ bounded by Σ^0, Σ^*, and by an outer cap Γ. We note that S meets the walls K_r in an angle $\gamma_S = (\pi/2) - \alpha$.

If Δ lies in the region Ω of definition of $u(\mathbf{x})$, we set $\Sigma^\alpha = \phi$, $\Sigma^\beta = \Sigma^* + \Gamma$; Theorem 6 then yields (since $\gamma_S \leqslant \gamma$ on Σ^*, $\boldsymbol{\nu} \cdot Tv \equiv 1$ on Γ) $H(\mathbf{x}) \equiv H_0$, $u(\mathbf{x}) \equiv v(\mathbf{x}) + \text{const}$. Thus, in this case the surface $u(\mathbf{x})$ is identically a lower hemisphere.

If $\Delta \not\subset \Omega$, we obtain the situation illustrated in Figure 8 a. We may redefine Ω so that it is bounded in part by portions Σ^β of Σ^* and Γ, by Σ^0 and by some set Σ^α on which $u(\mathbf{x})$ is, locally, bounded. If it is known that $u(\mathbf{x}) \leqslant M$ on all of Σ^α, Theorem 6 yields immediately $u(\mathbf{x}) \leqslant M + H_0^{-1}$ in Ω, as was to be proved.

Suppose $n=2$ and $u(\mathbf{x})$ is known to be bounded only on compact interior subsets of Ω. We construct a spherical cap of radius H_0^{-1} interior to Ω, as indicated in Figure 8 b. Letting $M^\alpha = \max_{\Gamma^\alpha} u(\mathbf{x})$, we find from Theorem 6 that $u(\mathbf{x}) < M^\alpha + H_0^{-1}$ throughout the cap.

a c b d

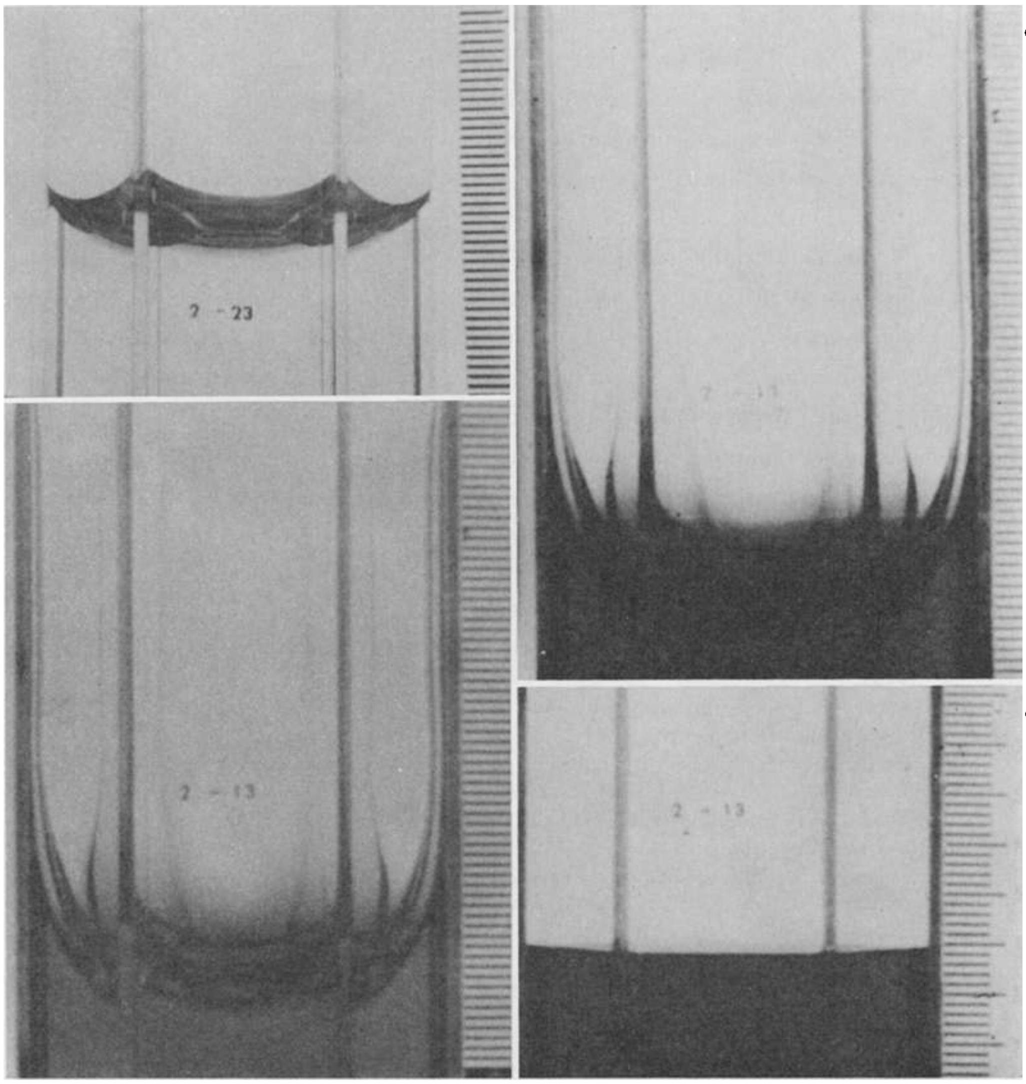


Fig. 9(*a*) $\alpha = 60°, \gamma = 48°$; (*b*) $\alpha = 60°, \gamma = 25°$; (*c*) $\alpha = 60°, \gamma = 0°$; (*d*) $\alpha = 60°, \gamma = 0°$.

Thus, if M_1^α is a bound for $u(\mathbf{x})$ on the remainder of Σ^α, Theorem 6 now yields $u(\mathbf{x}) \leqslant \max\{M^\alpha + H_0^{-1}, M_1^\alpha\} + H_0^{-1}$ in Ω, the desired result.

Remark. We note the bound in $\mathcal{K}$ depends on the bound on an $(n-1)$ dimensional compact subset of $\mathcal{K}$. Clearly there is no universal bound, as $u(\mathbf{x}) + C$ satisfies (11) whenever $u(\mathbf{x})$ does, for any constant C. It is not clear whether a significantly smaller set than the one introduced would suffice for the estimate; however, we point out here that a consideration of the example of § 3.8 shows that it does not suffice to know a bound at a single interior point.

3.11. Theorem 5 was tested experimentally by W. J. Masica at the NASA Lewis Zero Gravity Facility in Cleveland, using cylinders of polygonal cross section in a 142 meter drop-tower. As used in this experiment, the tower provided approximately five seconds during which the fluid contained in the cylinder experienced no gravitational acceleration. In Figure 9 the equilibrium configurations for an acrylic plastic cylinder of hexagonal section are compared, using fluids for which (a) $\gamma = 48°$, (b) $\gamma = 25°$, (c) $\gamma = 0°$. In Figure 9d the fluid of case (c) is shown at rest on the surface of the earth. The varying appearances of the fluid and surface are due to varying light conditions under which the photographs were made. In case (a) the solution appears to be part of a hemisphere, as our results predict, while in cases (b) and (c), to which Theorem 5 applies, the fluid rises in the edges and fills in the corners and edges at the top. This is in agreement with Theorem 5, according to which there can be no solution surface defined up to the edges and projecting simply onto the base.

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