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# ON CASTELNUOVO-MUMFORD REGULARITY OF PROJECTIVE CURVES

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Abstract. We give an effective method to compute the regularity of a saturated ideal $I$ defining a projective curve that also determines in which step of a minimal graded free resolution of $I$ the regularity is attained.

## Introduction

Let $S := K[x_0, \dots , x_n]$ be a polynomial ring over an algebraically closed field $K$, and let $I$ be a homogeneous ideal of $S$ defining a subscheme $\mathfrak{X}$ of projective $n$-space $\mathbb{P}^n_K$. The *Castelnuovo-Mumford regularity* (or simply *regularity*) of $I$, reg $I$, is defined as follows: if

$$(0.1) \qquad 0 \to \bigoplus_{j=1}^{\beta_p} S(-e_{pj}) \xrightarrow{\varphi_p} \cdots \xrightarrow{\varphi_1} \bigoplus_{j=1}^{\beta_0} S(-e_{0j}) \xrightarrow{\varphi_0} I \to 0$$

is a minimal graded free resolution of $I$, setting $e_i := \max\{e_{ij};\ 1 \le j \le \beta_i\}$, then reg $I := \max\{e_i - i;\ 0 \le i \le p\}$. In other words, reg $I$ is the smallest integer $m$ for which $I$ is $m$-regular, i.e. $e_{ij} \le m + i$ for all $i, j$ (see [2, Def. 3.2] for equivalent definitions). When $I$ is saturated (i.e. when it is the largest ideal defining $\mathfrak{X}$), we call this the *regularity* of $\mathfrak{X}$ (see [2, Sect. 1]).

The regularity is a numerical invariant of the ideal $I$ and is, as said in [6], "an important measure of how hard it will be to compute a free resolution". In fact, knowing it beforehand avoids unnecessary computation in large degrees while obtaining the minimal graded free resolution of $I$ through Buchberger's syzygy algorithm (see [3]).

In this paper, we shall essentially be concerned with the regularity of a saturated ideal $I$ defining a subscheme $\mathfrak{X}$ of $\mathbb{P}^n_K$ of dimension one.

In Section 1, we show a general property of finitely generated graded $S$-modules asserting that the regularity of $M$ is determined by the tail of the minimal graded free resolution (Proposition 1.1). As a consequence we obtain that, in our case, reg $I$ is equal to either $e_{n-1} - n + 1$ or $e_{n-2} - n + 2$, i.e. the regularity is always attained at one of the last two steps of the resolution.

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Assuming that $K[x_{n-1},x_n]$ is a Noether normalization of $S/I$, we give in Section 2 an effective method to compute the regularity of $I$ that does not require the knowledge of a minimal graded free resolution of $I$ (Theorem 2.7). The idea is to introduce an arithmetically Cohen-Macaulay curve whose regularity is closely related with that of $\mathfrak{X}$. For this reason, we first focus on the Cohen-Macaulay case (Theorem 2.4). These two theorems together with an effective criterion to determine whether $\mathfrak{X}$ is arithmetically Cohen-Macaulay (Proposition 2.1), give an algorithm to compute the regularity of $I$. Using Section 1, this algorithm also determines in which step of a minimal graded free resolution of $I$, reg $I$ is attained.

## 1. Where is the regularity attained?

Let $M$ be a finitely generated graded $S$-module and consider a minimal graded free resolution of $M$:

$$0 \to F_p \xrightarrow{\varphi_p} \cdots \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0\ , \tag{1.1}$$

with $F_i = \bigoplus_{j=1}^{\beta_i} S(-e_{ij})$. We denote by $e_i := \max\{e_{ij};\ 1 \le j \le \beta_i\}$.

Using spectral sequences, Schenzel proved that the regularity of $M$ is determined by the tail of (1.1) ([10, Thm. 3.11]). We propose here a different proof of this issue based on an observation of Herzog relating the vanishing of a row in some matrix in (1.1) and the regularity of $M$ when $M$ is Cohen-Macaulay ([11, Cor. B.4.1]). Our treatment is both elementary and carries some additional information.

**Proposition 1.1.** *Let $M$ be a finitely generated graded $S$-module and let* (1.1) *be a minimal graded free resolution of $M$. Denoting $c := n+1-\dim M$, one has:*

$$e_0 < e_1 < \cdots < e_c\ .$$

*Proof.* Assume the claim is false. Then for some $i$, $1 \le i \le c$, the matrix $M_i$ describing $\varphi_i : F_i \to F_{i-1}$ has a zero row.

Consider now the head of the minimal graded free resolution (1.1) of $M$:

$$F_c \xrightarrow{\varphi_c} F_{c-1} \xrightarrow{\varphi_{c-1}} \cdots \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

and apply $\mathrm{Hom}_S(.,S)$ to this complex. Setting $N := \mathrm{Coker}\ \varphi_c^\star$, one gets

$$F_0^\star \xrightarrow{\varphi_1^\star} F_1^\star \xrightarrow{\varphi_2^\star} \cdots \xrightarrow{\varphi_c^\star} F_c^\star \longrightarrow N \to 0 \tag{1.2}$$

which is a complex whose homology is $\mathrm{Ext}_S^i(M,S) = 0$ for $i < c$. Thus, (1.2) is the head of a minimal graded free resolution of $N$, contradicting the fact that the matrix describing $\varphi_i^\star$, the transpose of $M_i$, has a zero column. □

Consider a homogeneous ideal $I$ of $S$ and a minimal graded free resolution (0.1) of $I$. The following is a direct consequence of the above proposition.

**Corollary 1.2.** reg $I = \max\{e_i - i;\ n - \dim S/I \le i \le p\}$.

## 2. How to compute the regularity?

Let $I$ be a homogeneous ideal of $S$ defining a not necessarily reduced projective curve $\mathfrak{C}$ in $\mathbb{P}_K^n$. Assume that $K[x_{n-1},x_n]$ is a Noether normalization of $S/I$ (i.e. $K[x_{n-1},x_n] \hookrightarrow K[x_0,\ldots,x_n]/I$ is an integral ring extension). Monomials in $S$ will be denoted by $\mathbf{x}^\alpha := x_0^{\alpha_0} \cdots x_n^{\alpha_n}$, with $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$. Let $\operatorname{in}(I)$ denote the initial ideal of $I$ with respect to the reverse lexicographic order.

Consider the evaluation morphism $\theta$ (resp. $\chi$): $K[x_0, \dots, x_n] \to K[x_0, \dots, x_{n-2}]$ defined by $x_n \mapsto 0$ (resp. $x_n \mapsto 1$), $x_{n-1} \mapsto 0$ (resp. $x_{n-1} \mapsto 1$) and $x_i \mapsto x_i$ for $i \notin \{n-1, n\}$. Let $\tilde{I}$ be the ideal of $S$ generated by $\chi(\operatorname{in}(I))$. $\tilde{I}$ is a primary monomial ideal such that $\operatorname{in}(I) \subseteq \tilde{I}$ and $\tilde{I}$ defines a projective curve $\tilde{\mathfrak{C}} \subseteq \mathbb{P}_K^n$ of degree $\deg \tilde{\mathfrak{C}} = \deg \mathfrak{C}$ (see [5, Lemme 1]).

Denote by $I_0$ the ideal $I_0 := \theta(I)S \subset S$. As $\operatorname{in}(I_0) = \theta(\operatorname{in}(I))S$, then $\operatorname{in}(I_0) \subseteq \operatorname{in}(I)$ and so the degree of the curve $\mathfrak{C}_0 \subseteq \mathbb{P}_K^n$ defined by $I_0$ satisfies $\deg \mathfrak{C}_0 \geq \deg \mathfrak{C}$.

Define $F := \{\alpha = (\alpha_0, \dots, \alpha_{n-2}) \in \mathbb{N}^{n-1} \mid \mathbf{x}^{(\alpha,0,0)} \in \tilde{I} - \operatorname{in}(I_0)\} \subset \mathbb{N}^{n-1}$. As $K[x_{n-1}, x_n]$ is a Noether normalization of $S/I$, $F$ is finite (possibly empty). The following is a criterion to determine, in terms of $F$, whether $S/I$ is Cohen-Macaulay (i.e. whether $\mathfrak{C}$ is an arithmetically Cohen-Macaulay projective curve). It implies that $S/I$ is Cohen-Macaulay if and only if $S/\operatorname{in}(I)$ is Cohen-Macaulay, and that $S/I_0$ and $S/\tilde{I}$ are Cohen-Macaulay.

**Proposition 2.1.** *$S/I$ is Cohen-Macaulay if and only if* $F = \emptyset$.

*Proof.* Observe that $F = \emptyset$ is equivalent to $\operatorname{in}(I_0) = \operatorname{in}(I)$. As $S/I$ is Cohen-Macaulay if and only if $\{x_{n-1}, x_n\}$ is a regular sequence on $S/I$ ([9, Ch. 3, Prop. 4.4]), we shall prove that $\operatorname{in}(I_0) = \operatorname{in}(I)$ if and only if $\{x_{n-1}, x_n\}$ is a regular sequence on $S/I$.

Assume that $\operatorname{in}(I_0) = \operatorname{in}(I)$. Let $f \in (I : x_n)$. Then $f \in I$ because otherwise the remainder $r$ of the division of $f$ by a Gröbner basis of $I$ w.r.t. the reverse lexicographic order is nonzero and $\operatorname{in}(r) \notin \operatorname{in}(I)$. As $x_n \operatorname{in}(r) \in \operatorname{in}(I)$ and $\operatorname{in}(I) = \operatorname{in}(I_0)$, this is impossible. Similarly, let $f \in ((I, x_n) : x_{n-1})$. For the same reason as above, $f \in (I, x_n)$ because $\operatorname{in}(I, x_n) = (\operatorname{in}(I), x_n)$ and $\operatorname{in}(I) = \operatorname{in}(I_0)$.

Conversely, if $\{x_{n-1}, x_n\}$ is a regular sequence on $S/I$, then the monomials in a minimal set of generators of $\operatorname{in}(I)$ are not divisible by either $x_{n-1}$ or $x_n$. Thus, $\operatorname{in}(I_0) = \operatorname{in}(I)$. ∎

As already stated, $\mathfrak{C}_0$ is arithmetically Cohen-Macaulay by Proposition 2.1 and $\deg \mathfrak{C}_0 \geq \deg \mathfrak{C}$. The difference between $\deg \mathfrak{C}_0$ and $\deg \mathfrak{C}$ is indeed a measure of how far $\mathfrak{C}$ is from being arithmetically Cohen-Macaulay.

**Corollary 2.2.** *$\mathfrak{C}$ is arithmetically Cohen-Macaulay if and only if* $\deg \mathfrak{C} = \deg \mathfrak{C}_0$.

*Proof.* The difference $\deg \mathfrak{C}_0 - \deg \mathfrak{C}$ is equal to $\#F$. In fact, $\deg \mathfrak{C}_0$ is equal to $\#\{\alpha \in \mathbb{N}^{n-1} \mid \mathbf{x}^{(\alpha,0,0)} \notin \operatorname{in}(I_0)\}$ because the Hilbert polynomial of $S/I_0$ is $P_{I_0}(T) = \sum_{\alpha \notin E_0} (T + 1 - |\alpha|)$ where $E_0 = \{\alpha \in \mathbb{N}^{n-1} \mid \mathbf{x}^{(\alpha,0,0)} \in \operatorname{in}(I_0)\}$. By a similar argument $\deg \tilde{\mathfrak{C}} = \#\{\alpha \in \mathbb{N}^{n-1} \mid \mathbf{x}^{(\alpha,0,0)} \notin \tilde{I}\}$. ∎

Assume that $S/I$ is Cohen-Macaulay. We will give an effective method to compute $\operatorname{reg} I$ that does not require the knowledge of a minimal graded free resolution of $I$.

Set $E := \{(\alpha_0, \dots, \alpha_{n-2}) \in \mathbb{N}^{n-1} \mid \mathbf{x}^{(\alpha,0,0)} \in \operatorname{in}(I)\}$. As $K[x_{n-1}, x_n]$ is a Noether normalization of $S/I$, for $s \gg 0$ and $\alpha \in \mathbb{N}^{n-1}$ one has that $|\alpha| \geq s$ implies $\alpha \in E$. Define the *regularity* of $E$, $H(E)$, as the smallest integer $s$ satisfying this property.

Denote by $H(I)$ the *regularity of the Hilbert function* $H_I$ of $S/I$, i.e. the smallest integer $s_0$ such that for $s \geq s_0$, $H_I(s) = P_I(s)$ ($P_I(T)$ is the Hilbert polynomial of $S/I$).

**Lemma 2.3.** $H(E) = H(I) + 2$.

*Proof.* As the value at $s$ of $H_I$ is

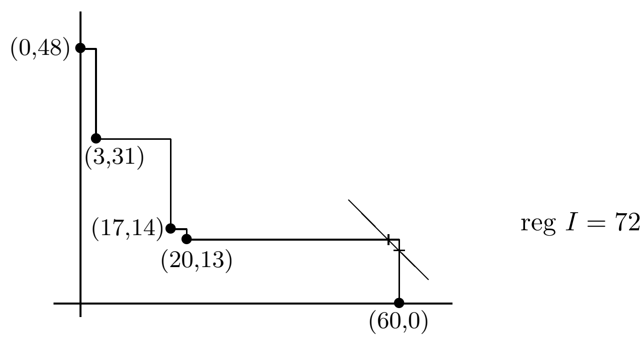
$$H_I(s) = \#\{(\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1} \,|\, \alpha_0 + \cdots + \alpha_n = s \text{ and } (\alpha_0, \dots, \alpha_{n-2}) \notin E\} ,$$

then $P_I(T) = \sum_{\alpha \notin E}(T + 1 - |\alpha|)$. Any element $\alpha \notin E$ satisfies $|\alpha| \leq H(E) - 1$ so $H(I) \leq H(E) - 1$. It is now easy to check that $H_I(s_0) = P_I(s_0)$ for $s_0 = H(E) - 2$ and that $H_I(s_0) > P_I(s_0)$ for $s_0 = H(E) - 3$. □

**Theorem 2.4.** *Let $I \subset S$ be the homogeneous defining ideal of an arithmetically Cohen-Macaulay projective curve $\mathfrak{C} \subset \mathbb{P}^n_K$. Then* reg $I = H(E)$.

*Proof.* By the previous lemma, one has to prove that reg $I = H(I) + 2$. From [6, Prop. 20.20], one gets that reg $I =$ reg $(I, x_{n-1}, x_n)$. As $\dim S/(I, x_{n-1}, x_n) = 0$, then reg $(I, x_{n-1}, x_n)$ coincides with the regularity $H(I, x_{n-1}, x_n)$ of the Hilbert function of $S/(I, x_{n-1}, x_n)$ ([3, Lemma 1.7]). The result now follows from the equality $H(I, x_{n-1}, x_n) = H(I) + 2$. □

**Example 2.5.** Consider the ideal $I \subset K[x, y, z, t]$ generated by $f_1 = x^{17}y^{14} - y^{31}$, $f_2 = x^{20}y^{13}$, $f_3 = x^{60} - y^{36}z^{24} - x^{20}z^{20}t^{20}$. The reduced Gröbner basis of $I$ w.r.t. the reverse lexicographic order is $\{f_1, f_2, f_3, y^{48}, x^3y^{31}\}$, so in $(I) = (x^{17}y^{14}, x^{20}y^{13}, x^{60}, y^{48}, x^3y^{31})$. Then $K[x, y, z, t]/I$ is Cohen-Macaulay (Proposition 2.1) and reg $I = 72$ (Theorem 2.4).




As already observed, $S/I$ is Cohen-Macaulay if and only if $S/$ in $(I)$ is Cohen-Macaulay. Thus, we get the following consequence of Theorem 2.4 which can also be obtained from [3, Thm. 2.4 (b)].

**Corollary 2.6.** *If $I$ satisfies the conditions of Theorem 2.4, then* reg $I =$ reg in $(I)$.

Let's assume now that $I$ is a saturated ideal defining a nonarithmetically Cohen-Macaulay projective curve $\mathfrak{C} \subset \mathbb{P}^n_K$. We shall give a relation between reg $I$ and reg $I_0$ to obtain, as in Theorem 2.4, an effective method to compute reg $I$ that does not require the knowledge of a minimal graded free resolution of $I$.

In this case $F \neq \emptyset$ (Proposition 2.1) and one has the partition introduced in [5]:

$$\{(\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1} \quad | \quad x_0^{\alpha_0} \cdots x_n^{\alpha_n} \notin \text{in}\,(I)\} = \{(\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1} \,|\, x_0^{\alpha_0} \cdots x_{n-2}^{\alpha_{n-2}} \notin \tilde{I}\} \cup \mathfrak{R},$$

where $\mathfrak{R} = \bigcup_{\alpha \in F} \{\alpha \times [\mathbb{N}^2 - E_\alpha]\}$ for $E_\alpha = \{(\alpha_{n-1}, \alpha_n) \in \mathbb{N}^2 \mid \mathbf{x}^{(\alpha, \alpha_{n-1}, \alpha_n)} \in \text{in}\,(I)\}$. Therefore, the value at $s \in \mathbb{N}$ of the Hilbert function $H_I$ of $S/I$ is

$$H_I(s) = H_{\tilde{I}}(s) + \#\{\beta \in \mathfrak{R} \,||\beta| = s\}\,,$$

where $\#\{\beta \in \mathfrak{R} \,||\beta| = s\}$ is constant for $s \gg 0$. Denote by $H(\mathfrak{R})$ (resp. $H(E_\alpha)$) the smallest integer $s_0$ such that for $s \geq s_0$, $\#\{\beta \in \mathfrak{R} \,||\beta| = s\}$ (resp. $\#\{(\alpha_{n-1}, \alpha_n) \in \mathbb{N}^2 - E_\alpha \mid \alpha_{n-1} + \alpha_n = s\}$) is constant. It is clear that

$$H(\mathfrak{R}) \leq \max{}_{\alpha \in F}\{|\alpha| + H(E_\alpha)\}.$$

**Theorem 2.7.** *Let $I \subset S$ be a saturated ideal defining a nonarithmetically Cohen-Macaulay projective curve $\mathfrak{C} \subset \mathbb{P}^n_K$. Then* $\text{reg}\ I = \max\{\text{reg}\ I_0, H(\mathfrak{R}) + 1\}$.

*Proof.* Since the field $K$ is infinite and $K[x_{n-1}, x_n]$ is a Noether normalization of $S/I$ and $I$ is a saturated ideal, then there exists $\kappa \in K - \{0\}$ such that $x_n - \kappa x_{n-1}$ is a nonzero divisor on $S/I$. If we denote by $I_\kappa$ the ideal $(I, x_n - \kappa x_{n-1})$ of $S$, then $\text{reg}\ I = \text{reg}\ I_\kappa$ by [6, Prop. 20.20].

On the other hand, if $(I_\kappa)^{sat}$ is the saturation of $I_\kappa$, one deduces from [3, Lemmas 1.6, 1.7, 1.8] that $\text{reg}\ I_\kappa = \max\{s_0,\, H(I_\kappa, h)\}$ where $h$ is a linear form which is a nonzero divisor on $S/(I_\kappa)^{sat}$, and $s_0$ is the smallest integer such that, for any $s \geq s_0$, $(I_\kappa : h)_s = (I_\kappa)_s$.

Since $S/(I_\kappa)^{sat}$ is a finite $K[x_n]$-module of dimension 1, then $K[x_n]$ is a Noether normalization of $S/(I_\kappa)^{sat}$ by [9, Ch. 2, Rem. 6.5.0]. Thus, $x_n$ is a nonzero divisor on $S/(I_\kappa)^{sat}$ and $\text{reg}\ I_\kappa = \max\{s_0,\, H(I_\kappa, x_n)\}$, $s_0$ being the smallest integer such that, for any $s \geq s_0$, $(I_\kappa : x_n)_s = (I_\kappa)_s$.

Let us prove now that $\text{reg}\ I_\kappa = \max\{H(I)+1, H(I_\kappa, x_n)\}$. Indeed, as for any $s$,

$$0 \to S_{s-1}/(I_\kappa : x_n)_{s-1} \xrightarrow{\cdot x_n} S_s/(I_\kappa)_s \xrightarrow{\varphi} S_s/(I_\kappa, x_n)_s \to 0$$

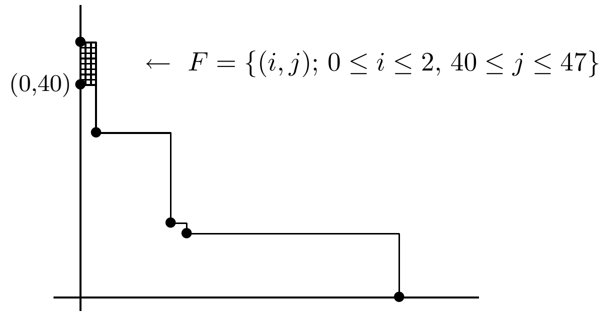
is an exact sequence, where $\varphi$ is the canonical morphism, and as $H(I_\kappa) = H(I)+1$, one has $\max\{s_0,\, H(I_\kappa, x_n)\} = \max\{H(I)+1,\, H(I_\kappa, x_n)\}$.

On the other hand, $H(I_\kappa, x_n) = \text{reg}\ I_0$ because $(I_\kappa, x_n) = (I_0, x_{n-1}, x_n)$ and $I_0$ defines an arithmetically Cohen-Macaulay curve (see proof of Theorem 2.4).

Finally, $\max\{H(I)+1, \text{reg}\ I_0\} = \max\{H(\mathfrak{R})+1, \text{reg}\ I_0\}$. Indeed, as in $(I_0) \subseteq \tilde{I}$, then $H(\tilde{I}) + 2 = \text{reg}\ \tilde{I} \leq \text{reg}\ I_0$ by Lemma 2.3, Theorem 2.4 and Corollary 2.6. If $H(\mathfrak{R})$ and $H(I)$ are smaller or equal to $H(\tilde{I})$, the result follows from the previous inequality. Otherwise, it is easy to check that $H(\mathfrak{R}) = H(I)$ and we are done. □

*Remark* 2.8. It is worth noting that knowledge of $\text{in}\,(I)$ and some extra combinatorial work give the value of $\text{reg}\ I$. In fact, $\text{in}\,(I_0)$ is generated by the minimal generators of $\text{in}\,(I)$ which are not divisible by either $x_n$ or $x_{n-1}$ because $\text{in}\,(I_0) = \theta(\text{in}\,(I))S$. Taking $E = \{\alpha \in \mathbb{N}^{n-1} \mid \mathbf{x}^{(\alpha,0,0)} \in \text{in}\,(I_0)\}$, one gets $\text{reg}\ I_0 = H(E)$ by Theorem 2.4. On the other hand, $H(\mathfrak{R})$ is also obtained from $\text{in}\,(I)$.

**Example 2.9.** For any $\ell \geq 1$, consider the saturated ideal $I_\ell = (f_1, f_2, f_3, h_\ell) \subset K[x, y, z, t]$ generated by $f_1, f_2, f_3$ of the Example 2.5 and by $h_\ell = y^{41}z^\ell - y^{40}z^{\ell+1}$. One can check that $\{f_1, f_2, f_3, h_\ell, y^{48}, x^3y^{31}, y^{40}z^{\ell+8}\}$ is the reduced Gröbner basis of $I_\ell$ w.r.t. the reverse lexicographic order. Then $\text{in}\,(I_\ell) = (x^{17}y^{14},\, x^{20}y^{13},\, x^{60},\, y^{41}z^\ell,\, y^{48},\, x^3y^{31},\, y^{40}z^{\ell+8})$. The set $F$ is not empty and independent of $\ell$. It is represented by the following diagram:




Then for $\ell \geq 1$, $K[x, y, z, t]/I_\ell$ is not Cohen-Macaulay by Proposition 2.1. Observe that for any $\ell \geq 1$, in $(I_\ell)_0$ coincides with in $(I)$, where $I$ is the ideal $(f_1, f_2, f_3)$ of the Example 2.5. The regularity of $(I_\ell)_0$ is then reg $(I_\ell)_0 = 72$. Now $E_\alpha = (\ell+8, 0)+\mathbb{N}^2$ for any $\alpha = (i, 40) \in F$, and $E_\alpha = (\ell, 0) + \mathbb{N}^2$ for any $\alpha = (i, j) \in F$ with $j \geq 41$. So $H(\mathfrak{R}) + 1 = \max_{\alpha \in F}\{|\alpha| + H(E_\alpha)\} + 1 = 50 + \ell$ and reg $I_\ell = \max\{72, 50 + \ell\}$ by Theorem 2.7.

*Remark* 2.10. Observe that in the previous example, in $(I_\ell)$ is a saturated ideal for any $\ell \geq 1$, but it is not true in general that $I = I^{sat}$ implies that in $(I) = $ in $(I)^{sat}$. For example, the ideal $I \subset K[x, y, z, t]$ generated by $x^2 - 3xy + 5xt$, $xy - 3y^2 + 5yt$, $xz - 3yz$, $2xt - yt$ and $y^2 - yz - 2yt$ is saturated since $z - t$ is a nonzero divisor on $K[x, y, z, t]/I$ and in $(I) = (yzt, y^2, xt, xz, xy, x^2)$ is not saturated because $z - \kappa t$ is a zero divisor on $K[x, y, z, t]/$ in $(I)$, for any $\kappa \in K$. In this example, reg $I \neq$ reg in $(I)$ as reg $I = 2$ by Theorem 2.7 (reg $I_0 = H(\mathfrak{R}) + 1 = 2$) and one can check with [4] that reg in $(I) = 3$. Nevertheless, if in $(I)$ is also saturated one gets directly from Theorem 2.7 that

$$\text{reg } I = \text{reg in}\,(I)\ .$$

In particular, if $x_n$ is a nonzero divisor on $S/I$, one has in $(I) = $ in $(I)^{sat}$ and the above equality also comes from [3, Thm. 2.4 (b)].

The last result of this section says that the method obtained from Theorems 2.4 and 2.7 to compute the regularity of $I$ also determines when the regularity is attained at the last step of a minimal graded free resolution of $I$.

**Corollary 2.11.** *Let $I \subset S$ be a saturated ideal defining a projective curve $\mathfrak{C} \subset \mathbb{P}^n_K$. Then* reg $I$ *is attained at the last step of a minimal graded free resolution of $I$ if and only if either $S/I$ is Cohen-Macaulay or* reg $I = H(\mathfrak{R}) + 1$.

*Proof.* When $S/I$ is Cohen-Macaulay, the result is a consequence of Corollary 1.2. Assume that $S/I$ is not Cohen-Macaulay. As a consequence of the proof of Theorem 2.7, one has that reg $I = H(\mathfrak{R}) + 1$ if and only if reg $I = H(I) + 1$. Let

$$0 \to \bigoplus_{j=1}^{\beta_{n-1}} S(-e_{n-1,j}) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{\beta_0} S(-e_{0j}) \longrightarrow I \to 0$$

be a minimal graded free resolution of $I$. The Hilbert series of $S/I$ is $\frac{Q(t)}{(1-t)^{n+1}}$ with

$$Q(t) = 1 - (t^{e_{01}} + \cdots + t^{e_{0\beta_0}}) + \cdots + (-1)^n(t^{e_{n-1,1}} + \cdots + t^{e_{n-1,\beta_{n-1}}})$$

and $\deg(Q(t)) = H(I) + n$. Since $\deg(Q(t)) \leq \text{reg } I + n - 1$, and equality holds if and only if $\text{reg } I + n - 1 = e_{n-1}$, and the result follows. □

In summary, avoiding the construction of a minimal graded free resolution of $I_\ell$, in Example 2.9, one can assert now that for any $\ell$, $1 \leq \ell \leq 21$, the regularity of $I_\ell$ is attained at the second step of a minimal graded free resolution of $I_\ell$ but not at the third step. For $\ell \geq 22$, the regularity of $I_\ell$ is attained at the third step of a minimal graded free resolution of $I$ but can also occur at the second step.

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