DOI: 10.15393/j3.art.2014.2441 **Issues of Analysis.** Vol. 3(21), No. 1, 2014

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ON CARLSON'S AND SHAFER'S INEQUALITIES

Abstract. In this paper the authors refine the Carlson's inequalities for inverse cosine function, and the Shafer's inequalities for inverse tangent function.

Key words: *Carlson's inequality, Shafer's inequality, inverse trigonometric functions.*

2010 Mathematical Subject Classification: *26D05, 26D07, 26D99.*

§ 1. Introduction

During the past fifteen years, numerous authors have studied various inequalities for trigonometric functions [1 − 12]. Thus, some classical and also more recent inequalities, such as inequalities of Jordan, Cusa–Huygens, Shafer–Fink, and Wilker have been refined and generalized. One of the key methods in these studies has been so called monotone l'Hospital Rule from [1] and an extensive survey of the related literature is given in [13]. This Rule is formulated as Lemma 1 and it will be also applied here. Motivated by these studies, in this paper we make a contribution to the topic by sharpening Carlson's and Shafer's inequalities, and our inequalities refine the existing results in literature.

We start our discussion with the following well-known inequalities,

$$\cos(t)^{1/3} < \frac{\sin(t)}{t} < \frac{\cos(t)+2}{3}, \tag{1}$$

for $0 < |t| < \pi/2$. The first inequality is due to by D. D. Adamović and D. S. Mitrinović [14, p. 238], while the second inequality was obtained by N. Cusa and C. Huygens [15]. These inequalities can be written as

$$\frac{3\sin(t)}{2+\cos(t)} < t < \frac{\sin(t)}{\cos(t)^{1/3}}.$$

For the further studies and refinements of inequalities in (1), e.g., see [2, 5–8, 13, 16] and the references therein. For the easy references we recall the following inequalities

$$\cos\left(\frac{t}{2}\right)^{4/3} < \frac{\sin(t)}{t} < \cos\left(\frac{t}{3}\right)^{3}, \tag{2}$$

the first inequality holds for $t \in (0, \pi/2)$ [6], while the second one is valid for $t \in (-\sqrt{27/5}, \sqrt{27/5})$, and was proved by Klén et al. [2]. The first inequality in (2) refines the following one

$$t < \frac{2\sin(t)}{1+\cos(t)}, \quad 0 < x < \frac{\pi}{2},$$

which was constructed in [17] by using Chebyshev's integral inequality.

Oppenheim's problem [14, 18, 19] states to determine the largest a_2 and the least a_3 as a function of $a_1 > 0$, such that the following inequalities

$$\frac{a_2\sin(x)}{1+a_1\cos(x)} < x < \frac{a_3\sin(x)}{1+a_1\cos(x)} \tag{3}$$

hold for all $x \in (0, \pi/2)$. A partial solution of this problem was given by Oppenheim and Carver [19], they showed that (3) holds for all $a_1 \in \in (0, 1/2)$ and $x \in (0, \pi/2)$ when $a_2 = 1 + a_1$ and $a_3 = \pi/2$. In 2007, Zhu [20, Theorem 7] solved Oppenheim's problem completely by proving that the inequalities in (3) hold if a_1, a_2 and a_3 are as follows:

1) if $a_1 \in (0, 1/2)$, then $a_2 = 1 + a_1$ and $a_3 = \pi/2$,

2) if $a_1 \in (1/2, \pi/2 - 1)$, then $a_2 = 4a_1(1 - a_1^2)$ and $a_3 = \pi/2$,

3) if $a_1 \in (\pi/2 - 1, 2/\pi)$, then $a_2 = 4a_1(1 - a_1^2)$ and $a_3 = 1 + a_1$,

4) if $a_1 > 2/\pi$, then $a_2 = \pi/2$ and $a_3 = 1 + a_1$,

where a_2 and a_3 are the best possible constants in (1) and (4), while a_3 is the best possible constant in (2) and (3). Thereafter, Carver's solution was extended to the Bessel functions for the further results by Baricz

[21, 22]. On the basis of computer experiments we came up that the following lower and upper bounds for x,

$$\frac{(\pi/2)\sin(x)}{1+(2/\pi)\cos(x)} < x < \frac{\pi\sin(x)}{2+(\pi-2)\cos(x)} \tag{4}$$

are the best possible bounds, and can be obtained from case (4) and (3), respectively.

Recently, Qi et al. [23] have given a new proof of Oppenheim's problem, and deduced the following inequalities,

$$\frac{(\pi/2)\sin(x)}{1+(2/\pi)\cos(x)} < x < \frac{(\pi+2)\sin(x)}{\pi+2\cos(x)}. \tag{5}$$

for $x \in (0, \pi/2)$. It is obvious that

$$((\pi-2)-4)(1-\cos(x)) < 0,$$

which is equivalent to

$$\frac{\pi\sin(x)}{2+(\pi-2)\cos(x)} < \frac{(\pi+2)\sin(x)}{\pi+2\cos(x)}.$$

This implies that the second inequality of (4) is better than the corresponding inequality of (5).

Our first main result, which refines the inequalities in (4), reads as follows.

Theorem 1. *For $x \in (0, \pi/2)$, we have*

$$C_\alpha < x < C_\beta, \tag{6}$$

where

$$C_\alpha = \frac{8\sin(x/2)-\sin(x)}{\alpha} \quad \textit{and} \quad C_\beta = \frac{8\sin(x/2)-\sin(x)}{\beta},$$

with the best possible constants $\alpha = 3$ and $\beta = (8\sqrt{2}-2)/\pi \approx 2.96465$.

By using Mathematica Software® [24], one can see that Theorem 1 refines the inequalities in (4) as follows:

$$Z_l(x) < C_\alpha(x), \quad \text{for} \quad x \in (0, 1.28966),$$

$$Z_u(x) < C_\beta(x), \quad \text{for} \quad x \in (0, 0.980316),$$

where Z_l and Z_u denote the lower and upper bound of (4), respectively. It is worth to mention that the first inequality in Theorem 1 was discovered heuristically by Huygens [25], here we have given a proof.

In 1970, Carlson [26] established the following inequalities,

$$\frac{6(1-x)^{1/2}}{2\sqrt{2}+(1+x)^{1/2}} < \arccos(x) < \frac{4^{1/3}(1-x)^{1/2}}{(1+x)^{1/6}}, \tag{7}$$

$0 < x < 1$. These inequalities are known as Carlson's inequalities in literature. Thereafter, several authors studied these inequalities, and gave some generalization and partial refinement, e. g., see [27 − 30]. It is interesting to observe that the Adamović-Mitrinović and Cusa-Huygens inequality (1) implies the second and the first inequality of (7), respectively, with the transformation $x = \arccos(t)$, $0 < t < \pi/4$.

For $0 < x < 1$, Guo and Qi [28, 29] gave the following inequalities,

$$\frac{\pi}{2}\frac{(1-x)^{1/2}}{(1+x)^{1/6}} < \arccos(x) < \frac{(1/2+\sqrt{2})(1-x)^{1/2}}{2\sqrt{2}+(1+x)^{1/2}}, \tag{8}$$

$$\frac{4^{1/\pi}(1-x)^{1/2}}{(1+x)^{(4-\pi)/(2\pi)}} < \arccos(x) < \frac{\pi(1-x)^{1/2}}{2(1+x)^{(4-\pi)/(2\pi)}}. \tag{9}$$

They concluded that these inequalities don't refine (7) in the whole interval $(0, 1)$ of x.

Chen et al. [27] established the lower bound for $\arccos(x)$ as follows,

$$\frac{\pi}{2}\frac{(1-x)^{(\pi+2)/\pi^2}}{(1+x)^{(\pi-2)/\pi^2}} < \arccos(x), \quad 0 < x < 1. \tag{10}$$

The inequality (10) refines the first inequality of (7) for $x \in (0, 0.345693)$.

In [30], Zhu proved that for $p \geq 1$ and $x \in (0, 1)$

$$\frac{2\cdot 3^{1/p}\sqrt{1-x}}{((2\sqrt{2})^p+(\sqrt{1+x})^p)^{\frac{1}{p}}} < \arccos(x) < \frac{2\pi\sqrt{1-x}}{((2\sqrt{2})^p+(\pi^p-2^p)(\sqrt{1+x})^p)^{\frac{1}{p}}}, \tag{11}$$

inequalities reverse for $p \in [0, 4/5]$.

We give the following theorem, which refines Carlson's inequality, see Figure 1.

Theorem 2. *For $x \in (0,1)$*

$$\frac{1}{3}\left(8\sqrt{2-\sqrt{2}\sqrt{1+x}}\right) < \arccos(x) < \frac{2^{11/6}\sqrt{1-x}}{(2+\sqrt{2}\sqrt{1+x})^{2/3}}. \tag{12}$$

We see that Theorem 2 refines the inequalities in (11) by using the Mathematica Software® [24].

In 1967, Shafer [31] proposed the following elementary inequality

$$\frac{3x}{1+2\sqrt{1+x^2}} < \arctan(x), \quad x > 0. \tag{13}$$

This inequality was proved by Grinstein, Marsh and Konhauser by different ways in [32].

In 2009, Qi et al. [33] refined the inequality (13) as follows,

$$\frac{(1+a)x}{a+\sqrt{1+x^2}} < \arctan(x) < \frac{(\pi/2)x}{4+\sqrt{1+x^2}},\ x > 0,\ -1 < a < 1/2, \tag{14}$$

$$\frac{4a(1+a^2)x}{a+\sqrt{1+x^2}} < \arctan(x) < \frac{\max\{\pi/2, 1+a\}x}{a+\sqrt{1+x^2}},\ x > 0,\ 1/2 < a < 2/\pi.$$

Recently, Alirezaei [34] has sharpened Shafer's inequality (13) by giving the following bounds for $\arctan(x)$,

$$\frac{x}{4/\pi^2+\sqrt{(1-4/\pi^2)^2+4x^2/\pi^2}} < \arctan(x) < \tag{15}$$

$$< \frac{x}{1-6/\pi^2+\sqrt{(6/\pi^2)^2+4x^2/\pi^2}},$$

for $x \in \mathbb{R}$. Graphically, it is shown that the maximum relative errors of the obtained bounds are approximately smaller than 0.27% and 0.23% for the lower and upper bound, respectively.

Our next result refines the bounds given in (15), which is illustrated in Figure 2.

Theorem 3. *For $x \in (0,1)$, we have*

$$\frac{1}{3}\left(4\sqrt{2}\sqrt{1-\frac{1}{\sqrt{1+x^2}}}-\frac{x}{\sqrt{1+x^2}}\right) < \arctan(x) < \tag{16}$$

$$< \frac{2^{2/3}x}{\sqrt{1+x^2}\left(1+1/\sqrt{1+x^2}\right)^{2/3}}.$$

§ 2. Preliminaries

For easy reference, we recall the the following Monotone l'Hôpital rule due to Anderson et al. [1, Theorem 2], which has been extremely used in literature.

Lemma 1. *For $-\infty < a < b < \infty$, let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, and be differentiable on (a, b). Let $g'(x) \neq 0$ on (a, b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \textit{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2. *The function*

$$f(x) = 4x\sin(x) + (4-x^2)\cos(x) - x^2$$

is strictly decreasing from $(0, \pi/2)$ onto $(a, 4)$, $a = \pi(8-\pi)/4 \approx 3.81578$. In particular,

$$\frac{\pi(8-\pi)/4 + x^2 - (4-x^2)\cos(x)}{4x^2} < \frac{\sin(x)}{x} < \frac{4+x^2-(4-x^2)\cos(x)}{4x^2}$$

for $x \in (0, \pi/2)$.

Proof. By differentiating and using the indentities $\sin(x) = 2\sin(x/2) \times$ $\times \cos(x/2)$ and $1-\cos(x) = 2\sin(x/2)^2$ we get

$$\begin{aligned} f'(x) &= x(2\cos(x) + x\sin(x) - 2) = \\ &= 2\sin(x/2)(x\cos(x/2) - 2\sin(x/2)) < 0. \end{aligned}$$

Hence f is strictly decreasing in $x \in (0, \pi/2)$, and the limiting values can be obtained easily. □

Lemma 3. *The following function*

$$f(x) = \frac{\sin(x) - x\cos(x)}{2\sin(x/2) - x\cos(x/2)}$$

is strictly decreasing from $(0, \pi/2)$ *onto* $(b, 4)$, $b = 2\sqrt{2}/(4-\pi) \approx 3.81578$. *In particular,*

$$\frac{2\sqrt{2}}{4-\pi}\left(2\sin\left(\frac{x}{2}\right) - x\cos\left(\frac{x}{2}\right)\right) < \sin(x) - x\cos(x) <$$
$$< 4\left(2\sin\left(\frac{x}{2}\right) - x\cos\left(\frac{x}{2}\right)\right),$$

for $x \in (0, \pi/2)$.

Proof. We get

$$\begin{aligned} f'(x) &= \frac{x\sin(x)}{2\sin(x/2) - x\cos(x/2)} - \frac{x\sin(x/2)(\sin(x) - x\cos(x))}{2(2\sin(x/2) - x\cos(x/2))^2} = \\ &= \frac{x\sin(x/2)(x(2+\cos(x)) - 3\sin(x))}{4 - (4x\sin(x) + (4-x^2)\cos(x) - x^2)}, \end{aligned}$$

which is negative by the second inequality of (1) and Lemma 2. This implies that f is strictly decreasing in $x \in (0, \pi/2)$, and by applying l'Hôpital rule we get the limiting values. □

Lemma 4. *The following function*

$$g(x) = \frac{8\sin(x/2) - \sin(x)}{x}$$

is strictly decreasing from $(0, \pi/2)$ *onto* $(\beta, 3)$, $\beta = (8\sqrt{2}-2)/\pi \approx 2.96465$. *Also, the function*

$$f(z) = \frac{8\sin(z)}{6z + \sin(2z)}$$

is strictly decreasing from $(0, \pi/4)$ *onto* $(1, \gamma)$, $\gamma = 8\sqrt{2}/(2+3\pi) \approx$ ≈ 0.99028.

Proof. We get

$$\begin{aligned} g'(x) &= \frac{4\cos(x/2) - \cos(x)}{x} - \frac{8\sin(x/2) - \sin(x)}{x^2} = \\ &= \frac{\sin(x) - x\cos(x) - 4(2\sin(x/2) - x\cos(x/2))}{x^2}, \end{aligned}$$

which is negative by Lemma 3. Thus, g is strictly decreasing in $x \in$ $\in (0, \pi/2)$, and the limiting values follow from the l'Hôpital rule.

Next, let $f = f_1(z)/f_2(z)$, $z \in (0, \pi/4)$, where $f_1(z) = 8\sin(z)$ and $f_2(z) = 6z + \sin(2z)$. We get

$$\frac{f_1'(z)}{f_2'(z)} = \frac{4\cos(z)}{1+\cos(z)^2} = f_3(z).$$

One has,

$$f_3'(z) = -\frac{\sin(z)^3}{(3+\cos(2z))^2} < 0.$$

Clearly, $f_1(0) = f_2(0) = 0$, hence by Lemma 1 f is strictly decreasing, and we get

$$\lim_{z\to\pi/4} f(z) = 8\sqrt{2}/(2+3\pi) \approx 0.99028 < f(z) < \lim_{z\to 0} f(z) = 1,$$

this implies the proof. □

§ 3. Proof of Theorems

Proof of Theorem 1. The proof follows easily from Lemma 4. □

Corollary. *For $x \in (0, \pi/2)$, we have*

$$\frac{8\sin(x/2) - \sin(x)}{\beta} < \frac{8\sin(x/2) - \beta\sin(x)}{\gamma},$$

where β and γ are as in Lemma 4.

Proof. For $x \in (0, \pi/2)$, let $f(x) = \sin(x/2)/\sin(x)$. One has

$$f'(x) = \frac{\sin(x/2)^3}{\sin(x)^2} > 0.$$

Hence, f is strictly increasing, and

$$\frac{1}{2} = \lim_{x\to 0} f(x) < f(x) < \lim_{x\to\pi/2} f(x) = \frac{1}{\sqrt{2}}.$$

We observe that

$$\frac{\sin(x/2)}{\sin(x)} < \frac{1}{\sqrt{2}} = -\frac{2-8\sqrt{2}+3\pi}{16-2\sqrt{2}-3\sqrt{2}\pi},$$

which is equivalent to

$$\frac{(16-2\sqrt{2}-3\sqrt{2}\pi)\sin(x/2)+(2-8\sqrt{2}+3\pi)\sin(x)}{24\sqrt{2}-6}>0.$$

This is equivalent to the desired inequality. □

Proof of Theorem 2. Let $x=\cos(2t)$ for $0<t<\pi/4$. Then $\arccos(x)/2=t$, and clearly $0<x<1$. From (2) and (6) we have

$$\frac{8\sin(t/2)-\sin(t)}{3}<t<\frac{2^{2/3}\sin(t)}{(1+\cos(t))^{2/3}}, \tag{17}$$

for $t\in(0,\pi/2)$. Replacing $\cos(t)$, $\sin(t)$ and t by $\sqrt{(1+x)/2}$, $\sqrt{(1-x)/2}$ and $\arccos(x)/2$, respectively, in (17), we get

$$\frac{8((1-\sqrt{(1+x)/2})/2)^{1/2}-\sqrt{(1-x)/2}}{3}<\frac{\arccos(x)}{2}<$$

$$<\frac{2^{2/3}\sqrt{(1-x)/2}}{(1+\sqrt{(1+x)/2})^{2/3}}.$$

After simplification we get the desired inequality. □

Proof of Theorem 3. Next, let $x=\tan(t)$, $t\in(0,\pi/2)$ and $x\in(0,1)$. Then $t=\arctan(x)$, and by using the identity $1+\tan(t)^2=\sec(t)^2$ we get

$$\sin(t)=\frac{x}{\sqrt{1+x^2}}=m,\quad \text{and}\quad \sin\left(\frac{t}{2}\right)=\left(\frac{\sqrt{1+x^2}-1}{2\sqrt{1+x^2}}\right)=n.$$

We get the desired inequalities if we replace, t, $\sin(t)$, $\sin(t/2)$ by $\arctan(x)$, m, n, respectively, in (17). □

For the comparison of the bounds of $\arccos(x)$ and $\arctan(x)$ given in (7) and (12) with the corresponding bounds appear in Theorem 2 and 3, we use the the graphical method, see Figure 1 and 2.

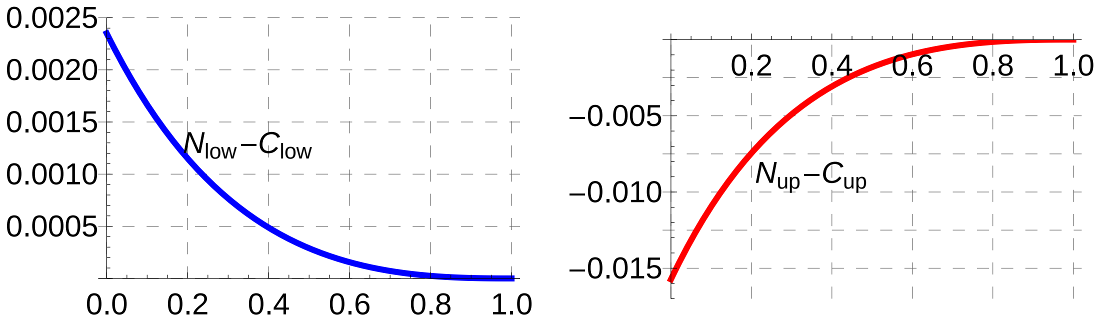

Figure 1: We denote the left-hand sides of (7) and (12) by C_{low} and N_{low}, respectively, while the right-hand sides by C_{up} and N_{up}, respectively. It is clear that (12) refines the Carlson's inequality (7)

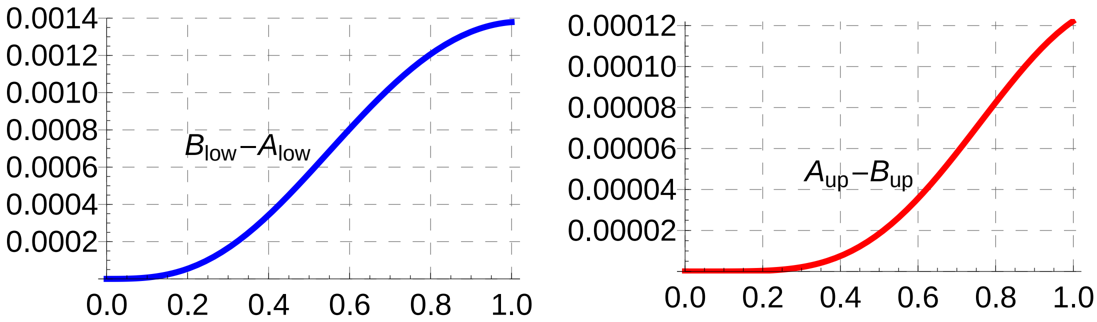

Figure 2: We denote the lower and upper bound of (16) by B_{low} and B_{up}, respectively, while the corresponding bounds of (15) are denoted by A_{low} and A_{up}. The differences $B_{low} - A_{low}$, $A_{up} - B_{up}$ are positive, this implies that the inequalities in (16) are better than the corresponding inequalities of (15)

References

[1] Anderson G. D., Vamanamurthy M. K., Vuorinen M. *Monotonicity Rules in Calculus*. Amer. Math. Month, 2006, vol. 113, no. 9, pp. 805–816.

[2] Klén R., Visuri M., Vuorinen M. *On Jordan type inequalities for hyperbolic functions*. J. Ineq. Appl, vol. 2010. Article ID 362548, pp. 14.

[3] Malesevic B. J. *One method for proving inequalities by computer*. J. Ineq. Appl, 2007, vol. 8. Article ID 78691.

[4] Malesevic B. J. *An application of* λ*-method on inequalities of Shafer–Fink type*. Math. Ineq. Appl., 2007, vol. 10 (3), pp. 529–534.

[5] Neuman E. and Sándor J. *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities*. Math. Inequal. Appl., 2010, vol. 13, no. 4, pp. 715–723.

[6] Neuman E. and Sándor J. *Optimal inequalities for hyperbolic and trigonometric functions*. Bull. Math. Analysis Appl., 2011, vol. 3, no. 3, pp. 177–181.

[7] Sándor J. *Two sharp inequalities for trigonometric and hyperbolic functions*. Math. Inequal. Appl., 2012, vol. 15, no. 2, pp. 409–413.

[8] Sándor J. *Sharp Cusa-Huygens and related inequalities*. Notes Number Theory Discrete Math., 2013, vol. 19, no. 1, pp. 50–54.

[9] Sándor J., and Oláh-Gál R. *On Cusa-Huygens type trigonometric and hyperbolic inequalities*. Acta Univ. Sapientiae, Mathematica, 2012, vol. 4, no. 2, pp. 145–153.

[10] Sándor J. *Trigonometric and hyperbolic inequalities*. http://arxiv.org/abs/1105.0859.

[11] Sun J., and Zhu L. *Six new Redheffer-type inequalities for circular and hyperbolic functions*. Comput. Math. Appl., 2008, vol. 56, no. 2, pp. 522–529.

[12] Zhu L. *Sharpening Redheffer-type inequalities for circular functions*. App. Math. Lett., 2009, vol. 22, pp. 743–748.

[13] Anderson G. D., Vuorinen M., and Zhang X. *Topics in special functions III*. http://arxiv.org/abs/1209.1696.

[14] Mitrinović D. S. *Analytic Inequalities*. Springer-Verlag, Berlin, 1970.

[15] Huygens C. *Oeuvres Completes 1888–1940*. Socié te Hollondaise des Science, Haga.

[16] Lv Y., Wang G. and Chua Y. *A note on Jordan type inequalities for hyperbolic functions*. Appl. Math. Lett., 2012, vol. 25, pp. 505–508.

[17] Qi F., Cui L.-H., and Xu S.-L. *Some inequalities constructed by Tchebysheff's integral inequality*. Math. Inequal. Appl., 1999, vol. 2, no. 4, pp. 517–528.

[18] Ogilvy C. S., Oppenheim A., Ivanoff V. F., Ford Jr. L. F., Fulkerson D. R., and Narayanan Jr. V. K. *Elementary problems and solutions: problems for solution: E1275-E1280*. Amer. Math. Monthly, 1957, vol. 64, no. 7, pp. 504–505.

[19] Oppenheim A. *E1277*. Amer. Math. Monthly, 1957, vol. 64, no. 6, pp. 504.

[20] Zhu L. *A solution of a problem of Oppeheim*. Math. Inequal. Appl., 2007, vol. 10, no. 1, pp. 57–61.

[21] Baricz Á. *Functional inequalities involving Bessel and modified Bessel functions of the first kind*. Exposition. Math., 2008, vol. 26, no. 3, pp. 279–293.

[22] Baricz Á. *Some inequalities involving generalized Bessel functions*. Math. Ineq. Appl., 2007, vol. 10, no. 4, pp. 827–842.

[23] Qi F., Luo Q-.M., and Guo B.-N. *A simple proof of Oppenheim's double inequality relating to the cosine and sine functions*. J. Math. Inequal., 2012, vol. 6, no. 4, pp. 645–654.

[24] Ruskeepää H. Mathematica®. Navigator. 3rd ed. Academic Press, 2009.

[25] Campan F. T. *The history of number pi (Romanian)*. 2nd ed. Albatros ed., Romania, 1977.

[26] Carlson B. C. *Inequality for a symmetric elliptic integral*. Proc. Amer. Math. Soc., 1970, vol. 25 (3), pp. 698–703.

[27] Chen C.-P., Cheung W.-S., and Wang W. *On Shafer and Carlson Inequalities*. J. Ineq. Appl., 2011, vol. 2011. Article ID 840206, 10 p.

[28] Guo B.-N., and Qi F. *Sharpening and generalizations of Carlson's inequality for the arc cosine function*. Hacettepe Journal of Mathematics and Statistics, 2010, vol. 39 (3), pp. 403–409.

[29] Zhao J.-L., Wei C.-F., Guo B.-N., and Qi F. *Sharpening and generalizations of Carlson's double inequality for the arc cosine function*. Hacettepe Journal of Mathematics and Statistics, 2012, vol. 41, no. 2, pp. 201–209.

[30] Zhu L. *A source of inequalities for circular functions*. Comput. Math. Appl., 2009, vol. 58, pp. 1998–2004.

[31] Shafer R. E. *Problem E1867*. Amer. Math. Monthly, 1967, vol. 74, no. 6, pp. 726–727.

[32] Shafer R. E., Grinstein L. S., Marsh D. C. B., and Konhauser J. D. E. *Problems and solutions: solutions of elementary problems: E1867*. Amer. Math. Monthly, 1967, vol. 74, no. 6, pp. 726–727.

[33] Qi F., Zhang Sh.-Q., and Guo B.-N. *Sharpening and generalizations of Shafers inequality for the arc tangent function*. J. Ineq. Appl., 2009. Article ID 930294. 10 p.

[34] Alirezaei G. *A sharp double inequality for the inverse tangent Function.* http://arxiv.org/pdf/1307.4983.pdf.

[35] Abramowitz M., and Stegun I., eds. *Handbook of mathematical functions with formulas, graphs and mathematical tables.* National Bureau of Standards, Dover, New York, 1965.

[36] Anderson G. D., Vamanamurthy M. K., and Vuorinen M. *Inequalities of quasiconformal mappings in the space.* Pacific J. Math., 1993, vol. 160, no. 1, pp. 1–20.

[37] Baricz Á., and Zhu L. *Extension of Oppenheims problem to Bessel functions.* J. Inequal. Appl., 2007. Article ID 82038. 7 p.

[38] Guo B.-N., Luob Q.-M., and Qi F. *Sharpening and generalizations of Shafer–Finks double inequality for the arc sine function.* Filomat, 2013, vol. 27:2, pp. 261–265. DOI 10.2298/FIL1302261G.

[39] Larsson L. *A new Carlson type inequality.* Math. Ineq. Appl., 2003, vol. 6, no. 1, pp. 55–79.

[40] Oppenheim A. and Carver W. B. *Elementary problems and solutions: solutions: E1277.* Amer. Math. Monthly, 1958, vol. 65, no. 3, pp. 206–209.

[41] Qi F., Zhang S.-Q., and Guo B.-N. *Sharpening and generalizations of Shafers inequality for the arc tangent function.* J. Ineq. Appl, vol. 2009.

[42] Redheffer R. *Problem 5642.* Amer. Math. Monthly, 1969, vol. 76, pp. 422.

[43] Sándor J. *On new refinements of Kober's and Jordan's trigonometric inequalities.* Notes Number Theory Discrete Math., 2013, vol. 19, no. 1, pp. 73–83.

[44] Zhu L. *On Shafer–Fink inequalities.* Math. Ineq. Appl., 2005, vol. 8 (4), pp. 571–574.

[45] Zhu L. *On Shafer–Fink-type inequality.* J. Ineq. Appl., 2007, vol. 4. Article ID 67430.

The work is received on May 7, 2014.

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