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On Cartan's Conformally Deformable Hypersurfaces

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Starting in 1916, E. Cartan devoted five years to the study of isometric, conformal, and projective deformations of submanifolds by the use of the method of moving frames. In the first of a series of papers ([2]; see also [9]), he locally classified the hypersurfaces M^n ($n \geq 3$) in flat Euclidean space $\mathbb{R}^{n+1}$ that are isometrically deformable. Shortly after, he followed with a long and more difficult paper [3] where he classified conformally deformable Euclidean hypersurfaces of dimension $n \geq 5$. The special cases $n = 4, 3$ were subsequently treated by Cartan in [4] and [5]. In all cases, it turns out that hypersurfaces are generically conformally rigid.

Quite similarly to the isometric case, conformally deformable hypersurfaces of dimension $n \geq 5$, other than the conformally flat ones, can be separated into four classes: surface-like, conformally ruled, those having precisely a continuous 1-parameter family of deformations, and those that admit only one deformation. Cartan's main result is a parametric description of the hypersurfaces in the last two classes as envelopes of 2-parameter families of spheres determined by a certain partial differential equation together with an additional condition.

Our first and main achievement is a nonparametric classification of all conformally deformable Euclidean hypersurfaces of dimension $n \geq 5$ by means of a rather simple geometric construction. Roughly speaking, we show that any hypersurface M^n in $\mathbb{R}^{n+1}$ ($n \geq 5$) that admits a conformal deformation $\tilde{M}^n$ can be locally characterized as the intersection $M^n = N^{n+1} \cap \mathbb{V}$ of a flat $(n+1)$-dimensional Riemannian submanifold of the standard flat Lorentzian space $\mathbb{L}^{n+3}$ with the light cone $\mathbb{V}$ of $\mathbb{L}^{n+3}$. Moreover, $\tilde{M}^n$ is obtained by projecting M^n onto the standard model of $\mathbb{R}^{n+1}$ as an embedded hypersurface of $\mathbb{V}$. In addition, we characterize how the conformally deformable hypersurfaces that are conformally congruent to isometrically deformable ones can be produced by the procedure just described. They are the ones obtained from the flat Riemannian submanifolds whose relative nullity leaves are open subsets of affine subspaces in $\mathbb{L}^{n+3}$ with a common point in $\mathbb{V}$.

For reasons we can only guess (perhaps uncertainty about the very existence of examples), Cartan's statement in the introduction of [3] completely ignores the discrete last class (although the possibility of its existence arises in his proof; see Sec. 41). This raises the question of whether the discrete class is nonempty. The

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similar problem for the isometric case was considered in [9], where many hypersurfaces admitting only one isometric deformation were explicitly described. It follows from our main result that such hypersurfaces admit no further conformal deformations, thus giving a positive answer to the question.

We continue the paper with our own version of Cartan's classification. Our result provides a parametric description of conformally deformable hypersurfaces in the spirit of the one given by Sbrana [17] for the isometric case. In particular, this allows us to produce explicit examples admitting 1-parameter families of deformations. A classification of the Euclidean hypersurfaces that admit conformal deformations preserving the Gauss map has been given in [13].

Further results are given in the last section. First, we derive the Sbrana–Cartan classification of isometrically deformable hypersurfaces from our main results. Then, we characterize the flat Riemannian submanifolds of Lorentzian space that give rise to the different types of conformally deformable hypersurfaces in Cartan's classification. Finally, we show that the n-dimensional conformally deformable hypersurfaces obtained from flat Riemannian submanifolds that are compositions of flat hypersurfaces are, precisely, the ones that arise as intersections between flat and conformally flat hypersurfaces in either $\mathbb{R}^{n+2}$ or $\mathbb{L}^{n+2}$. Moreover, we prove that this last class is invariant under conformal deformations.

1. The Nonparametric Classification

In the standard flat Lorentzian space $\mathbb{L}^{n+3}$ with inner product

$$\langle X, Y\rangle = -x_1y_1 + x_2y_2 + \cdots + x_{n+3}y_{n+3},$$

we consider its upper light cone

$$\mathbb{V} = \{X \in \mathbb{L}^{n+3} : \langle X, X\rangle = 0,\ x_1 > 0\}$$

endowed with the induced degenerate metric. The intersection $\mathbb{E}_w = \mathcal{H}_w \cap \mathbb{V}$ with the affine hyperplane

$$\mathcal{H}_w = \{X \in \mathbb{L}^{n+3} : \langle X, w\rangle = 1\}$$

that is orthogonal to $w \in \mathbb{V}$ gives rise to the *standard model* of the $(n+1)$-dimensional Euclidean space into the light cone $\mathbb{V}$. It is the image of the isometric embedding $j_{\mathcal{B}_w} : \mathbb{R}^{n+1} \to \mathbb{V}$ given by $j_{\mathcal{B}_w}(x) = (1, x, \|x\|^2)$ with respect to a *pseudo-orthonormal* basis $\mathcal{B}_w = \{e_1, \dots, e_{n+3} = -w/2\}$ of $\mathbb{L}^{n+3}$ such that

$$\begin{aligned} &\|e_1\| = 0 = \|e_{n+3}\|, \quad \langle e_1, e_{n+3}\rangle = -1/2, \\ &\text{and} \quad \langle e_i, e_j\rangle = \delta_{i,j} \quad \text{if } i \neq 1,\ n+3. \end{aligned} \tag{1}$$

If $\tilde{\mathcal{B}}_w = \{\tilde{e}_1, \dots, \tilde{e}_{n+3} = -w/2\}$ is another such basis, let $T \in \mathbb{O}_1(n+3)$ be the orthogonal transformation of $\mathbb{L}^{n+3}$ given by $Te_i = \tilde{e}_i$ for $1 \le i \le n+3$. Then T maps $\mathbb{E}_w$ onto $\mathbb{E}_w$ and hence induces an isometry $\mathcal{T}$ of $\mathbb{R}^{n+1}$ such that $T \circ j_{\mathcal{B}_w} = j_{\mathcal{B}_w} \circ \mathcal{T}$. Thus $j_{\tilde{\mathcal{B}}_w} = T \circ j_{\mathcal{B}_w} = j_{\mathcal{B}_w} \circ \mathcal{T}$. For this reason, from now on we use the shorter notation j_w to stand for $j_{\mathcal{B}_w}$ for any basis $\mathcal{B}_w$ as just described. Notice that, given another vector $\tilde{w} \in \mathbb{V}$, there is a $T \in \mathbb{O}_1(n+3)$ mapping w to $\tilde{w}$ such that $j_{\tilde{w}} = T \circ j_w$. Since T restricts to an isometry of $\mathbb{V}$, it follows that j_w and $j_{\tilde{w}}$ are *congruent* isometric immersions into $\mathbb{V}$. Recall that any isometry of (an open subset of) $\mathbb{V}$ is the restriction of a orthogonal transformation of $\mathbb{L}^{n+3}$.

Let $h\colon M^m \to \mathbb{V}$ $(m \leq n+1)$ be a conformal immersion of a Riemannian manifold. Denote by $\varphi_h > 0$ its conformal factor, which is given by

$$\langle h_* X, h_* Y\rangle = \varphi_h^2 \langle X, Y\rangle.$$

Using that $\langle h, h\rangle = 0$ and hence that $\langle h_* X, h\rangle = 0$ for any $X \in TM$, it follows that for any smooth function $\lambda \in C^\infty(M)$ the map λh is also conformal with conformal factor $\lambda\varphi_h$. In particular, any conformal immersion $g\colon M^m \to \mathbb{R}^{n+1}$ can be made into an isometric immersion $\mathcal{L}_w(g)\colon M^m \to \mathbb{V}$ by setting

$$\mathcal{L}_w(g) = (1/\varphi_g) j_w \circ g,$$

where $w \in \mathbb{V}$ is arbitrary. The observation in the preceding paragraph shows that different choices of $w \in \mathbb{V}$ give rise to congruent isometric immersions into $\mathbb{V}$.

Conversely, let $G\colon M^m \to \mathbb{V}$ $(m \leq n+1)$ be an isometric immersion. For $m = n+1$, we assume the existence of a vector $w \in \mathbb{V}$ with $\langle G, w\rangle > 0$ everywhere; that is, $G(M)$ does not intersect $\mathbb{R}_w = \{tw;\ t > 0\}$. If $m \leq n$, there always exists such a vector. Otherwise, the immersion $\bar{G}\colon M^m \times \mathbb{R}_+ \to \mathbb{V}$ given by $\bar{G}(x,t) = tG(x)$ would be surjective, a contradiction. Let us define $\mathcal{C}_w(G)\colon M^m \to \mathbb{R}^{n+1}$ by

$$j_w \circ \mathcal{C}_w(G) = \Pi_w \circ G,$$

where $\Pi_w\colon \mathbb{V} \setminus \mathbb{R}_w \to \mathbb{V}$ is the projection onto $\mathbb{E}_w$ given by $\Pi_w(x) = x/\langle x, w\rangle$. Since Π_w is conformal with conformal factor $\varphi_{\Pi_w}(x) = 1/\langle x, w\rangle$, it follows that $\mathcal{C}_w(G)$ is also conformal with conformal factor $\varphi_{\Pi_w} \circ G = \langle G, w\rangle^{-1}$. For any vector $\tilde{w} \in \mathbb{V}$ with $G(M) \cap \mathbb{R}_{\tilde{w}} = \emptyset$, define $T\colon \mathbb{V} \setminus \mathbb{R}_{\tilde{w}} \to \mathbb{V}$ by $T(x) = \langle x, w\rangle \Pi_{\tilde{w}}(x)$. Then T is conformal and maps $\mathbb{E}_w$ onto $\mathbb{E}_{\tilde{w}}$. Hence, T induces a conformal transformation C of $\mathbb{R}^{n+1}$ (i.e., an inversion up to a dilation and a rigid motion), so that $j_{\tilde{w}} \circ C = T \circ j_w$. Then,

$$\begin{aligned} j_{\tilde{w}} \circ C \circ \mathcal{C}_w(G) &= T \circ j_w \circ \mathcal{C}_w(G) \\ &= T \circ \Pi_w \circ G = \Pi_{\tilde{w}} \circ G = j_{\tilde{w}} \circ \mathcal{C}_{\tilde{w}}(G). \end{aligned}$$

Since $j_{\tilde{w}}$ is an embedding, it follows that $\mathcal{C}_{\tilde{w}}(G) = C \circ \mathcal{C}_w(G)$, that is, $\mathcal{C}_{\tilde{w}}(G)$ is *conformally congruent* to $\mathcal{C}_w(G)$. Observe that $\mathcal{L}_w(\mathcal{C}_w(G)) = G$ and $\mathcal{C}_w(\mathcal{L}_w(g)) = g$ for any conformal immersion $g\colon M^m \to \mathbb{R}^{n+1}$ and any isometric immersion $G\colon M^m \to \mathbb{V}$ with $G(M) \cap \mathbb{R}_w = \emptyset$.

Given a hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$, a conformal immersion $g\colon M^n \to \mathbb{R}^{n+1}$ not conformally congruent to f is said to be a *conformal deformation* of f. We say that g is *nowhere conformally congruent* to f if it is not conformally congruent to f on any open subset of M^n. If any other conformal immersion g is conformally congruent to f, then f is said to be *conformally rigid*. A similar terminology is used in the isometric case.

We say that an isometric immersion $H\colon V \to \mathbb{L}^{n+3}$ of an open subset $V \subset \mathbb{R}^{n+1}$ is of *trivial type* if there exist a smooth real function $\phi \in C^\infty(V)$ and a basis of $\mathbb{L}^{n+3}$ as in (1) such that H is given parametrically by

$$H(x) = \phi(x)e_1 + e_{n+3} + \sum_{i=1}^{n+1}(x_i + k_i)e_{i+1},$$

where $x = (x_1, \dots, x_{n+1})$ and $k_i \in \mathbb{R}$.

The pointwise structure of the second fundamental form $\alpha_H\colon TV \times TV \to T^\perp V$ of an isometric immersion $H\colon V \to \mathbb{L}^{n+3}$ of an open subset $V \subset \mathbb{R}^{n+1}$ was determined in [15, Thm. 2]. It follows from this result (cf. Lemma 19 in this paper) and Proposition 20(a) that if H is not of trivial type on any open subset of V then it has *relative nullity* $\nu_H \geq n-1$ on an open dense subset of V. Recall that $\nu_H(x)$ is the dimension of the kernel $\Delta_H(x)$ of $\alpha_H(x)$. It is then a standard fact that Δ_H is an integrable distribution with totally geodesic leaves on any open subset where ν_H is constant. Moreover, the leaves are mapped by H onto open subsets of affine subspaces of $\mathbb{L}^{n+3}$. We say that an isometric immersion $H\colon V \to \mathbb{L}^{n+3}$ with constant relative nullity and the light cone $\mathbb{V}$ are in *general position* if the relative nullity leaves of H through any point of $H(V) \cap \mathbb{V}$ are transversal to $\mathbb{V}$.

We are now in position to state our main result. For an embedding $H\colon V \subset \mathbb{R}^{n+1} \to \mathbb{L}^{n+3}$, we write H^{-1} to denote the inverse of $H\colon V \to H(V)$.

THEOREM 1. *Let $H\colon V \subset \mathbb{R}^{n+1} \to \mathbb{L}^{n+3}$ be an isometric embedding with constant relative nullity ν_H that is not of trivial type on any open subset. Assume that H and $\mathbb{V}$ are in general position and set $M^n = H(V) \cap \mathbb{V}$. Endow M^n with the Riemannian metric induced by the inclusion $i\colon M^n \to \mathbb{V}$ and set $f_H = H^{-1}\big|_{M^n}\colon M^n \to \mathbb{R}^{n+1}$. Then, for any $w \in \mathbb{V}$ such that $M^n \cap \mathbb{R}_w = \emptyset$, the conformal immersion $g_H = \mathcal{C}_w(i)$ is nowhere conformally congruent to f_H.*

Conversely, let $f\colon M^n \to \mathbb{R}^{n+1}$ $(n \geq 5)$ be a hypersurface with no flat points and let $g\colon M^n \to \mathbb{R}^{n+1}$ be a conformal immersion. Then there is a dense union of open subsets $\mathcal{U} = \bigcup_{i=1}^3 \mathcal{U}_i$ such that f has a principal curvature of multiplicity at least $n-2$ on $\mathcal{U}_2 \cup \mathcal{U}_3$ that vanishes on $\mathcal{U}_2$ and such that:

(i) *g is conformally congruent to f on any connected component of $\mathcal{U}_1$;*
(ii) *g is conformally congruent to an isometric deformation of f on any connected component of $\mathcal{U}_2$; and*
(iii) *at each point of $\mathcal{U}_3$ there is an open neighborhood $U \subset \mathcal{U}_3$, an isometric embedding $H\colon V \to \mathbb{L}^{n+3}$ transversal to $\mathbb{V}$ of an open subset $V \subset \mathbb{R}^{n+1}$, and an isometry $\mathcal{T}\colon U \to H(V) \cap \mathbb{V}$ such that*

$$f\big|_U = f_H \circ \mathcal{T} \quad \text{and} \quad g\big|_U = g_H \circ \mathcal{T}.$$

REMARK 2. The assumption that H is an embedding and not just an immersion could be avoided by defining $M^n = H^{-1}(\mathbb{V})$, f_H as the inclusion map of M^n into $\mathbb{R}^{n+1}$, and $g_H = \mathcal{C}_w\big(H\big|_{M^n}\big)$. Our choice, however, was made so as not to lose the geometrical nature of our characterization of conformally deformable hypersurfaces of $\mathbb{R}^{n+1}$ as *intersections* of flat $(n+1)$-dimensional submanifolds of $\mathbb{L}^{n+3}$ with $\mathbb{V}$.

The following criterion (due to Cartan [3]) for conformal rigidity is a consequence of Theorem 1.

COROLLARY 3. *A hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ $(n \geq 5)$ is conformally rigid if all principal curvatures have multiplicity less than $n-2$ everywhere.*

It is a classical result due to Schouten (cf. [7]) that an n-dimensional Euclidean hypersurface has a principal curvature of multiplicity at least $n-1$ everywhere if and only if it is conformally flat and hence highly conformally deformable. By Corollary 3, if a Euclidean hypersurface of dimension $n \geq 5$ has principal curvatures of multiplicity less than $n-1$ everywhere and admits a conformal nowhere conformally congruent deformation, then it must have a principal curvature λ of constant multiplicity $n-2$ everywhere. We call it a *Cartan hypersurface* if, in addition, λ is nowhere zero. Then, it is a standard fact that the corresponding eigenspaces form an integrable distribution whose leaves are open subsets of round spheres in $\mathbb{R}^{n+1}$. By Theorem 1, Cartan hypersurfaces are precisely those f_H with $\nu_H = n-1$ on an open neighborhood of M^n, where the spherical leaves correspondent to the principal curvature of multiplicity $n-2$ are the intersections of the relative nullity leaves of H with the light cone.

The hypersurfaces that play the role of Cartan hypersurfaces in the isometric case were named *Sbrana–Cartan hypersurfaces* in [9]. They all have everywhere a zero principal curvature of multiplicity $n-2$, and the corresponding eigenspaces form an integrable distribution whose leaves are open subsets of affine subspaces of $\mathbb{R}^{n+1}$.

COROLLARY 4. *A hypersurface is conformally but not isometrically congruent to a Sbrana–Cartan hypersurface if and only if it is a Cartan hypersurface such that the spheres in $\mathbb{R}^{n+1}$ containing the spherical leaves correspondent to the principal curvature of multiplicity $n-2$ have a common point. Moreover, any conformal* (*nowhere conformally congruent*) *deformation of the hypersurface is conformally congruent to an isometric* (*nowhere congruent*) *deformation of the Sbrana–Cartan hypersurface.*

Many explicit examples of Sbrana–Cartan hypersurfaces in the discrete class of Sbrana–Cartan's classification were constructed in [9]. It follows from Corollary 4 that any hypersurface conformally congruent to one of these examples also belongs to the discrete class in Cartan's classification.

Corollary 4 also yields the following characterization of the isometric immersions H in Theorem 1 that give rise to hypersurfaces conformally congruent to Sbrana–Cartan hypersurfaces.

COROLLARY 5. *The hypersurface f_H constructed in Theorem 1 is conformally but not isometrically congruent to a Sbrana–Cartan hypersurface if and only if* (a) $\nu_H = n-1$ *on an open neighborhood W of M^n and* (b) *the relative nullity leaves of H on W are open subsets of affine subspaces in $\mathbb{L}^{n+3}$ with a common point in the light cone $\mathbb{V}$.*

2. Proof of Theorem 1

We first prove the following basic result.

PROPOSITION 6. *Let $f, g\colon M^m \to \mathbb{R}^{n+1}$ be conformal immersions, and set $F = \mathcal{L}_v(f)$ and $G = \mathcal{L}_w(g)$ for $v, w \in \mathbb{V}$. Then there exists a conformal transformation ν of $\mathbb{R}^{n+1}$ such that $g = \nu \circ f$ if and only if there exists an isometry Λ of $\mathbb{V}$ such that $G = \Lambda \circ F$.*

Proof. Assume first that $g = \nu \circ f$ for some conformal transformation ν of $\mathbb{R}^{n+1}$. Then the conformal factors of f, g, and ν are related by $\varphi_g = (\varphi_\nu \circ f)\varphi_f$. Since Π_v is conformal with conformal factor $\varphi_{\Pi_v}(x) = 1/\langle x, v\rangle$, the map

$$j_w \circ \nu \circ j_v^{-1} \circ \Pi_v \colon \mathbb{V} \setminus \mathbb{R}_v \to \mathbb{V} \setminus \mathbb{R}_w$$

is also conformal with conformal factor $(\varphi_\nu \circ j_v^{-1} \circ \Pi_v)\varphi_{\Pi_v}$. Therefore, the map $\Lambda\colon \mathbb{V} \setminus \mathbb{R}_v \to \mathbb{V} \setminus \mathbb{R}_w$ given by

$$\Lambda(x) = (\langle x, v\rangle/(\varphi_\nu \circ j_v^{-1} \circ \Pi_v)(x))(j_w \circ \nu \circ j_v^{-1} \circ \Pi_v)(x)$$

is an isometry, which extends to an isometry $\Lambda\colon \mathbb{V} \to \mathbb{V}$ by setting $\Lambda(tv) = tw$ for any $t > 0$. Moreover,

$$\Lambda \circ F = \Lambda \circ ((1/\varphi_f) j_v \circ f) = (1/(\varphi_\nu \circ f)\varphi_f) j_w \circ \nu \circ f = (1/\varphi_g) j_w \circ g = G.$$

Suppose that $\Lambda \circ F = G$ for some isometry $\Lambda\colon \mathbb{V} \to \mathbb{V}$. Set $V = j_v^{-1}(\mathbb{V} \setminus \mathbb{R}_w)$ and define $\nu\colon V \to \mathbb{R}^{n+1}$ by

$$j_w \circ \nu = \Pi_w \circ \Lambda \circ j_v.$$

Then ν is conformal with $\varphi_\nu = \varphi_{\Pi_w} \circ \Lambda \circ j_v = 1/\langle \Lambda \circ j_v, w\rangle$. Moreover,

$$\begin{aligned} j_w \circ \nu \circ f &= \Pi_w \circ \Lambda \circ j_v \circ f = \varphi_f \Pi_w \circ \Lambda \circ F \\ &= \varphi_f \Pi_w \circ G = (\varphi_f/\varphi_g)\Pi_w \circ j_w \circ g = j_w \circ g; \end{aligned}$$

hence $g = \nu \circ f$. $\square$

Proof of Theorem 1. Assume that g_H is conformally congruent to f_H on some open subset $U \subset M^n$. Set $F = \mathcal{L}_w(f_H)$ and $G = \mathcal{L}_w(g_H)$. Then $F = j_w \circ f_H$ because f_H is isometric and $G = \mathcal{L}_w(\mathcal{C}_w(i)) = i$. By Proposition 6, there exists an isometry $\Lambda\colon \mathbb{V} \to \mathbb{V}$ such that $\Lambda \circ F\big|_U = G\big|_U$. As already pointed out, the isometry Λ is the restriction of a orthogonal transformation T of $\mathbb{L}^{n+3}$. Let us still denote by F and G the maps $k \circ F$ and $k \circ G$, where $k\colon \mathbb{V} \to \mathbb{L}^{n+3}$ is the inclusion map. Then

$$H \circ f_H\big|_U = G\big|_U = T \circ F\big|_U = j_v \circ f_H\big|_U,$$

where $v = T(w)$. Hence, $U = H(f_H(U)) = j_v(f_H(U)) \subset \mathbb{E}_v$.

Let $W \subset V$ be an open neighborhood of $H^{-1}(U)$ such that $H(W) \cap \mathbb{V} \subset U$. For each leaf of relative nullity σ of H in W, we have that $H(\sigma)$ intersects $\mathbb{V}$ transversally and

$$H(\sigma) \cap \mathbb{V} \subset U \subset \mathcal{H}_v.$$

Since $H(\sigma)$ is an open subset of an affine subspace of $\mathbb{L}^{n+3}$, it follows that $H(\sigma) \subset \mathcal{H}_v$. Thus $H(W) \subset \mathcal{H}_v$. The following result yields a contradiction with our assumption that H is nowhere of trivial type and so completes the proof of the direct statement.

PROPOSITION 7. *An isometric immersion $H : V \subset \mathbb{R}^{n+1} \to \mathbb{L}^{n+3}$ is of trivial type if and only if there exists $v \in \mathbb{V}$ such that $H(V) \subset \mathcal{H}_v$.*

Proof. Assume that $H(V) \subset \mathcal{H}_v$ for some $v \in \mathbb{V}$. Let $e_1 = -v/2, \dots, e_{n+3}$ be a basis of $\mathbb{L}^{n+3}$ as in (1). Then we may write

$$H(x) = \phi_1(x)e_1 + e_{n+3} + \sum_{i=1}^{n+1} \phi_i(x)e_{i+1}, \quad x = (x_1, \dots, x_{n+1}),$$

for some $\phi_1, \dots, \phi_{n+1} \in C^\infty(V)$. Since H is isometric, it is easily seen that $e_2, \dots, e_{n+2}$ can be chosen so that $\phi_i(x) = x_i + k_i$ ($k_i \in \mathbb{R}$) for all $1 \le i \le n+1$. The converse is trivial. $\square$

We shall now prove the converse statement of Theorem 1. Given a hypersurface $f : M^n \to \mathbb{R}^{n+1}$ ($n \ge 5$) and a conformal immersion $g : M^n \to \mathbb{R}^{n+1}$, set $G = \mathcal{L}_w(g)$ for $w \in \mathbb{V}$. For the rest of the proof, we regard G as a map into $\mathbb{L}^{n+3}$; that is, we write G for $k \circ G$, where $k : \mathbb{V} \to \mathbb{L}^{n+3}$ is the inclusion map. Set $\delta = j_{w*}N_g$, where N_g is a smooth unit vector field normal to g. Differentiating $G = \varphi_g^{-1} j_w \circ g$ twice, we easily obtain that

$$\langle \alpha_G(X,Y), \delta \rangle = \varphi_g^{-1} \langle \tilde{\nabla}_Y j_{w*} g_* X, j_{w*} N_g \rangle = \varphi_g^{-1} \langle \alpha_g(X,Y), N_g \rangle$$

for any $X, Y \in TM$, where $\tilde{\nabla}$ denotes the derivative in $\mathbb{L}^{n+3}$. Hence, the shape operators A^G_δ and $A^g_{N_g}$ are related by

$$A^G_\delta = \varphi_g^{-1} A^g_\delta. \tag{2}$$

On the other hand, differentiating $\langle G, G \rangle = 0$ implies that the position vector G is a normal vector field. Differentiating once more shows that

$$\langle \alpha_G, G \rangle = -\langle \cdot, \cdot \rangle. \tag{3}$$

Moreover, since $\tilde{\nabla}_X G = G_* X$ is a tangent vector, it follows that G is parallel in the normal connection.

Let us denote by $L^2 \subset T^\perp_G M$ the Lorentzian plane subbundle orthogonal to δ. Since the position vector $G \in L^2$ is null, one can easily verify that there exists a unique smooth orthonormal frame $\{\xi, \eta\}$ of L^2 with $\|\xi\| = -1$ such that $G = \xi + \eta$. For each $x \in M^n$, let $W^{2,2} = T^\perp_f M \oplus \operatorname{span}\{\xi\} \oplus \operatorname{span}\{\eta\} \oplus \operatorname{span}\{\delta\}$ be endowed with the natural inner product $\langle\langle \cdot, \cdot \rangle\rangle$ of type $(2, 2)$ and define a symmetric bilinear form $\beta : T_x M \times T_x M \to W$ by

$$\beta(X,Y) = \langle A^f_N X, Y \rangle N - \langle \alpha_G(X,Y), \xi \rangle \xi + \langle \alpha_G(X,Y), \eta \rangle \eta + \langle A^G_\delta X, Y \rangle \delta,$$

where N is a smooth unit vector field normal to f. Since $\beta = \langle A^f_N \cdot, \cdot \rangle N \oplus \alpha_G$, it follows from the Gauss equations for f and G that β is *flat*, that is,

$\langle\langle\beta(X,Y),\beta(Z,W)\rangle\rangle - \langle\langle\beta(X,W),\beta(Z,Y)\rangle\rangle = 0$ for all $X,Y,Z,W \in T_xM$.

Let $\mathcal{V}_1$ be the closed subset of points $x \in M^n$ where β is *null*, that is,

$$\langle\langle\beta(X,Y),\beta(Z,W)\rangle\rangle = 0 \quad \text{for all } X,Y,Z,W \in T_xM.$$

LEMMA 8. (i) *At each point $x \in \mathcal{V}_1$, G extends to a pseudo-orthonormal basis G, ζ_1, ζ_2 of $T^\perp_{G(x)}M$ with $\langle G,\zeta_2\rangle = 1$ and $\|\zeta_2\| = 0$ such that*

$$\alpha_G(X,Y) = \langle A^f_N X, Y\rangle\zeta_1 - \langle X,Y\rangle\zeta_2. \tag{4}$$

(ii) *At each point of $\mathcal{V}_2 := M\backslash\mathcal{V}_1$, f and g have principal curvatures $\lambda, \tilde\lambda$ having the same eigenspace Δ with* $\dim\Delta \ge n-2$. *Moreover*:

(a) *for each x in the closed subset $\mathcal{V}_3 \subset \mathcal{V}_2$ where λ vanishes, there exists $\rho \in \mathbb{V}$ with $\langle\rho, G\rangle = 1$ such that $\langle\alpha_G(X,Y),\rho\rangle = 0$;*

(b) *for each $x \in \mathcal{U}_3 := \mathcal{V}_2\backslash\mathcal{V}_3$ there exist $\mu \in T^\perp_{G(x)}M$ of unit length as well as a flat bilinear form $\gamma\colon T_xM \times T_xM \to (\mathrm{span}\{\mu\})^\perp$ with $\ker\gamma = \Delta$ such that*

$$\alpha_G(X,Y) = \langle A^f_N X, Y\rangle\mu + \gamma(X,Y). \tag{5}$$

Proof. (i) Write $\beta = \bar\beta \oplus \tilde\beta$, where $\bar\beta$ denotes the span$\{N,\xi\}$-component of β and $\tilde\beta$ the span$\{\eta,\delta\}$-component. That β is null means that

$$\langle\bar\beta(X,Y),\bar\beta(Z,W)\rangle = \langle\tilde\beta(X,Y),\tilde\beta(Z,W)\rangle.$$

Hence, there exists a linear isometry $T\colon \mathrm{span}\{N,\xi\} \to \mathrm{span}\{\eta,\delta\}$ such that

$$\tilde\beta = T\circ\bar\beta. \tag{6}$$

Thus, for some $\theta \in [0,2\pi)$ we may assume that

$$T(N) = \sin\theta\eta - \cos\theta\delta, \qquad T(\xi) = \cos\theta\eta + \sin\theta\delta. \tag{7}$$

By (3), we have

$$\langle\alpha_G,\eta\rangle + \langle\alpha_G,\xi\rangle = \langle\alpha_G, G\rangle = -\langle\cdot,\cdot\rangle. \tag{8}$$

It follows from (6), (7), and (8) that

$$\langle\alpha_G,\eta\rangle(1-\cos\theta) = \cos\theta\langle\cdot,\cdot\rangle + \sin\theta\langle A^f_N\cdot,\cdot\rangle. \tag{9}$$

In particular, $1-\cos\theta \ne 0$. From (6)–(9) we obtain

$$A^G_\delta = \frac{\sin\theta}{1-\cos\theta}\,\mathrm{I} + A^f_N, \tag{10}$$

where I denotes the identity map. We conclude from (8)–(10) that (4) holds for

$$\zeta_1 = \delta + \frac{\sin\theta}{1-\cos\theta}(\xi+\eta), \qquad \zeta_2 = \frac{1}{\cos\theta - 1}(\sin\theta\delta + \xi + \cos\theta\eta).$$

(ii) We make use of the following lemma from [6] or [7].

LEMMA 9. *Let $\beta\colon V \times V \to W^{2,2}$ be a nonull flat symmetric bilinear form with* $\dim\ker(\beta) < \dim V - 4$. *Then $W^{2,2}$ admits an orthogonal direct sum decomposition into Lorentzian planes $W = W_1 \oplus W_2$ such that the W_1-component β_0*

of β is nonzero but null and the W_2-component γ of β is nonzero and flat with $\dim\ker(\gamma) \geq \dim V - 2$.

We have from (8) that $\beta(X, X) \neq 0$ for $X \neq 0$. Moreover, $n - 4 > 0$ by assumption. By Lemma 9, at any $x \in \mathcal{V}_2$ there is an orthogonal direct sum decomposition $\beta = \beta_0 \oplus \gamma$ such that the kernel Δ of γ satisfies $\dim \Delta \geq n - 2$. Then, there exist $\theta, \psi \in [0, 2\pi)$ and a bilinear form $\phi\colon T_xM \times T_xM \to \mathbb{R}$ such that

$$\beta_0 = \phi(\cos\theta N + \sin\theta\xi + \cos\psi\delta + \sin\psi\eta).$$

Thus, for $T \in \Delta$ we have that

$$\begin{aligned} &\text{(i)} \ \ \langle A^f_N X, T\rangle = \cos\theta\phi(X, T),\\ &\text{(ii)} \ \ \langle \alpha_G(X, T), \xi\rangle = -\sin\theta\phi(X, T),\\ &\text{(iii)} \ \ \langle A^G_\delta X, T\rangle = \cos\psi\phi(X, T),\\ &\text{(iv)} \ \ \langle \alpha_G(X, T), \eta\rangle = \sin\psi\phi(X, T). \end{aligned} \tag{11}$$

From (8), (11)(ii), and (11)(iv) we have that $\langle X, T\rangle = (\sin\theta - \sin\psi)\phi(X, T)$. Hence, $\sin\theta - \sin\psi \neq 0$. Then (i), (iii), and (iv) of (11) yield

$$\langle A^f_N X, T\rangle = \frac{\cos\theta}{\sin\theta - \sin\psi}\langle X, T\rangle, \tag{12}$$

$$\langle A^G_\delta X, T\rangle = \frac{\cos\psi}{\sin\theta - \sin\psi}\langle X, T\rangle, \tag{13}$$

$$\langle \alpha_G(X, T), \eta\rangle = \frac{\sin\psi}{\sin\theta - \sin\psi}\langle X, T\rangle. \tag{14}$$

The subspace Δ is an eigenspace for f and g from (2), (12), and (13). At a point $x \in \mathcal{V}_3$ where the principal curvature $\lambda = \cos\theta/(\sin\theta - \sin\psi)$ of f vanishes, we derive from (8), (13), and (14) that

$$\alpha_G(X, T) = -\langle X, T\rangle\rho, \quad \text{where } \ \rho = (\sin\psi - 1)^{-1}(\cos\psi\delta + \xi + \sin\psi\eta), \tag{15}$$

for all $T \in \Delta$. Then $\|\rho\| = 0$, $\langle\rho, G\rangle = 1$, and (12) and (15) yield

$$\langle \alpha_G(X, T), \alpha_G(Y, T)\rangle = 0 = \langle A^f_N X, T\rangle\langle A^f_N Y, T\rangle.$$

For $T \in \Delta$ of unit length, the Gauss equations for f and G give

$$\langle \alpha_G(X, Y), \rho\rangle = -\langle \alpha_G(X, Y), \alpha_G(T, T)\rangle = -\langle A^f_N X, Y\rangle\langle A^f_N T, T\rangle = 0.$$

Finally, let $x \in \mathcal{U}_3$. Then (8), (13), and (14) yield

$$\alpha_G(X, T) = \lambda\langle X, T\rangle\mu, \quad \text{where } \ \mu = (1/\cos\theta)(\sin\theta\xi + \cos\psi\delta + \sin\psi\eta).$$

Since $\|\mu\| = 1$, we obtain using (12) that

$$\langle \alpha_G(X, T), \alpha_G(Y, T)\rangle = \langle A^f_N X, T\rangle\langle A^f_N Y, T\rangle \quad \text{for all } \ X, Y \in T_xM.$$

Choosing $T \in \Delta$ of unit length, the Gauss equations for f and G give

$$\lambda\langle \alpha_G(X, Y), \mu\rangle = \langle \alpha_G(X, Y), \alpha_G(T, T)\rangle = \langle A^f_N X, Y\rangle\langle A^f_N T, T\rangle = \lambda\langle A^f_N X, Y\rangle.$$

Hence $A^G_\mu = A^f_N$. Since $\beta_0 = \cos\theta\phi(N+\mu)$, we have

$$\alpha_G = \beta - \langle A^f_N\cdot, \cdot\rangle N = (\cos\theta\phi - \langle A^f_N\cdot, \cdot\rangle)N + \cos\theta\phi\mu + \gamma.$$

Then the N component must vanish, and (5) follows. $\square$

We now show that (i) holds on the interior $\mathcal{U}_1$ of $\mathcal{V}_1$. It is easily seen from (4) that ζ_1, ζ_2 can be chosen to be smooth vector fields along any connected component U of $\mathcal{U}_1$. Comparing the Codazzi equations for f and G for $A^f_N = A^G_{\zeta_1}$ yields

$$A^G_{\nabla^\perp_X \zeta_1} Y = A^G_{\nabla^\perp_Y \zeta_1} X \quad \text{for all } \; X, Y \in TU. \tag{16}$$

But $\nabla^\perp_X \zeta_1 = \langle \nabla^\perp_X \zeta_1, \zeta_2\rangle G$, since G is parallel in the normal connection. We conclude from (16) that $\{\zeta_1, \zeta_2, G\}$ is a parallel normal frame.

Set $F = \mathcal{L}_v(f) = j_v \circ f : M^n \to \mathbb{V} \subset \mathbb{L}^{n+3}$ for $v \in \mathbb{V}$, regarded as a map into $\mathbb{L}^{n+3}$. Then

$$\alpha_F(X, Y) = \langle A^f_N X, Y\rangle j_{v*} N - \langle X, Y\rangle v, \tag{17}$$

where the pseudo-orthonormal frame $\{j_{v*}N, v, F\}$ is parallel in the normal connection of F. Define a parallel vector bundle isometry $\tau : T^\perp_F U \to T^\perp_G U$ by setting $\tau(j_{v*}N) = \zeta_1$, $\tau(v) = \zeta_2$, and $\tau(F) = G$. Then $\alpha_G = \tau \circ \alpha_F$ from (4) and (17). By the fundamental theorem for submanifolds there exists an isometry Λ of $\mathbb{L}^{n+3}$, preserving $\mathbb{V}$, such that $G = \Lambda \circ F$. By Proposition 6, there is a conformal transformation ν of $\mathbb{R}^{n+1}$ such that $g\big|_U = \nu \circ f\big|_U$.

Now, let $\mathcal{U}_2$ be the interior of $\mathcal{V}_3$. It follows from (15) that ρ is smooth on any connected component U of $\mathcal{U}_2$. The Codazzi equation for A^G_ρ $(=0)$ yields

$$\langle \nabla^\perp_X \rho, \bar\delta\rangle A^G_{\bar\delta} Y = \langle \nabla^\perp_Y \rho, \bar\delta\rangle A^G_{\bar\delta} X, \tag{18}$$

where $\bar\delta$ is a smooth unit vector field orthogonal to the Lorentzian plane bundle spanned by ρ and G. Assume that the linear functional $X \mapsto \langle \nabla^\perp_X \rho, \bar\delta\rangle$ is nonzero at some point $x \in \mathcal{U}_2$, and let Z be a vector spanning the orthogonal complement of its kernel K. Applying (18) to $X \in K$ and Z yields $A^G_{\bar\delta} X = 0$. Hence $\alpha_G(X, Y) = -\langle X, Y\rangle\rho + \langle A^G_{\bar\delta} X, Y\rangle\bar\delta$ is flat, which is a contradiction. This proves that ρ is constant in $\mathbb{L}^{n+3}$. Now an easy argument shows that $G(U)$ is contained in an affine hyperplane in $\mathbb{L}^{n+3}$ orthogonal to ρ, say, $G(U) \subset \mathbb{E}_\rho$. Hence there exists an isometric immersion $\bar f : M^n \to \mathbb{R}^{n+1}$ such that $G = j_\rho \circ \bar f = \mathcal{L}_\rho(\bar f)$. We conclude from Proposition 6 that $\bar f$ is conformally congruent to g.

Finally, we prove that (iii) holds on $\mathcal{U}_3$. Let $U \subset \mathcal{U}_3$ be an open subset where Δ has constant dimension. Then μ and γ in (5) are smooth on U. Moreover, standard arguments (see e.g. [16]) show that Δ is an integrable distribution, μ is parallel along Δ in the normal connection, and the leaves of Δ are totally umbilic submanifolds of both $\mathbb{R}^{n+1}$ and $\mathbb{L}^{n+3}$. We denote the Riemannian connections in M^n and $\mathbb{L}^{n+3}$ by ∇ and $\tilde\nabla$, respectively, and consider the smooth line bundle $\pi_G : L^G \to U$ with fibers

$$L^G = \text{span}\{(\tilde\nabla_T T)_\Omega \text{ for all } T \in \Delta\}, \quad \text{where } \; \Omega = \Delta^\perp \oplus \text{span}\{\mu\}.$$

Notice that $L^G \not\subset TU$ because $\lambda \neq 0$. The Codazzi equation for $\xi \perp \mu$ yields

$$\nabla_T A^G_\xi X - A^G_\xi [X, T] + A^G_{\nabla^\perp_X \xi} T = 0 \quad \text{for all } T \in \Delta,\ X \in TU.$$

Taking the inner product with T gives $\langle \tilde\nabla_T T, \tilde\nabla_X \xi\rangle = 0$. Since $\langle \tilde\nabla_T T, \xi\rangle = 0$, we obtain that $\tilde\nabla_X \tilde\nabla_T T \in TU \oplus \operatorname{span}\{\mu\}$. Therefore,

$$\tilde\nabla_X \tilde\nabla_T T = \tilde\nabla_X(\nabla_T T + \lambda\mu) = \nabla_X \nabla_T T - \lambda A^G_\mu X + \langle A^G_\mu \nabla_T T + \nabla\lambda, X\rangle \mu,$$

which easily implies that

$$\tilde\nabla_X (\tilde\nabla_T T)_\Omega = \nabla_X (\nabla_T T)_{\Delta^\perp} - \lambda A^G_\mu X + \langle A^G_\mu (\nabla_T T)_{\Delta^\perp} + \nabla\lambda, X\rangle \mu. \tag{19}$$

Denote by $\pi_f \colon L^f \to U$ the line bundle similarly defined as L^G and by $\tau \colon L^f \to L^G$ the obvious bundle isometry. Restricting U, if necessary, so that $f\big|_U$ is an embedding, the map $\bar f \colon L^f \to \mathbb{R}^{n+1}$ given by

$$\bar f(\zeta) = f(x) + \zeta, \quad x = \pi_f(\zeta),$$

is a diffeomorphism of an open neighborhood N^{n+1} of the zero section of L^f onto a tubular neighborhood of $f(U)$. A similar calculation shows that (19) also holds for f when we replace μ by N. We easily conclude using $A^f_N = A^G_\mu$ that the map $\tilde G \colon L^f \to \mathbb{L}^{n+3}$ given by

$$\tilde G(\zeta) = G(x) + \tau(\zeta), \quad x = \pi_G(\zeta),$$

is isometric with respect to the flat metric induced by $\bar f$. Therefore, the map

$$H = \tilde G\big|_{N^{n+1}} \circ \left(\bar f\big|_{N^{n+1}}\right)^{-1} \colon V \to \mathbb{L}^{n+3}, \quad V = \bar f(N^{n+1}),$$

is an isometric immersion and $G\big|_U = H \circ f\big|_U$. Moreover, by restricting N^{n+1} if necessary, we may also assume that H is an embedding and that $\tilde G(\zeta) \in \mathbb{V}$ if and only if $\zeta = 0$, that is, $H(V) \cap \mathbb{V} = G(U)$. Thus, $\mathcal{T} = G\big|_U \colon U \to G(U) = H(V) \cap \mathbb{V}$ is an isometry. Moreover, $f_H \circ \tau = H^{-1} \circ G\big|_U = f\big|_U$ and

$$\begin{aligned} j_w \circ g_H \circ \tau &= j_w \circ \mathcal{C}_w(i) \circ \tau = \Pi_w \circ i \circ \tau = \Pi_w \circ G\big|_U = j_w \circ \mathcal{C}_w\big(G\big|_U\big) \\ &= j_w \circ \mathcal{C}_w\big((\mathcal{L}_w(g))\big|_U\big) = j_w \circ g\big|_U; \end{aligned}$$

hence $g_H \circ \tau = g\big|_U$. □

Proof of Corollary 4. Let $f \colon M^n \to \mathbb{R}^{n+1}$ be a hypersurface such that $\tilde f = i \circ f$ is a Sbrana–Cartan hypersurface for some inversion I on $\mathbb{R}^{n+1}$. Then f has a principal curvature λ of multiplicity $n-2$ everywhere, the spherical leaves correspondent to λ being the images by i of the relative nullity leaves of $\tilde f$. Therefore, all spheres containing such leaves pass through the image by i of the point at infinity. Let $\tilde g \colon M^n \to \mathbb{R}^{n+1}$ be an isometric nowhere congruent deformation of $\tilde f$. Then $\tilde g$ is conformal to f. Assume that $\tilde g\big|_U$ is conformally congruent to $f\big|_U$ for some open subset $U \subset M^n$; then $\tilde g\big|_U$ is also conformally congruent to $\tilde f\big|_U$. Since $\tilde g$ and $\tilde f$ are isometric, this implies that $\tilde g\big|_U$ and $\tilde f\big|_U$ are isometrically congruent. Therefore, any isometric nowhere congruent deformation of $\tilde f$ is also a conformal nowhere conformally congruent deformation of f. Thus f is a Cartan hypersurface.

Conversely, let $f\colon M^n \to \mathbb{R}^{n+1}$ be a Cartan hypersurface such that the spheres in $\mathbb{R}^{n+1}$ containing the spherical leaves correspondent to the principal curvature of multiplicity $n-2$ have a common point $P_0 \in \mathbb{R}^{n+1}$. Let i be an inversion with pole at P_0. Then $\tilde f = i \circ f$ has a zero principal curvature of multiplicity $n-2$. Let $g\colon M^n \to \mathbb{R}^{n+1}$ be a conformal nowhere conformally congruent deformation of f. Then g is also a conformal nowhere conformally congruent deformation of $\tilde f$. By the converse of Theorem 1 applied to $\tilde f$, g is conformally congruent to an isometric nowhere isometrically congruent deformation of $\tilde f$. This proves that $\tilde f$ is a Sbrana–Cartan hypersurface and also shows the last assertion. □

3. Cartan's Classification

In this section we give our own version of Cartan's classification obtained in [3] by characterizing in the following result from [1] the pairs $\{\psi, r\}$ that give rise to Cartan hypersurfaces in the two interesting classes.

PROPOSITION 10. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a hypersurface with Gauss map N and a principal curvature $\lambda > 0$ of multiplicity $n-2$. Then the focal map $\Psi = f + rN$, $r = 1/\lambda$, induces an (isometric) immersion $\psi\colon L^2 \to \mathbb{R}^{n+1}$ such that $\|\nabla r\| < 1$, and f can be locally parametrized along the unit normal bundle $T_1^\perp L$ of ψ by*

$$\mathcal{X}(\phi) = \psi - r\big(\psi_*\nabla r + \sqrt{1-\|\nabla r\|^2}\,\phi\big). \tag{20}$$

Conversely, given a surface $\psi\colon L^2 \to \mathbb{R}^{n+1}$ and an $r \in C^\infty(L^2)$ positive whose gradient satisfies $\|\nabla r\| < 1$, the parametrized hypersurface (20) *determined by the pair $\{\psi, r\}$ has, on the open subset of regular points, a nonzero principal curvature $\lambda = 1/r$ of multiplicity $n-2$.*

First, however, we briefly discuss some special classes of surfaces in the unit Lorentzian sphere $\mathbb{S}_1^m = \{x \in \mathbb{L}^{m+1} : \|x\| = 1\}$; we refer to [9] for further details. By a pair $\{\varphi, (u, v)\}$ we denote a surface $\varphi\colon V^2 \to \mathbb{S}_1^m$, with Riemannian induced metric, carrying a global system (u, v) of either real or complex conjugate coordinates. Recall that the coordinates (u, v) being *real conjugate* means that the second fundamental form of φ satisfies everywhere the condition

$$\alpha_\varphi(\partial_u, \partial_v) = 0 \tag{21}$$

for the coordinate vector fields. The coordinates are *complex conjugate* when condition (21) holds for the complexified coordinate vector fields, that is, when

$$\alpha_\varphi(\partial_z, \partial_{\bar z}) = \alpha_\varphi(\partial_u, \partial_u) + \alpha_\varphi(\partial_v, \partial_v) = 0. \tag{22}$$

For φ regarded as an $\mathbb{L}^{m+1}$-valued map, (21) takes the form

$$\varphi_{uv} - \Gamma^1\varphi_u - \Gamma^2\varphi_v + \langle\partial_u, \partial_v\rangle\varphi = 0,$$

where Γ^1, Γ^2 are the Christoffel symbols of the Levi–Civita connection ∇' of the metric induced on V^2 by φ; that is, $\nabla'_{\partial_u}\partial_v = \Gamma^1\partial_u + \Gamma^2\partial_v$. Assume that there exists an everywhere positive solution, other than the trivial one $\tau \equiv 1$, of the system

$$\begin{cases} \tau_u = 2\Gamma^2\tau(1-\tau), \\ \tau_v = 2\Gamma^1(1-\tau), \end{cases} \tag{23}$$

whose integrability condition is

$$(1-\tau)[(\Gamma^2_v - 2\Gamma^1\Gamma^2)\tau - \Gamma^1_u + 2\Gamma^1\Gamma^2] = 0. \tag{24}$$

The surface $\{\varphi, (u, v)\}$ is called of *first species* when, in addition,

$$\Gamma^1_u = \Gamma^2_v = 2\Gamma^1\Gamma^2, \tag{25}$$

that is, when (24) is trivially satisfied. It is called of *second species* if it is not of first species and hence $\tau = (\Gamma^2_v - 2\Gamma^1\Gamma^2)/(\Gamma^1_u - 2\Gamma^1\Gamma^2)$ is the (necessarily unique) nontrivial positive solution of (23).

When $\{\varphi, (u, v)\}$ has complex conjugate coordinates, we define a complex-valued connection function $\Gamma = \Gamma(z, \bar z)$ by $\nabla'_{\partial_z}\partial_{\bar z} = \Gamma\partial_z + \bar\Gamma\partial_{\bar z}$. Then (22) takes the form

$$\varphi_{uu} + \varphi_{vv} - 2\Gamma^1\varphi_u - 2\Gamma^2\varphi_v + (\langle\partial_u, \partial_u\rangle + \langle\partial_v, \partial_v\rangle)\varphi = 0,$$

where $\Gamma = \Gamma^1 + i\Gamma^2$. In this case, the differential equation to consider is

$$\rho_{\bar z} + \Gamma(\rho - \bar\rho) = 0, \tag{26}$$

where $\rho = \rho(z, \bar z)$ takes values in the unit circle. The surface $\{\varphi, (u, v)\}$ is called of first species when the integrability condition

$$\operatorname{Im}\rho(\Gamma_z - 2\Gamma\bar\Gamma) = 0 \tag{27}$$

of (26) is trivially satisfied; that is, when $\Gamma_z\ (= \bar\Gamma_{\bar z}) = 2\Gamma\bar\Gamma$, which is the complex analog of (25). It is said to be of second species when it is not of first species and (26) has a unique solution determined by (27).

Proposition 11. *For a surface of first species with real* (*resp., complex*) *conjugate coordinates, system* (23) (*resp.,* (26)) *has a* 1*-parameter family of local solutions.*

Proof. See [9] or [17]. □

Consider a surface $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_{n+2})\colon L^2 \to \mathbb{S}^{n+2}_1 \subset \mathbb{L}^{n+3}$ given in a pseudo-orthonormal basis as in (1). When $\varphi_0 \neq 0$ everywhere, we may associate to φ a map $\psi\colon L^2 \to \mathbb{R}^{n+1}$ and an $r \in C^\infty(L^2)$ given by

$$\psi = r(\varphi_1, \ldots, \varphi_{n+1}) \quad \text{and} \quad r = 1/\varphi_0. \tag{28}$$

Clearly, φ can be recovered from (ψ, r) by

$$\varphi = r^{-1}(1, \psi, \|\psi\|^2 - r^2). \tag{29}$$

LEMMA 12. *L^2 is Riemannian if and only if ψ is an immersion and the gradient $\nabla^\psi r$ of r in the metric induced by ψ satisfies $\|\nabla^\psi r\| < 1$.*

Proof. Set $\lambda = \varphi_0$. We have

$$\|\varphi_u\|^2\|\varphi_v\|^2 - \langle\varphi_u, \varphi_v\rangle^2 = \lambda^4(\|\psi_u\|^2\|\psi_v\|^2 - \langle\psi_u, \psi_v\rangle^2) - \|\lambda_v\psi_u - \lambda_u\psi_v\|^2.$$

Assume that φ induces a Riemannian metric. Then ψ is an immersion, and a straightforward computation now yields

$$\|\varphi_u\|^2\|\varphi_v\|^2 - \langle\varphi_u, \varphi_v\rangle^2 = \lambda^4(1 - \|\nabla^\psi\lambda^{-1}\|^2)(\|\psi_u\|^2\|\psi_v\|^2 - \langle\psi_u, \psi_v\rangle^2). \quad (30)$$

Hence $\|\nabla^\psi r\| < 1$. The converse follows from (30). $\square$

Given a surface $\varphi\colon L^2 \to \mathbb{R}^3_+$ contained in the upper half-space, we denote by $\varphi_{\mathrm{hyp}}\colon L^2_{\mathrm{hyp}} \to \mathbb{H}^3(-1)$ the surface φ with the metric induced from the standard hyperbolic metric on $\mathbb{R}^3_+$.

We are now ready to state our version of Cartan's classification.

THEOREM 13. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ $(n \geq 5)$ be a Cartan hypersurface. Then there exists an open dense subset $\mathcal{V} \subset M^n$ such that one of the following holds on any connected component U of $\mathcal{V}$.*

- (I) *Up to a conformal transformation of $\mathbb{R}^{n+1}$, either:* (a) $f(U) \subset L^2 \times \mathbb{R}^{n-2}$, *where $L^2 \subset \mathbb{R}^3$; or* (b) $f(U) \subset CL^2 \times \mathbb{R}^{n-3}$, *where $CL^2 \subset \mathbb{R}^4$ is a cone over $L^2 \subset \mathbb{S}^3$; or* (c) *f is a rotation hypersurface over $L^2 \subset \mathbb{R}^3$.*
- (II) *f is conformally ruled, that is, foliated by open subsets of codimension-*1 *round spheres in $\mathbb{R}^{n+1}$.*
- (III) *In terms of parameterization* (20), *f is determined by a pair $\{\psi, r\}$ associated by* (28) *to a surface φ of first species.*
- (IV) *In terms of parameterization* (20); *f is determined by a pair $\{\psi, r\}$ associated by* (28) *to a surface φ of second species.*

Conversely, any simply connected hypersurface that can be described as in (I)(c), (II), (III), *and* (IV), *or that differs by an inversion from a hypersurface as in* (I)(a) *or* (I)(b), *is a Cartan hypersurface.*

Moreover, all deformations are type-preserving. For hypersurfaces of type (I)(a) *and* (I)(b), *all deformations are given by isometric deformations of L^2, whereas deformations of hypersurfaces of type* (I)(c) *are given by isometric deformations of L^2_{hyp}. The set of deformations of a hypersurface of type* (II), (III), *or* (IV) *that is not of type* (I) *is, respectively, parametrized by all smooth functions in an interval, a continuous* 1*-parameter family, or contains only one other immersion.*

Proof. Let $g\colon M^n \to \mathbb{R}^{n+1}$ be a conformal immersion that is nowhere conformally congruent to f, and set $G = \mathcal{L}_w(g)$. By Theorem 1, there exist an open dense subset $\mathcal{U} = \mathcal{U}_3 \subset M^n$, a smooth unit vector field $\mu \in T_G^\perp\mathcal{U}$, and a smooth flat bilinear form $\gamma\colon T\mathcal{U} \times T\mathcal{U} \to (\mathrm{span}\{\mu\})^\perp$ such that (5) holds everywhere. Furthermore, $\Delta = \ker\gamma$ is an eigenbundle of f and g of rank $n-2$.

We have an orthogonal splitting $T\mathcal{U} = \Delta \oplus \Delta^\perp$ and write $X = X^v + X^h$ accordingly for any $X \in T\mathcal{U}$. Recall that the *splitting tensor* C of Δ assigns to each $T \in \Delta$ the endomorphism C_T of $\Delta^\perp$ given by

$$C_T X = -\nabla^h_X T.$$

Define $\zeta \in T^\perp_G \mathcal{U}$ by $\zeta = \lambda G + \mu$. We have from (5) that

$$-1 = \langle \alpha_G(T,T), G\rangle = \langle A^f_N T, T\rangle \langle \mu, G\rangle = \lambda \langle \mu, G\rangle$$

for any unit vector $T \in \Delta$. Hence $\|\zeta\| = -1$, $\langle \mu, \zeta\rangle = 0$, and

$$A_\zeta = A_\mu - \lambda I. \tag{31}$$

Differentiating ζ and taking the normal component yields

$$\nabla^\perp_X(\mu - \zeta) = \lambda^{-1} X(\lambda)(\mu - \zeta) \quad \text{for all } \; X \in T\mathcal{U}. \tag{32}$$

Extend μ, ζ to an orthonormal frame $\{\mu, \zeta, \bar\zeta\}$, and let $A_\mu, A_\zeta, A_{\bar\zeta}$ also denote the restrictions of the shape operators to $\Delta^\perp$. We get from (32) that

$$\nabla^\perp_X \bar\zeta = \omega(X)(\mu - \zeta) \quad \text{and} \quad \nabla^\perp_X \zeta = -\lambda^{-1} X(\lambda)\mu - \omega(X)\bar\zeta, \tag{33}$$

where $\omega(X) = \langle \nabla^\perp_X \bar\zeta, \mu\rangle$. Moreover, we easily conclude that ζ and $\bar\zeta$ are parallel along Δ.

The Codazzi equation for $A := A^f_N$ yields

$$\nabla^h_T A = (A - \lambda \mathrm{I}) C_T. \tag{34}$$

Similarly, the Codazzi equation for $A_{\bar\zeta}$ gives

$$\nabla^h_T A_{\bar\zeta} = A_{\bar\zeta} C_T. \tag{35}$$

It follows from (34) and (35) that

$$(A - \lambda \mathrm{I}) C_T = C^*_T (A - \lambda \mathrm{I}) \quad \text{and} \quad A_{\bar\zeta} C_T = C^*_T A_{\bar\zeta}, \tag{36}$$

where C^*_T stands for the adjoint operator of C_T.

LEMMA 14. *The endomorphism* $D := (A - \lambda \mathrm{I})^{-1} A_{\bar\zeta} \colon \Delta^\perp \to \Delta^\perp$ *satisfies*:

(i) $\det D = 1$;
(ii) $[D, C_T] = 0$; *and*
(iii) $\nabla^h_T D = 0$ *for* $T \in \Delta$.

Proof. (i) Flatness of γ implies that $\det A_{\bar\zeta} = \det A_\zeta = \det(A - \lambda \mathrm{I})$.

(ii) Using (36), we have

$$(A - \lambda \mathrm{I}) D C_T = A_{\bar\zeta} C_T = C^*_T A_{\bar\zeta} = C^*_T (A - \lambda \mathrm{I}) D = (A - \lambda \mathrm{I}) C_T D.$$

(iii) Equation (34) yields $(A - \lambda \mathrm{I}) C_T D = (\nabla^h_T A) D$, whereas (35) gives

$$(A - \lambda \mathrm{I}) D C_T = A_{\bar\zeta} C_T = \nabla^h_T A_{\bar\zeta} = \nabla^h_T (A - \lambda \mathrm{I}) D = \nabla^h_T (AD) - \lambda \nabla^h_T D.$$

Hence, $(A - \lambda \mathrm{I})[D, C_T] = (A - \lambda \mathrm{I}) \nabla^h_T D$, and the proof follows from (ii). □

LEMMA 15. *For any Cartan hypersurface,* $\dim \operatorname{coker} C \leq 2$. *Moreover, if equality holds, then either C_T is symmetric for all $T \in \Delta$ or there exists $S \in \operatorname{coker} C$ such that $C_S = \mu \mathrm{I}$.*

Proof. The first assertion is an immediate consequence of Lemma 14(ii). When equality holds, by dimension reasons we know there exists an $\bar{S} \in \operatorname{coker} C$ such that $C_{\bar{S}}$ is symmetric. The last assertion then follows easily using again Lemma 14(ii). □

Let $M_0 \subset \mathcal{U}$ be the interior of the subset in which there exists $T \in \Delta$ of unit length such that $\operatorname{coker} C = \operatorname{span}\{T\}$ and $C_T = \mu \mathrm{I}$. This is clearly equivalent to $\Delta^\perp$ being a totally umbilic distribution. We conclude from the main result in [10] that f is as in (I) of the theorem on each connected component.

Now let $M_1 \subset \mathcal{U}$ be the interior of the set for which $\dim \ker C$ is locally constant and there exists an $S \in \operatorname{coker} C$ such that C_S is not symmetric and has one eigenvalue of multiplicity 2. It follows from Lemma 15 that there exist a smooth vector field $S \in \operatorname{coker} C$ and a unique (up to signs) orthonormal frame $\{Z, W\}$ in $\Delta^\perp$ with respect to which

$$C_S = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix}, \quad b \neq 0. \tag{37}$$

Since $(A - \lambda \mathrm{I})C_S$ is symmetric by (34), we have

$$\begin{aligned} a\langle (A - \lambda \mathrm{I})W, Z\rangle &= \langle (A - \lambda \mathrm{I})C_S W, Z\rangle = \langle W, (A - \lambda \mathrm{I})C_S Z\rangle \\ &= a\langle (A - \lambda \mathrm{I})W, Z\rangle + b\langle (A - \lambda \mathrm{I})W, W\rangle, \end{aligned}$$

which implies that

$$\langle AW, W\rangle = \lambda. \tag{38}$$

A similar argument using symmetry of $A_{\bar{\zeta}} C_S$ shows that

$$\langle A_{\bar{\zeta}} W, W\rangle = 0. \tag{39}$$

We conclude from (31), (38), and (39) that there exist smooth functions α, β, θ such that

$$A_\zeta = \begin{bmatrix} \alpha & \beta \\ \beta & 0 \end{bmatrix}, \qquad A_{\bar{\zeta}} = \begin{bmatrix} \alpha + \theta & \beta \\ \beta & 0 \end{bmatrix}. \tag{40}$$

We claim that the distribution $x \mapsto \operatorname{span}\{W(x)\} \oplus \Delta(x)$ is totally umbilical. Comparing the Codazzi equations for $A_N^f = A_\mu = A$ for f and G, we have

$$A_{\nabla^\perp_X \mu} Y = A_{\nabla^\perp_Y \mu} X \quad \text{for all } X, Y \in T\mathcal{U}. \tag{41}$$

Equations (33) and (41) yield

$$\lambda \omega(W) + W(\lambda) = 0, \qquad \lambda \omega(Z) + Z(\lambda) + \theta \beta^{-1} W(\lambda) = 0. \tag{42}$$

A straightforward computation using (33) and (42) shows that the Codazzi equation for $A_{\bar{\zeta} - \zeta}$ is equivalent to

$$\langle \nabla_W T, Z\rangle = 0 = \langle \nabla_T W, Z\rangle \tag{43}$$

and

$$\langle \nabla_T T, Z \rangle = -\beta^{-1} W(\lambda) = \langle \nabla_W W, Z \rangle \tag{44}$$

and that

$$W(\theta) = \langle \nabla_Z Z, W \rangle \theta \quad \text{and} \quad T(\theta) = \langle \nabla_Z Z, T \rangle \theta \tag{45}$$

for all $T \in \Delta$. The claim follows from (43) and (44). Moreover, the images by f of the leaves are also totally umbilical in $\mathbb{R}^{n+1}$ by (38), so $f\big|_{M_1}$ is conformally ruled.

Consider now the open subset $M_2 \subset \mathcal{U}$ for which there exists an $S \in \Delta$ such that C_S has two distinct real eigenvalues. By (i) and (ii) of Lemma 14, there exists a unique (up to signs) frame $\{Y_1, Y_2\}$ of unit eigenvectors of C_T, for all $T \in \Delta$, with respect to which D has the form

$$D = \begin{bmatrix} \theta & 0 \\ 0 & 1/\theta \end{bmatrix}, \quad \theta \neq 0, \pm 1. \tag{46}$$

Set $\Psi = f + \lambda^{-1} N$. We need the following result.

LEMMA 16. (i) *There are smooth functions* μ_1, μ_2 *such that the frame* $\{X_1 = \mu_1 Y_1,\ X_2 = \mu_2 Y_2\}$ *satisfies*:

(a) $\tilde{\nabla}_T \Psi_* X_j = 0$, $1 \leq j \leq 2$;
(b) $[X_1, X_2] \in \Delta$.

(ii) *The following equations hold:*

$$\tilde{\nabla}_{X_1} \lambda \theta^{-1} \Psi_* X_2 + \theta^{-1} X_2(\lambda) \Psi_* X_1 = \tilde{\nabla}_{X_2} \lambda \theta \Psi_* X_1 - \theta X_1(\lambda) \Psi_* X_2; \tag{47}$$

$$X_2(\lambda^{-1} \theta X_1(\lambda)) - X_1((\theta\lambda)^{-1} X_2(\lambda)) - \lambda^2 \theta^{-1}(1 - \theta^2) \langle \Psi_* X_1, \Psi_* X_2 \rangle = 0. \tag{48}$$

Proof. (i) We have

$$\Psi_* X_j = -\lambda^{-1}(A - \lambda \mathrm{I}) X_j + X_j(\lambda^{-1}) N. \tag{49}$$

Using the Codazzi equation for f yields

$$\tilde{\nabla}_T \Psi_* X_j = -\lambda^{-1}(A - \lambda \mathrm{I})[T, X_j] + [T, X_j](\lambda^{-1}) N.$$

Therefore, (i)(a) is equivalent to $[X_j, T] \in \Delta$ $(1 \leq j \leq 2)$ for all $T \in \Delta$. Since $\nabla_T^h Y_j = 0$ $(1 \leq j \leq 2)$ by Lemma 14(iii), in order to prove Lemma 16(i) it suffices to prescribe each μ_j arbitrarily along an integral curve γ of Y_j, and then extend it along each integral curve of Y_i $(i \neq j)$ and each geodesic of Δ through γ as a solution of the linear differential equations of first order

$$T(\mu_j) + b_j \mu_j = 0, \qquad Y_i(\mu_j) + r_j \mu_j = 0,$$

where $C_T Y_j = b_j Y_j$ and $[Y_1, Y_2] + r_1 Y_1 - r_2 Y_2 \in \Delta$.

(ii) We obtain from

$$\langle A_{\bar{\xi}} Y_1, Y_2 \rangle = \langle (\lambda \mathrm{I} - A) D Y_1, Y_2 \rangle = \theta \langle (\lambda \mathrm{I} - A) Y_1, Y_2 \rangle = \theta^2 \langle A_{\bar{\xi}} Y_1, Y_2 \rangle$$

that $\langle A_{\bar{\xi}} Y_1, Y_2 \rangle = 0$ and that

$$\langle (A - \lambda \mathrm{I}) Y_1, Y_2 \rangle = 0. \tag{50}$$

On the other hand, equations (33) and (41) yield

$$\lambda\omega(X_1) = -\theta X_1(\lambda) \quad \text{and} \quad \lambda\theta\omega(X_2) = -X_2(\lambda). \tag{51}$$

Using (33), (i)(b), and (51), the Codazzi equation for $A_{\bar{\zeta}}$ gives

$$\nabla_{X_1}(A - \lambda \mathrm{I})\theta^{-1}X_2 + \theta X_1(\lambda)X_2 = \nabla_{X_2}(A - \lambda \mathrm{I})\theta X_1 + \theta^{-1}X_2(\lambda)X_1,$$

which can be written using (50) as

$$\begin{aligned}\tilde{\nabla}_{X_1}&(A - \lambda \mathrm{I})\theta^{-1}X_2 - \theta^{-1}\langle(A - \lambda \mathrm{I})X_1, (A - \lambda \mathrm{I})X_2\rangle N + \theta X_1(\lambda)X_2 \\ &= \tilde{\nabla}_{X_2}(A - \lambda \mathrm{I})\theta X_1 - \theta\langle(A - \lambda \mathrm{I})X_1, (A - \lambda \mathrm{I})X_2\rangle N + \theta^{-1}X_2(\lambda)X_1. \end{aligned} \tag{52}$$

Using (49), it is easily seen that (52) is equivalent to

$$\begin{aligned}\big(X_2(\lambda^{-1}&\theta X_1(\lambda)) - X_1((\theta\lambda)^{-1}X_2(\lambda)) - \lambda^2\theta^{-1}(1 - \theta^2)\langle\Psi_* X_1, \Psi_* X_2\rangle\big)N \\ &= \tilde{\nabla}_{X_1}\lambda\theta^{-1}\Psi_* X_2 + \theta^{-1}X_2(\lambda)\Psi_* X_1 - \tilde{\nabla}_{X_2}\lambda\theta\Psi_* X_1 - \theta X_1(\lambda)\Psi_* X_2. \end{aligned} \tag{53}$$

A straightforward computation using (33) and (51) shows that the Ricci equation $\langle R^{\perp}(X_1, X_2)\mu, \bar{\xi}\rangle = \langle[A_\mu, A_{\bar{\xi}}]X_1, X_2\rangle$ is equivalent to the vanishing of the left-hand side of (53). □

Let $\pi: M_2 \to L^2$ be the quotient map onto the space of leaves of Δ. By Proposition 10, the focal map Ψ induces an immersion $\psi: L^2 \to \mathbb{R}^{n+1}$ so that $\psi \circ \pi = \Psi$. By Lemma 16(i), there exists a coordinate system (u_1, u_2) on L^2 such that $\psi_* \partial_{u_j} = \Psi_* X_j$. It follows from Lemma 14(iii) that $T(\theta) = 0$ for any $T \in \Delta$; hence θ can be regarded as a function on L^2. Then, equations (47) and (48) can be rewritten as

$$\tilde{\nabla}_{\partial_{u_1}}\lambda\theta^{-1}\partial_{u_2} + \lambda_{u_2}\theta^{-1}\partial_{u_1} = \tilde{\nabla}_{\partial_{u_2}}\lambda\theta\partial_{u_1} + \lambda_{u_1}\theta\partial_{u_2} \tag{54}$$

and

$$(\lambda\lambda_{u_1u_2} - \lambda_{u_1}\lambda_{u_2} + \lambda^4\langle\partial_{u_1}, \partial_{u_2}\rangle)\theta^{-1}(1 - \theta^2) - \lambda(\lambda_{u_1}\theta_{u_2} + \theta^{-2}\lambda_{u_2}\theta_{u_1}) = 0. \tag{55}$$

It follows easily from (54) that

$$\alpha_\psi(\partial_{u_1}, \partial_{u_2}) = 0 \tag{56}$$

and

$$\begin{cases} \theta_{u_1} = \theta(1 - \theta^2)(\Gamma^2 + \lambda^{-1}\lambda_{u_1}), \\ \theta_{u_2} = \theta^{-1}(1 - \theta^2)(\Gamma^1 + \lambda^{-1}\lambda_{u_2}). \end{cases} \tag{57}$$

Then (55) takes the form

$$\lambda_{u_1u_2} - \Gamma^1\lambda_{u_1} - \Gamma^2\lambda_{u_2} - 3\lambda^{-1}\lambda_{u_1}\lambda_{u_2} + \lambda^3\langle\partial_{u_1}, \partial_{u_2}\rangle = 0. \tag{58}$$

Define $\varphi: L^2 \to \mathbb{S}_1^{n+2} \subset \mathbb{L}^{n+3}$ by (29), where $r = \lambda^{-1}$. It follows from Proposition 10 and Lemma 12 that the metric induced by φ is Riemannian. A straightforward computation using (57) and (58) yields

$$\varphi_{u_1u_2} = (\Gamma^1 + \lambda^{-1}\lambda_{u_2})\varphi_{u_1} + (\Gamma^2 + \lambda^{-1}\lambda_{u_1})\varphi_{u_2} - \langle\varphi_{u_1}, \varphi_{u_2}\rangle\varphi. \tag{59}$$

Hence, the coordinates (u_1, u_2) are real-conjugate for φ, and

$$\tilde{\Gamma}^1 = \Gamma^1 + \lambda^{-1}\lambda_{u_2}, \qquad \tilde{\Gamma}^2 = \Gamma^2 + \lambda^{-1}\lambda_{u_1}$$

are the Christoffel symbols of the induced metric. We conclude from (57) that $\tau = \theta^2$ satisfies (23) for $\tilde{\Gamma}^1$ and $\tilde{\Gamma}^2$. Thus, φ is a surface of first or second species along any connected component of an open dense subset of L^2.

Finally, let $M_3 \subset \mathcal{U}$ be the open subset for which there exists $S \in \Delta$ such that C_S has two complex conjugate eigenvalues. One can verify using similar arguments as in the real case that equations (54) and (55) now take the form

$$\tilde{\nabla}_{\partial_z} \lambda \bar{\rho} \partial_{\bar{z}} + \lambda_{\bar{z}} \bar{\rho} \partial_z = \tilde{\nabla}_{\partial_{\bar{z}}} \lambda \rho \partial_z + \lambda_z \rho \partial_{\bar{z}} \tag{60}$$

and

$$(\lambda \lambda_{z\bar{z}} - \lambda_z \lambda_{\bar{z}} + \lambda^4 \langle \partial_z, \partial_{\bar{z}} \rangle)(\bar{\rho} - \rho) + \lambda(\lambda_{\bar{z}} \bar{\rho}_z - \lambda_z \rho_{\bar{z}}) = 0. \tag{61}$$

It follows from (60) that

$$\alpha_\varphi(\partial_z, \partial_{\bar{z}}) = 0 \quad \text{and} \quad \rho_{\bar{z}} = (\bar{\rho} - \rho)(\Gamma + \lambda^{-1} \lambda_{\bar{z}}). \tag{62}$$

Then (61) becomes

$$\lambda_{z\bar{z}} - \Gamma \lambda_z - \bar{\Gamma} \lambda_{\bar{z}} - 3\lambda^{-1} \lambda_z \lambda_{\bar{z}} + \lambda^3 \langle \partial_z, \partial_{\bar{z}} \rangle = 0. \tag{63}$$

We can now easily check using (63) that the surface φ given by (29) satisfies

$$\varphi_{z\bar{z}} = (\Gamma + \lambda^{-1} \lambda_{\bar{z}}) \varphi_z + (\bar{\Gamma} + \lambda^{-1} \lambda_z) \varphi_{\bar{z}} - \langle \varphi_z, \varphi_{\bar{z}} \rangle \varphi. \tag{64}$$

It follows from (62) and (64) that φ is a surface of first or second species along any connected component of an open dense subset of L^2. We have shown that the statement of the theorem holds on $\mathcal{V} = \bigcup_{i=0}^3 M_i$.

We now prove the converse. First observe that the distributions associated to the principal curvatures of multiplicity $n-2$ of a Cartan hypersurface $f : M^n \to \mathbb{R}^{n+1}$ and any of its conformal deformations $g : M^n \to \mathbb{R}^{n+1}$ coincide. Then, it is easily seen that the splitting tensors of f and g are related by

$$C_T^g = C_T^f - (1/2) T(\log \mu) \mathrm{I}, \tag{65}$$

where μ is the conformal factor. By the proof of the direct statement, the type of f is determined by the structure of its splitting tensor. Since the splitting tensors of f and g have the same structure by (65), it follows that f and g are necessarily of the same type.

Let $f : M^n \to \mathbb{R}^{n+1}$ be as in parts (I)(a) or (b). It was shown in [9] that isometric deformations of f are given by isometric deformations of L^2. The assertion on the set of conformal deformations of f is then a consequence of Corollary 4.

It follows from these considerations that any conformal deformation g of a Cartan hypersurface f as in part (I)(c) must also be as in part (I)(c). Consider the isometric embedding $i \times j : \mathbb{H}^3 \times \mathbb{S}_{n-2} \to \mathbb{L}^4 \times \mathbb{R}^{n-1} = \mathbb{L}^{n+3}$, where $i : \mathbb{H}^3 \to \mathbb{L}^4$ and $j : S^{n-2} \to \mathbb{R}^{n-1}$ are the standard inclusions. Choose a pseudo-orthonormal basis $\mathcal{B} = \{e_1, \ldots, e_{n+3}\}$ of $\mathbb{L}^{n+3}$ such that

$$\|e_1\| = 0 = \|e_4\|, \quad \langle e_1, e_4 \rangle = -1/2, \quad \text{and} \quad \langle e_i, e_j \rangle = \delta_{ij} \ \text{ if } i \neq 1, 4.$$

It is easily seen that $(i \times j)(\mathbb{H}^3 \times \mathbb{S}^{n-2}) \subset \mathbb{V} \setminus \mathbb{R}_w$, where $w = -2e_4$. Hence, $i \times j$ induces a conformal diffeomorphism $\mathcal{C}_w(i \times j) : \mathbb{H}^3 \times \mathbb{S}^{n-2} \to \mathbb{R}^{n-1}$, whose inverse is easily checked to be given with respect to $\mathcal{B}$ by

$$\Theta(a_1, \dots, a_{n+1}) = \left(\sum_{j=3}^{n+1} a_j^2\right)^{-1/2} \left(1, a_1, a_2, \sum_{i=1}^{n+1} a_j^2, a_3, \dots, a_{n+3}\right).$$

Since f is as in part (I)(c), it can be parametrized by $f\colon L^2 \times \mathbb{S}^{n-2} \to \mathbb{R}^{n+1}$,

$$f(x,t) = (\varphi_1(x), \varphi_2(x), \varphi_3(x)\phi(t)),$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ parametrizes L^2 in $\mathbb{R}^3_+$ and ϕ parametrizes the unit sphere. Then $\Theta \circ f\colon L^2 \times \mathbb{S}^{n-2} \to \mathbb{H}^3 \times \mathbb{S}^{n-2}$ is given by $\Theta \circ f = (\Phi \circ \varphi) \times \mathrm{I}$, where $\Phi\colon \mathbb{R}^3_+ \to \mathbb{H}^3 \subset \mathbb{L}^4$ given by

$$\Phi(x_1, x_2, x_3) = x_3^{-1}(1, x_1, x_2, x_1^2 + x_2^2 + x_3^2)$$

is an isometry between the half-space and hyperboloidal models of $\mathbb{H}^3$. Denote by $\psi\colon L^2 \to \mathbb{R}^3_+$ the profile of g. Then $(\Phi \circ \varphi) \times \mathrm{I}$ and $(\Phi \circ \psi) \times \mathrm{I}$ are conformal and hence isometric because they agree on the second factor. It follows that $\Phi \circ \varphi$ and $\Phi \circ \psi$ are isometric; that is, φ_{hyp} and ψ_{hyp} are isometric.

We now show that the set of conformal deformations of a simply connected conformally ruled hypersurface is parametrized by the set of smooth functions in an interval. Let $\{Z, W\}$ be an orthonormal frame of $\Delta^\perp$ with Z orthogonal to the rulings. Equations (38) and (43) hold because f is conformally ruled. Let α, β be smooth functions such that

$$A = \begin{bmatrix} \alpha + \lambda & \beta \\ \beta & \lambda \end{bmatrix} \tag{66}$$

with respect to $\{Z, W\}$. Then (44) follows from the Codazzi equation and the fact that f is conformally ruled.

We need only prove that a function θ satisfying (45) gives rise to a conformal deformation of f, since each such function is completely determined once an initial condition is chosen along a fixed orthogonal trajectory to the rulings. Consider the trivial vector bundle $\mathcal{E} = M^n \times \mathbb{L}^3$, where $\mathbb{L}^3 = \operatorname{span}\{\mu, \zeta, \bar{\zeta}\}$ is endowed with the Lorentzian inner product that makes $\{\mu, \zeta, \bar{\zeta}\}$ into an orthonormal frame such that $\|\zeta\| = -1$. Next, define a 1-form ω by requiring $\omega(Z)$ and $\omega(W)$ to be given by (42) and $\omega = 0$ on Δ. Finally, endow $\mathcal{E}$ with the connection ∇' determined by (33), and define $\alpha \in C^\infty(\operatorname{Hom}(TM \times TM, \mathcal{E}))$ by $\alpha = \langle A\cdot, \cdot\rangle\mu - \langle A_\zeta\cdot, \cdot\rangle\zeta + \langle A_{\bar{\zeta}}\cdot, \cdot\rangle\bar{\zeta}$, where A_ζ and $A_{\bar{\zeta}}$ are determined by (40). We claim that $(\mathcal{E}, \nabla', \alpha)$ satisfies the Gauss, Codazzi, and Ricci equations for an isometric immersion into $\mathbb{L}^{n+3}$.

The Gauss equation is trivial. Using the Codazzi equation for $A = A_N^f$, the Codazzi equation for $A = A_\mu$ reduces to (41), which follows easily using (33) and (42). To verify the Codazzi equation for A_ζ, it suffices to do the same for $A_{\mu-\zeta} = \lambda\mathrm{I}$. But the latter is trivially satisfied by (33). Similarly, the Codazzi equation for $A_{\bar{\zeta}}$ holds if and only if the same is true for $A_{\bar{\zeta}-\zeta}$. As before, this is equivalent to (43), (44), and (45).

It remains to verify the Ricci equations. An easy calculation using (33) shows that the left-hand side of $\langle R^\perp(Z,W)\mu, \zeta\rangle = \langle [A_\mu, A_\zeta]Z, W\rangle$ vanishes, and the same holds for the right-hand side because A and A_ζ commute. On the other hand, a long but straightforward computation making use of (33), (40), (42), (44), and (45) shows that the Ricci equations for $\mu, \bar{\zeta}$ and $\zeta, \bar{\zeta}$ reduce to

$$W(\beta^{-1}W(\lambda)) + \lambda\beta = 0.$$

This equality follows by using (43) and (44) to compute the left-hand side of the Gauss equation $\langle R(W, T)T, Z\rangle = \lambda\beta$, which concludes the proof of the claim.

By the fundamental theorem of submanifolds, $\mathcal{E}$, ∇', and α are (respectively) the normal bundle, normal connection, and second fundamental form of an isometric immersion $F_\theta \colon M^n \to \mathbb{L}^{n+3}$. Set $\kappa = (1/\lambda)(\zeta - \mu)$. Then κ is a null vector field satisfying $A_\kappa = (1/\lambda)(A_\zeta - A_\kappa) = \mathrm{I}$. Moreover, κ is parallel with respect to ∇' by (33). This implies that $F_\theta(M^n) \subset \mathbb{V}$; thus, F_θ induces a conformal immersion $f_\theta \colon M^n \to \mathbb{R}^{n+1}$, as we wished.

Finally, assume that f is of type (III) or (IV). We argue for the real case; the proof for the complex case is similar. Assume that f is given in terms of parameterization (20) by a pair $\{\psi, r\}$ associated by (28) to a surface of first or second species endowed with real conjugate coordinates. We show that each positive solution τ of (23) gives rise to a conformal deformation of f, and the proof then follows from Proposition 11.

Set $\tau = \theta^2$. Going backwards in the proof of the direct statement shows that the function $\lambda = r^{-1}$ satisfies (54) and (55). Defining vector fields X_1, X_2 by $\pi_* X_j = \partial/\partial u_j$, it follows that $\Psi_* X_j = \psi_* \partial/\partial u_j$ and then (54) and (55) yield (53), which is equivalent to (52). Equality of the N-components implies that X_1, X_2 are conjugate directions for $A - \lambda\mathrm{I}$ (i.e., $\langle (A - \lambda\mathrm{I})X_1, X_2\rangle = 0$). Define D by (46) with respect to this frame. Consider the trivial vector bundle $\mathcal{E} = M^n \times \mathbb{L}^3$, where $\mathbb{L}^3 = \operatorname{span}\{\mu, \zeta, \bar{\zeta}\}$ is endowed with the Lorentzian inner product that makes $\{\mu, \zeta, \bar{\zeta}\}$ into an orthonormal frame with $\|\zeta\| = -1$. Define a 1-form ω by requiring $\omega(X_1), \omega(X_2)$ to be given by (51) and $\omega = 0$ on Δ. Finally, endow $\mathcal{E}$ with the connection ∇' determined by (33), and define $\alpha \in C^\infty(\mathrm{Hom}(TM \times TM, \mathcal{E}))$ by $\alpha = \langle A\cdot, \cdot\rangle\mu - \langle A_\zeta\cdot, \cdot\rangle\zeta + \langle A_{\bar{\zeta}}\cdot, \cdot\rangle\bar{\zeta}$, where A_ζ and $A_{\bar{\zeta}}$ are determined by (31) and $A_{\bar{\zeta}} = A_\zeta \circ D$. We claim that $(\mathcal{E}, \nabla', \alpha)$ satisfies the Gauss, Codazzi, and Ricci equations for an isometric immersion into $\mathbb{L}^{n+3}$.

The Gauss equation is trivial. Using the Codazzi equation for $A = A_N^f$, the Codazzi equation for $A = A_\mu$ reduces to (41), which follows easily using (33) and (51). To verify the Codazzi equation for A_ζ, it suffices to do the same for $A_{\mu-\zeta} = \lambda\mathrm{I}$. But the latter is also trivially satisfied by (33) and (51). The Codazzi equation for $A_{\bar{\zeta}}$ follows by taking tangential components in (52). An easy calculation using (33), Lemma 16(i)(b), and (51) shows that the left-hand side of the Ricci equation $\langle R^\perp(Z, W)\mu, \zeta\rangle = \langle [A_\mu, A_\zeta]Z, W\rangle$ vanishes, and the same holds for the right-hand side since A and A_ζ commute. Finally, the Ricci equations for $\mu, \bar{\zeta}$ and $\zeta, \bar{\zeta}$ are easily seen to be equivalent to (55). This proves the claim. The same argument as used in the ruled case completes the proof. $\square$

REMARKS 17. (i) In Cartan's terminology, a hypersurface of type (III) is completely determined by a set of $n + 3$ *homogeneous coordinate functions* $\alpha^j =$

$\alpha^j(u, v)$, $0 \leq j \leq n+2$, all of which satisfy for $M = M(u, v)$ the (same) differential equation of type either

$$\alpha^j_{uv} + M\alpha^j = 0, \quad 0 \leq j \leq n+2,$$

and the condition $\sum_{j=1}^{n+1}(\alpha^j)^2 - \alpha^0\alpha^{n+2} = U + V$, where $U = U(u)$ and $V = V(v)$ are arbitrary functions of one variable, or

$$\alpha^j_{uu} + \alpha^j_{vv} + M\alpha^j = 0, \quad 0 \leq j \leq n+2,$$

and the condition $\sum_{j=1}^{n+1}(\alpha^j)^2 - \alpha^0\alpha^{n+2} = \phi$, where ϕ satisfies $\phi_{uu} + \phi_{vv} = 0$. An argument similar to the one in [9, Rem. 4] shows that Cartan's characterization is equivalent to ours. It turns out that the deformable hypersurface is given by (20) for the pair $\{\psi, r\}$ defined as $\alpha^0\psi = (\alpha^1, \dots, \alpha^{n+1})$ and $\alpha^0 r = \big(\sum_{j=1}^{n+1}(\alpha^j)^2 - \alpha^0\alpha^{n+2}\big)^{-1/2}$.

(ii) Explicit examples of deformable hypersurfaces of class (III) can be constructed by applying the procedure of Theorem 13 to the surfaces of first species given by [9, Prop. 15].

4. Further Results

We first show that the parametric classification of the Sbrana–Cartan hypersurfaces (due to Sbrana [17] and Cartan [3]) can be derived from Theorem 13 together with Corollary 4, but only for dimension $n \geq 5$.

THEOREM 18 [9]. *Let $f : M^n \to \mathbb{R}^{n+1}$ $(n \geq 3)$ be a Sbrana–Cartan hypersurface. Then there is an open dense subset $\mathcal{U} \subset M^n$ such that one of the following holds on any connected component U of $\mathcal{U}$.*

(I) (a) *$f(U) \subset L^2 \times \mathbb{R}^{n-2}$ where L^2 is a surface in $\mathbb{R}^3$, or*
(b) *$f(U) \subset CL^2 \times \mathbb{R}^{n-3}$ where CL^2 is a cone over a surface $L^2 \subset \mathbb{S}^3 \subset \mathbb{R}^4$.*

(II) *f is ruled, that is, foliated by open subsets of codimension-1 affine subspaces in $\mathbb{R}^{n+1}$.*

(III) *The Gauss image $\nu : V^2 \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ of f is a surface of first species, and f has a Gauss parameterization*

$$\Psi(x, w) = (\gamma\nu + \nu_* \operatorname{grad}\gamma)(x) + w \tag{67}$$

along the normal bundle of ν, where γ satisfies the same differential equation that any one of the coordinate functions of ν does.

(IV) *The Gauss image ν of f is a surface of second species and f has a Gauss parameterization* (67), *where γ is as in* (III).

Conversely, any simply connected hypersurface that can be described as in (I), (II), (III), *or* (IV) *is Sbrana–Cartan. Moreover, any deformation of a hypersurface of type* (I) *is given by a deformation of the surface L^2, whereas the set of deformations of a hypersurface of type* (II), (III), *or* (IV) *that is not of type* (I) *is, respectively, parametrized by all smooth functions in an interval, a continuous*

1-*parameter family, or contains only one other immersion. In all cases, deformations are always of the same type.*

Proof for $n \geq 5$. Set $\bar{f} = i \circ f$, where i is an inversion in $\mathbb{R}^{n+1}$ whose pole, which we may assume to be the origin, does not lie on $f(M^n)$. By Corollary 4, $\bar{f}$ is a Cartan hypersurface whose spherical leaves, correspondent to the principal curvature of multiplicity $n-2$, are open subsets of spheres in $\mathbb{R}^{n+1}$ through the origin. By Theorem 13, for any connected component U of an open dense subset $\mathcal{U} \subset M^n$, $\bar{f}\big|_U$ is of one of the types (I) to (IV). Class (I)(c) is ruled out because the spheres containing the spherical leaves have a common point. Moreover, if $\bar{f}$ is conformally congruent to a hypersurface in one of the classes (I)(a) or (I)(b), then $f\big|_U$ must be in the corresponding isometric congruence class.

Now assume that $\bar{f}\big|_U$ is in one of the classes (III) or (IV) in Theorem 13. Then $\bar{f}\big|_U$ is given, in terms of parameterization (20), by a pair $\{\psi, r\}$ associated by (28) to a surface $\varphi\colon L^2 \to \mathbb{S}_1^{n+2} \subset \mathbb{L}^{n+3}$ of first or second species, respectively. We consider the case where ψ has real conjugate coordinates; the case of complex conjugate coordinates is similar. We have that $\varphi = r^{-1}(1, \psi, \|\psi\|^2 - r^2)$, and $r \in C^\infty(L^2)$ is a solution of

$$r \operatorname{Hess} r(\partial_u, \partial_v) + r_u r_v - \langle \partial_u, \partial_v \rangle = 0. \tag{68}$$

Since the spherical leaves of $\bar{f}\big|_U$ are open subsets of spheres in $\mathbb{R}^{n+1}$ through the origin, there exists a vector field $\mu \in T_h^\perp L$ of length

$$\|\mu\|^2 = r^2(1 - \|\nabla r\|^2) \tag{69}$$

such that

$$\psi - r\nabla r + \mu = 0. \tag{70}$$

We obtain from (70) that $\langle \psi, \psi_u \rangle - r r_u = 0 = \langle \psi, \psi_v \rangle - r r_v$. Hence, $\|\psi\|^2 - r^2$ is a constant, which by (69) and (70) must vanish. Hence there exists a $\nu\colon L^2 \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+1}$ such that $\psi = r\nu$. It is now easy to verify that ν and φ are isometric and have the same conjugate coordinates. In particular, ν is a surface of first or second species. A long but straightforward computation now shows that, after applying the inversion i, equation (20) takes the form (67) with $\gamma = r^{-1}$. Finally, it is easy to check that r satisfies (68) with respect to the metric induced by ψ if and only if $\gamma = r^{-1}$ satisfies the same differential equation that any one of the coordinate functions of ν does with respect to the metric induced by ν, namely,

$$\operatorname{Hess} \gamma(\partial_u, \partial_v) + \langle \partial_u, \partial_v \rangle \gamma = 0. \tag{71}$$

Therefore, $f\big|_U = i \circ \bar{f}\big|_U$ is as in parts (III) or (IV) in the statement.

Going backwards in the preceding argument shows that if f is as in parts (III) or (IV) in the statement and if i is an inversion whose pole does not lie on $f(M^n)$, then $\bar{f} = i \circ f$ is in one of the classes (III) or (IV) in Theorem 13. Using this, the converse follows immediately from Theorem 13 and Corollary 4. $\square$

Now we recall from [15] the possible pointwise structures of the second fundamental form of an isometric immersion $H\colon V \subset \mathbb{R}^{n+1} \to \mathbb{L}^{n+3}$.

LEMMA 19. *At any $x \in V$ one of the following possibilities holds*:

(i) *there exist a null vector $v \in T_x^\perp V$ and a symmetric bilinear form h on T_xV such that*

$$\alpha_H(X, Y) = h(X, Y)v;$$

(ii) *there exist a vector $\xi \in T_x^\perp V$ with $\|\xi\| = \pm 1$ and a symmetric bilinear form h on T_xV of rank* 1 *such that*

$$\alpha_H(X, Y) = h(X, Y)\xi;$$

(iii) $\nu_H(x) = n-1$ *and there are bases X_1, X_2 of $\Delta_H^\perp$ and ξ, η of $T_x^\perp V$ such that*
(a) $\alpha_H(X_1, X_1) = \xi$, $\alpha_H(X_1, X_2) = \eta$, *and* $\alpha_H(X_2, X_2) = 0$,
(b) $\alpha_H(X_1, X_1) = \xi$, $\alpha_H(X_1, X_2) = 0$, *and* $\alpha_H(X_2, X_2) = \eta$,
(c) $\alpha_H(X_1, X_1) = \xi$, $\alpha_H(X_1, X_2) = \eta$, *and* $\alpha_H(X_2, X_2) = -\xi$,
where ξ, η is pseudo-orthonormal in case (a) *and orthonormal otherwise.*

The next result characterizes isometric immersions $H\colon V \subset \mathbb{R}^{n+1} \to \mathbb{L}^{n+3}$ whose second fundamental forms have one of the structures (i), (ii), or (iii)(a) in Lemma 19.

PROPOSITION 20. *With the notation from Lemma 19, we have*:

(a) α_H *is everywhere as in* (i) *if and only if H is of trivial type*;
(b) α_H *is everywhere as in* (ii) *if and only if H is a composition $H = j \circ i$, where i is an isometric immersion of V into an open subset W of either $\mathbb{R}^{n+2}$ or $\mathbb{L}^{n+2}$, j is an isometric immersion of W into $\mathbb{L}^{n+3}$, and either i or j is totally geodesic*;
(c) α_H *is everywhere as in* (iii)(a) *if and only if H is ruled*.

Proof. (a) Let v be a smooth null vector field and h a smooth symmetric bilinear form such that

$$\alpha_H(X, Y) = h(X, Y)v.$$

Differentiating $\langle v, v\rangle = 0$, we get $\nabla_X^\perp v = \omega(X)v$ for some 1-form ω. The Ricci equation yields $d\omega = 0$ and thus $\omega = d\rho$ locally for some $\rho \in C^\infty(V)$. It follows that $e_1 = e^{-\rho}v$ is constant in $\mathbb{L}^{n+3}$, so $H(V)$ is contained in an affine degenerate hyperplane of $\mathbb{L}^{n+3}$ orthogonal to e_1. Therefore, H is of trivial type by Proposition 7. The converse is trivial.

(b) This statement follows from [12, Thm. 1], which can easily be checked to hold also for isometric immersions into Lorentzian space.

(c) Let ξ, η be a smooth pseudo-orthonormal frame of $T^\perp V$, and let X_1, X_2 be smooth unit vector fields on V spanning $\Delta^\perp$ such that

$$\alpha_H(X_1, X_1) = \xi, \quad \alpha_H(X_1, X_2) = \eta, \quad \text{and} \quad \alpha_H(X_2, X_2) = 0.$$

We need only show that the distribution $\mathcal{D} = \Delta_H \oplus \operatorname{span}\{X_2\}$ is totally geodesic. Taking the ξ-component of the Codazzi equation

$$(\nabla_Z^\perp \alpha_H)(X, Y) = (\nabla_X^\perp \alpha_H)(Z, Y) \tag{72}$$

applied to $Z = X_1$ and $X = Y = X_2$ gives $\Gamma^1_{22} = 0$, where the Γ^k_{ij} are defined by $\nabla_{X_i} X_j = \sum \Gamma^k_{ij} X_k$ $(1 \le i, j, k \le 2)$. Hence, $\nabla_{X_2} X_2 \in \mathcal{D}$. On the other hand, taking the η-component of (72) applied to $Z \in \Delta_H$ and $X = Y = X_2$ yields that the X_1-components α and β of $\nabla_{X_2} Z$ and $\nabla_Z X_2$ (respectively) are related by $\alpha = 2\beta$, whereas taking the ξ-component of (72) applied to $Z \in \Delta_H$, $X = X_2$, and $Y = X_1$ gives $\alpha = \beta$. Therefore $\alpha = \beta = 0$, and we conclude that $\nabla_{X_2} Z, \nabla_Z X_2 \in \mathcal{D}$. □

Notice that if the second fundamental form α_H of an isometric immersion $H\colon V \subset \mathbb{R}^{n+1} \to \mathbb{L}^{n+3}$ is as in (iii)(c) of Lemma 19 at some point $x \in M^n$, then $Z = X_1 + iX_2$ and $\bar{Z} = X_1 - iX_2$ form a complex conjugate diagonalizing basis for $\alpha_H(x)$. We call H of *real* (resp., *complex*) type if α_H is everywhere as in (iii)(b) (resp., (iii)(c)). The easiest way to construct isometric immersions of real type is through a *composition*. By that we mean composing two flat hypersurfaces, where the first one lies in either $\mathbb{R}^{n+1}$ or $\mathbb{L}^{n+1}$. We say accordingly that H is of *first* or *second composition type.* Observe that this construction can be done parametrically because any flat hypersurface can be locally given by means of the Gauss parameterization (see [11]). Many isometric immersions of real type that are not of composition type can be obtained by a parametric construction similar to the one given in the last section of [8].

If α_H is everywhere as in (ii) or (iii)(a) of Lemma 19, then it follows immediately from parts (ii) and (iii) of Proposition 20 that f_H is, respectively, conformally flat and conformally ruled. On the other hand, we have the following result.

PROPOSITION 21. *Let $H\colon V \subset \mathbb{R}^{n+1} \to \mathbb{L}^{n+3}$ be as in Theorem 1. Then the following statements hold*:

- (i) *f_H is a rotation hypersurface over a surface $L^2 \subset \mathbb{R}^3$ if and only if $H(V) \subset N^2 \times \mathbb{R}^{n-1}$, where N^2 is a surface in $\mathbb{L}^4$;*
- (ii) *f_H is of type* (I) *but not a rotation hypersurface over a surface $L^2 \subset \mathbb{R}^3$ if and only if $H(V) \subset N^3 \times \mathbb{R}^{n-2}$, where N^3 is the cone over a surface in a totally umbilical hypersurface of $\mathbb{L}^4$;*
- (iii) *f_H is a Cartan hypersurface of real or complex type if and only if H is, respectively, of real or complex type.*

Proof. Let $\hat{C}$ and C denote the splitting tensors associated (respectively) to the relative nullity distribution Δ_H of H and to the eigenbundle Δ of f_H correspondent to the principal curvature of multiplicity $n-2$. Since the spherical leaves of Δ are the intersections of the leaves of Δ_H with the light cone, it follows that $\Delta_H^\perp = \Delta^\perp$ and

$$C_T = \hat{C}_T \quad \text{for all } T \in \Delta. \tag{73}$$

In particular, $\hat{C}_T$ is identically zero for all $T \in \Delta$ or there exist $T_0 \in \Delta$ and a smooth function μ such that $\hat{C}_T = \mu\langle T, T_0\rangle \mathrm{I}$ for all $T \in \Delta$ if and only if the same holds for C_T. This is equivalent to saying that the distribution $\Delta_H^\perp$ is totally geodesic or totally umbilical if and only if the same holds for $\Delta^\perp$. The assertions in

(i) and (ii) now follow from the main theorem of [10], which is also valid for isometric immersions into Lorentzian space. The statement in (iii) follows from (73) and

$$\alpha_H(\hat{C}_T X, Y) = \alpha_H(X, \hat{C}_T Y) \quad \text{for all } \; X, Y \in \Delta_H^\perp, \tag{74}$$

which is an easy consequence of the Codazzi equation. □

Let $H_i\colon U_i \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$ $(1 \le i \le 2)$ be isometric embeddings that are free of totally geodesic points. Assume that H_1 and H_2 are in *general position*; that is, assume their Gauss maps η_1, η_2 satisfy $0 < \langle \eta_1, \eta_2 \rangle < 1$ along $M^n = H_1(U_1) \cap H_2(N^{n+1})$ and that the relative nullity spaces of H_1 and H_2 are transversal at any point of M^n. Here and in the following, by H_i^{-1} we mean the inverse of the map $H_i\colon U_i \to H_i(U_i)$. Define $f_{H_i}\colon M^n \to \mathbb{R}^{n+1}$ by $f_{H_i} = H_i^{-1}\big|_{M^n}$. It was shown in [9] that f_{H_1} and f_{H_2} are, generically, isometric nowhere congruent Sbrana–Cartan hypersurfaces that admit no further isometric deformations. Moreover, an explicit parameterization for the hypersurfaces obtained by this construction was provided.

The procedure just described can be adapted to construct Cartan hypersurfaces. Let $H_1\colon U \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$ be an isometric embedding that is free of totally geodesic points and let $H_2\colon N^{n+1} \to \mathbb{R}^{n+2}$ be a conformally flat hypersurface free of umbilic points. We assume that N^{n+1} is globally conformal to an open subset $V \subset \mathbb{R}^{n+1}$ by a conformal diffeomorphism $\Phi\colon N^{n+1} \to V$. Assume that H_1 and H_2 are in *general position*; that is, the Gauss maps η_1, η_2 of H_1, H_2 (respectively) satisfy $0 < \langle \eta_1, \eta_2 \rangle < 1$ along $M^n = H_1(U) \cap H_2(N^{n+1})$, and the relative nullity leaves of H_1 and the spherical leaves of the eigenbundle of H_2 correspondent to the principal curvature with multiplicity n are transversal at any point of M^n. Then, we may produce a pair (f_{H_1}, f_{H_2}) of conformal immersions of M^n into $\mathbb{R}^{n+1}$ by letting $f_{H_1}\colon M^n \to \mathbb{R}^{n+1}$ be the isometric immersion $f_{H_1} = H_1^{-1}\big|_M$ and setting $f_{H_2} = \Phi \circ H_2^{-1}\big|_M$.

We say that a pair (f, g), where $f\colon M^n \to \mathbb{R}^{n+1}$ is a hypersurface and $g\colon M^n \to \mathbb{R}^{n+1}$ a conformal deformation of f, is of *first intersection type* if there are isometric embeddings $H_1\colon U \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$ and $H_2\colon N^{n+1} \to \mathbb{R}^{n+2}$ as before and an isometry $\tau\colon M^n \to H_1(U) \cap H_2(N^{n+1})$ such that $f = f_{H_1} \circ \tau$ and $g = f_{H_2} \circ \tau$. We call the pair (f, g) of *second intersection type* if $\mathbb{R}^{n+2}$ is replaced by $\mathbb{L}^{n+2}$ in the construction.

THEOREM 22. *Let $H\colon V \subset \mathbb{R}^{n+1} \to \mathbb{L}^{n+3}$ be as in Theorem 1. Suppose further that H is of first or second composition type. Then the pair (f_H, g_H) is of first or second intersection type, respectively.*

Conversely, if a hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ and a conformal deformation $g\colon M^n \to \mathbb{R}^{n+1}$ of f form a pair of first or second intersection type, then there exist an isometric embedding $H\colon V \subset \mathbb{R}^{n+1} \to \mathbb{L}^{n+3}$ as in Theorem 1 of first or second composition type (respectively) and an isometry $\tau\colon M^n \to H(V) \cap \mathbb{V}$ such that $f = f_H \circ \tau$ and $g = g_H \circ \tau$.

Proof. Assume $H = K \circ H_1$, where $H_1 \colon V \to U \subset \mathbb{R}^{n+2}$ and $K \colon U \to \mathbb{L}^{n+3}$ are isometric embeddings. Set $N^{n+1} = K(U) \cap \mathbb{V}$ and let $i \colon N^{n+1} \to \mathbb{V}$ denote the inclusion map. Restricting U if necessary, we may assume that $N^{n+1} \cap \mathbb{R}_w = \emptyset$ for some $w \in \mathbb{V}$. Then $\mathcal{C}_w(i)$ is a conformal diffeomorphism of N^{n+1} onto an open subset of $\mathbb{R}^{n+1}$. Hence, N^{n+1} is conformally flat and admits an isometric embedding $H_2 = K^{-1}\big|_{N^{n+1}} \colon N^{n+1} \to U \subset \mathbb{R}^{n+2}$. Set $M^n = H(V) \cap \mathbb{V}$, $\tilde{M}^n = H_2(M^n) = H_1(V) \cap H(N^{n+1})$, and $\tau = H_2\big|_{M^n} \colon M^n \to \tilde{M}^n$. Then

$$f_{H_1} \circ \tau = H_1^{-1} \circ K^{-1}\big|_{M^n} = H^{-1}\big|_{M^n} = f_H$$

and

$$f_{H_2} \circ \tau = \big(\mathcal{C}_w(i) \circ H_2^{-1}\big|_{\tilde{M}^n}\big) \circ H_2\big|_{M^n} = \mathcal{C}_w(i)\big|_{M^n} = \mathcal{C}_w\big(i\big|_{M^n}\big) = g_H.$$

Conversely, assume that there exist a conformally flat manifold N^{n+1} that admits a global conformal diffeomorphism Φ onto an open subset of $\mathbb{R}^{n+1}$, isometric embeddings $H_1 \colon V \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$ and $H_2 \colon N^{n+1} \to \mathbb{R}^{n+2}$ in general position, and an isometry $\tau_1 \colon M^n \to H_1(V) \cap H_2(N^{n+1})$ such that $f = f_{H_1} \circ \tau_1$ and $g = f_{H_2} \circ \tau_1$. By [8, Thm. 2.1], there exist an isometric embedding $K \colon W \subset \mathbb{R}^{n+2} \to \mathbb{L}^{n+3}$ of an open subset $W \supset H_2(N^{n+1})$ and an isometry $\tau_2 \colon N^{n+1} \to K(W) \cap \mathbb{V}$ such that $H_2 = K^{-1} \circ \tau_2$ and $\Phi = \mathcal{C}_w(\tau_2)$ for some $w \in \mathbb{V}$. We can assume that $H_1(V) \subset W$ and set $H = K \circ H_1$, $\tau = K \circ \tau_1$, and $\tilde{M}^n = H(V) \cap \mathbb{V}$. Then

$$f_H \circ \tau = H^{-1} \circ \tau = H_1^{-1} \circ K^{-1} \circ \tau = f_{H_1} \circ \tau_1 = f$$

and

$$g_H \circ \tau = \mathcal{C}_w(i) \circ K \circ \tau_1 = \mathcal{C}_w(\tau_2) \circ (\tau_2^{-1} \circ K) \circ \tau_1 = \Phi \circ H_2^{-1} \circ \tau_1 = f_{H_2} \circ \tau_1 = g,$$

and this concludes the proof. □

Finally, we show that the class of conformally deformable hypersurfaces of intersection type is invariant under conformal deformations.

PROPOSITION 23. *Let $H \colon V \subset \mathbb{R}^{n+1} \to \mathbb{L}^{n+3}$ be an isometric immersion of composition type and let $\bar{f} \colon M^n := H(V) \cap \mathbb{V} \to \mathbb{R}^{n+1}$ be a conformal deformation of f_H. Then there is an isometric immersion of composition type $\bar{H} \colon \bar{V} \subset \mathbb{R}^{n+1} \to \mathbb{L}^{n+3}$ such that $\bar{f} = f_{\bar{H}}$.*

We need the following result.

LEMMA 24. *Let $H \colon V \subset \mathbb{R}^{n+1} \to \mathbb{L}^{n+3}$ be an isometric immersion of real type. Let ξ, η be an orthonormal normal frame of H as in Lemma 19(iii)(b) and set $\Delta_\xi = \ker A_\xi$ and $\Delta_\eta = \ker A_\eta$. Then H is locally of composition type if and only if either Δ_ξ or Δ_η is totally geodesic in V.*

Proof. Define a 1-form ω on V by $\omega(X) = \langle \nabla_X^\perp \xi, \eta \rangle$. We have from [12] that H is of composition type if and only if either $\ker \omega \subset \ker A_\xi$ or $\ker \omega \subset \ker A_\eta$. We show that $\ker \omega \subset \ker A_\xi$ if and only if Δ_η is totally geodesic in V. In fact, let $X, Y \in TV$ be unit length eigenvectors of A_ξ and A_η (respectively) correspondent to the nonzero principal curvature; that is, $A_\xi X = \lambda X$ and $A_\eta Y = \mu Y$, where $\lambda, \mu \neq 0$. The Codazzi equation gives

$$\mu\langle \nabla_S S', Y\rangle = \lambda\langle S', X\rangle(\omega(Y)\langle S, X\rangle - \omega(S)\langle Y, X\rangle) \quad \text{for all } \ S, S' \in \Delta_\eta,$$

and the proof follows easily. □

Proof of Proposition 23. As before, let $\hat{C}$ and C denote the splitting tensors associated (respectively) to the relative nullity distribution Δ_H of H and to the eigenbundle Δ of f_H correspondent to the principal curvature of multiplicity $n-2$. With the notation of Lemma 24, by (74) we have that $A_\xi \hat{C}_T = \hat{C}_T^* A_\xi$ and $A_\eta \hat{C}_T = \hat{C}_T^* A_\eta$ for any $T \in \Delta_H$. It follows that the vectors X, Y are eigenvectors of $\hat{C}_T^*$ for any $T \in \Delta_H$. By (73), the same holds for C_T^* for any $T \in \Delta$. In particular, the unit vector $Y^\perp$ orthogonal to Y in $\Delta_H^\perp$ is an eigenvector of C_T for any $T \in \Delta$. Using that $\nabla_T^h Y = 0$ and $\langle \nabla_{Y^\perp}^h T, Y\rangle = -\langle C_T Y^\perp, Y\rangle = 0$, we conclude that $\Delta_\eta = \Delta \oplus \operatorname{span}\{Y^\perp\}$ is totally geodesic if and only if $\nabla_{Y^\perp}^h Y^\perp = 0$, which is an intrinsic condition, and the proof follows. □

References

[1] A. Asperti and M. Dajczer, *N-dimensional submanifolds of* $\mathbb{R}^{N+1}$ and $\mathbb{S}^{N+2}$, Illinois J. Math. 28 (1984), 621–645.

[2] E. Cartan, *La déformation des hypersurfaces dans l'espace euclidien réel a n dimensions,* Bull. Soc. Math. France 44 (1916), 65–99.

[3] ———, *La déformation des hypersurfaces dans l'espace conforme réel a $n \geq 5$ dimensions,* Bull. Soc. Math. France 45 (1917), 57–121.

[4] ———, *Sur certains hypersurfaces de l'espace conforme réel a cinq dimensions,* Bull. Soc. Math. France 46 (1918), 84–105.

[5] ———, *Sur le problème général de la déformation*, C. R. Congrès Strasbourg (1920), 397–406.

[6] M. Dajczer, *A characterization of complex hypersurfaces in C^m*, Proc. Amer. Math. Soc. 105 (1989), 425–428.

[7] M. Dajczer, et al., *Submanifolds and isometric immersions,* Math. Lecture Ser., 13, Publish or Perish, Houston, TX, 1990.

[8] M. Dajczer and L. Florit, *On conformally flat submanifolds,* Comm. Anal. Geom. 4 (1996), 261–284.

[9] M. Dajczer, L. Florit, and R. Tojeiro, *On deformable hypersurfaces in space forms,* Ann. Mat. Pura Appl. (4) 147 (1998), 361–390.

[10] ———, *On a class of submanifolds carrying an extrinsic totally umbilical foliation*, Israel J. Math. (to appear).

[11] M. Dajczer and D. Gromoll, *Gauss parameterizations and rigidity aspects of submanifolds,* J. Differential Geom. 22 (1985), 1–12.

[12] M. Dajczer and R. Tojeiro, *Submanifolds with nonparallel first normal bundle,* Canad. Math. Bull. 37 (1994), 330–337.

[13] M. Dajczer and E. Vergasta, *Conformal hypersurfaces with the same Gauss map,* Trans. Amer. Math. Soc. 347 (1995), 2437–2450.

[14] M. do Carmo and M. Dajczer, *Conformal rigidity,* Amer. J. Math. 109 (1987), 963–985.

[15] J. D. Moore, *Submanifolds of constant positive curvature I,* Duke Math. J. 44 (1977), 449–484.

[16] H. Reckziegel, *On the eigenvalues of the shape operator of an isometric immersion into a space of constant curvature,* Math. Ann. 243 (1979), 71–82.

[17] V. Sbrana, *Sulla varietá ad $n-1$ dimensioni deformabili nello spazio euclideo ad n dimensioni,* Rend. Circ. Mat. Palermo 27 (1909), 1–45.

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