Hu et al. *Journal of Inequalities and Applications* (2020) 2020:8
https://doi.org/10.1186/s13660-020-2283-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

On Cauchy–Schwarz inequality for N-tuple diamond-alpha integral

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Abstract
In this paper, we present some new Cauchy–Schwarz inequalities for *N*-tuple diamond-alpha integral on time scales. The obtained results improve and generalize some Cauchy–Schwarz type inequalities given by many authors.

MSC: 26D15

Keywords: Cauchy–Schwarz inequality; Diamond-alpha integral; Time scales

1 Introduction

To unify discrete and continuous analysis and generalize the discrete and continuous theories to cases "in between", Stefan Hilger in [1] initiated the notions of a time scale and a delta derivative of a function defined on the time scale. The author then presented the calculus on time scales. If the time scale is an interval, the calculus is reduced to the classical calculus; if the time scale is discrete, the calculus is reduced to the calculus of finite differences. Since then, research in the area of the theory of time scales introduced by Stefan Hilger has exceeded by far a thousand publications, and numerous applications to all branches of science, such as operations research, engineering, economics, physics, finance, statistics, and biology, have been proposed. For more details on time scales theory, the interested readers may consult [2–12] and the references therein.

As we all know, inequality plays a very important and basic role in all mathematic areas, and it is also an indispensable and basic tool in engineering technology (see [13–27]). The classic Cauchy–Schwarz inequality is an important cornerstone in some branches of mathematic areas. It is also a bridge to help solve problems into depth. In [28], Agarwal et al. first gave the following Cauchy–Schwarz inequality for Δ-integral on time scale.

Theorem A *Let $\mathbb{T}$ be a time scale, $t_1, t_2 \in \mathbb{T}$ with $t_1 < t_2$, and let $s, t \in C_{rd}([t_1, t_2], \mathbb{R})$. Then*

$$\int_{t_1}^{t_2} \left|s(x)t(x)\right| \Delta x \leq \left(\int_{t_1}^{t_2} s^2(x)\Delta x\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} t^2(x)\Delta x\right)^{\frac{1}{2}}. \tag{1}$$

Later, Wong et al. [29] presented the extension of inequality (1).

Theorem B *Let $\mathbb{T}$ be a time scale, $t_1, t_2 \in \mathbb{T}$ and $t_1 < t_2$, and let $s, t, \lambda \in C_{rd}([t_1, t_2], \mathbb{R})$. Then*

$$\int_{t_1}^{t_2} |\lambda(x)||s(x)t(x)|\Delta x \le \left(\int_{t_1}^{t_2} |\lambda(x)|s^2(x)\Delta x\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |\lambda(x)|t^2(x)\Delta x\right)^{\frac{1}{2}}.$$

In 2008, Özkan et al. [30] introduced the time scale versions of (1) for the ∇-integral and $\diamond_\alpha$-integral, respectively.

Theorem C *Let $\mathbb{T}$ be a time scale, $t_1, t_2 \in \mathbb{T}$ and $t_1 < t_2$, and let $s, t, \lambda \in C_{ld}([t_1, t_2], \mathbb{R})$. Then*

$$\int_{t_1}^{t_2} |\lambda(x)||s(x)t(x)|\nabla x \le \left(\int_{t_1}^{t_2} |\lambda(x)|s^2(x)\nabla x\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |\lambda(x)|t^2(x)\nabla x\right)^{\frac{1}{2}}. \tag{2}$$

Theorem D *Let $\mathbb{T}$ be a time scale, $t_1, t_2 \in \mathbb{T}$ with $t_1 < t_2$, and let $s, t, \lambda : [t_1, t_2] \to \mathbb{R}$ be $\diamond_\alpha$-integrable functions. Then*

$$\int_{t_1}^{t_2} |\lambda(x)||s(x)t(x)| \diamond_\alpha x \le \left(\int_{t_1}^{t_2} |\lambda(x)|s^2(x) \diamond_\alpha x\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |\lambda(x)|t^2(x) \diamond_\alpha x\right)^{\frac{1}{2}}. \tag{3}$$

Remark 1.1 Taking $\alpha = 0$ in Theorem D, inequality (3) is reduced to inequality (2). Taking $\alpha = 1$ in Theorem D, inequality (3) is reduced to inequality (1).

In 2018, Tian [3] gave the triple diamond-alpha integral and proved that the Cauchy–Schwarz inequality holds for the triple diamond-alpha integral on the time scale.

Theorem E *Let $\lambda(x_1, x_2, x_3), s(x_1, x_2, x_3), t(x_1, x_2, x_3) : [a_i, b_i]^3_{\mathbb{T}} \to \mathbb{R}$ be $\diamond_\alpha$-integrable functions with $\lambda(x_1, x_2, x_3) \ge 0$. Then*

$$\begin{aligned}
&\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \lambda(x_1,x_2,x_3)s(x_1,x_2,x_3)t(x_1,x_2,x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^2 \\
&\quad\le \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \lambda(x_1,x_2,x_3)s^2(x_1,x_2,x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right) \\
&\qquad\times \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \lambda(x_1,x_2,x_3)t^2(x_1,x_2,x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right).
\end{aligned}$$

In 2019, Tian et al. [4] introduced the notion of n-tuple diamond-alpha integral for a function of n variables and established the Cauchy–Schwarz inequality for n-tuple diamond-alpha integral as follows.

Theorem F *Let $\lambda(\mathbf{x}), s(\mathbf{x}), t(\mathbf{x}) : [a_i, b_i]^n_{\mathbb{T}} \to \mathbb{R}$ be $\diamond_\alpha$-integrable functions with $\lambda(\mathbf{x}) \ge 0$. Then*

$$\begin{aligned}
&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right)^2 \\
&\quad\le \left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\
&\qquad\times \left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right),
\end{aligned} \tag{4}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

Moreover, Yeh et al. in [31] presented some interesting complements of the Cauchy–Schwarz inequality via delta integral. Motivated by the above results, in the paper, by using methods similar to that in [31], we shall give some new variants, generalizations, and refinements of the Cauchy–Schwarz inequality for n-tuple diamond-alpha integral on time scales.

2 Main results

Throughout the paper, we use $\mathbb{T}$ to denote a time scale, denote $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $[a_i, b_i]^n_{\mathbb{T}} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, where $x_i, a_i, b_i \in \mathbb{T}$ with $a_i < b_i$, $i = 1, 2, \ldots, n$.

Proposition 2.1 ([4]) *Let $s(\mathbf{x})$, $t(\mathbf{x})$ be $\diamond_\alpha$-integrable functions on $[a_i, b_i]^n_{\mathbb{T}}$ ($i = 1, 2, \ldots, n$).*

(P1) *If $s(\mathbf{x}) \geq 0$ for $\mathbf{x} \in [a_i, b_i]^n_{\mathbb{T}}$, then*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} s(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n \geq 0.$$

(P2) *If $s(\mathbf{x}) \leq t(\mathbf{x})$ for $\mathbf{x} \in [a_i, b_i]^n_{\mathbb{T}}$, then*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} s(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n \leq \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} t(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n.$$

(P3) *If $f(\mathbf{x}) \geq 0$ for $\mathbf{x} \in [a_i, b_i]^n_{\mathbb{T}}$, then $s(\mathbf{x}) = 0$ if and only if*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} s(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n = 0.$$

Lemma 2.2 ((AG inequality) [32]) *Let $r_i > 0$ ($i = 1, 2, \ldots, n$), and let $\theta_1, \theta_2, \ldots, \theta_n \in (1, +\infty)$ such that $\sum_{i=1}^n \frac{1}{\theta_i} = 1$. Then*

$$\prod_{i=1}^n r_i \leq \sum_{i=1}^n \frac{r_i^{\theta_i}}{\theta_i}. \tag{5}$$

Remark 2.3 The Cauchy–Schwarz inequality for n-tuple diamond-alpha integral has the following variants.

(V1) Let $t(\mathbf{x}) > 0$, $p, q \in \mathbb{N}^+$, and let $s(\mathbf{x})$ and $t(\mathbf{x})$ be replaced by $s^p(\mathbf{x})/\sqrt{t^q(\mathbf{x})}$ and $\sqrt{t^q(\mathbf{x})}$ in (4), respectively. Then

$$\begin{aligned}
&\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) s^p(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right)^2 \\
&\quad \leq \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \frac{h(\mathbf{x}) s^{2p}(\mathbf{x})}{t^q(\mathbf{x})} \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\
&\qquad \times \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) t^q(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right).
\end{aligned}$$

(V2) Let $s(\mathbf{x})$ and $t(\mathbf{x})$ be replaced by $(s^{2p}(\mathbf{x}) + t^{2q}(\mathbf{x}))^{\frac{1}{2}}$ and $s^p(\mathbf{x})t^q(\mathbf{x})/(s^{2p}(\mathbf{x}) + t^{2q}(\mathbf{x}))^{\frac{1}{2}}$ in (4), respectively, where $p, q \in \mathbb{N}^+$. Then we get

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^p(\mathbf{x})t^q(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\&\quad\le\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\bigl(s^{2p}(\mathbf{x})+t^{2q}(\mathbf{x})\bigr)\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\&\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{\lambda(\mathbf{x})s^{2p}(\mathbf{x})t^{2q}(\mathbf{x})}{s^{2p}(\mathbf{x})+t^{2q}(\mathbf{x})}\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right).\end{aligned}$$

(V3) Let $s(\mathbf{x})$ and $t(\mathbf{x})$ be replaced by $\sqrt{t^q(\mathbf{x})/s^p(\mathbf{x})}$ and $\sqrt{s^p(\mathbf{x})t^q(\mathbf{x})}$ in (4), respectively, where $s^p(\mathbf{x})t^q(\mathbf{x}) \ge 0$, $s(\mathbf{x}) \neq 0$, and $p, q \in \mathbb{N}^+$. Then we get

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^q(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\&\quad\le\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{\lambda(\mathbf{x})t^q(\mathbf{x})}{s^p(\mathbf{x})}\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\&\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^p(\mathbf{x})t^q(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right).\end{aligned}$$

Theorem 2.4 *Let $\lambda(\mathbf{x}), s(\mathbf{x}), t(\mathbf{x}) : [a_i, b_i]^n_{\mathbb{T}} \to \mathbb{R}$ be $\diamond_\alpha$-integrable functions with $\lambda(\mathbf{x}) \ge 0$, and let there be constants $m, M, h, H \in \mathbb{R}$ such that*

$$\bigl(Ms(\mathbf{x}) - ht(\mathbf{x})\bigr)\bigl(Ht(\mathbf{x}) - ms(\mathbf{x})\bigr) \ge 0. \tag{6}$$

Then

$$\begin{aligned}&mM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\quad+hH\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\le(mh+MH)\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\le|mh+MH|\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^{1/2}\\&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^{1/2}.\end{aligned}\tag{7}$$

Specially, if $Mm > 0$, $Hh > 0$, then

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\&\le\frac{(hm+HM)^2}{4hmHM}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2.\end{aligned}\tag{8}$$

Proof It is easy to find from (6) that

$$\lambda(\mathbf{x})\big(Ms(\mathbf{x}) - ht(\mathbf{x})\big)\big(Ht(\mathbf{x}) - ms(\mathbf{x})\big) \geq 0.$$

Then

$$\begin{aligned}
&\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} MH\lambda(\mathbf{x})t(\mathbf{x})s(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n \\
&\quad - \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} hH\lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n \\
&\quad - mM \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n \\
&\quad + mh \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})t(\mathbf{x})s(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n \geq 0.
\end{aligned} \tag{9}$$

From (9) and Cauchy–Schwarz inequality (4), we find that (7) holds.

Moreover, from $mM > 0$, $hH > 0$, and

$$\begin{aligned}
&\Bigg[\bigg(hH \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\bigg)^{\frac{1}{2}} \\
&\quad - \bigg(mM \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\bigg)^{\frac{1}{2}}\Bigg]^2 \geq 0,
\end{aligned} \tag{10}$$

we have

$$\begin{aligned}
&hH \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n \\
&\qquad + mM \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n \\
&\quad \geq 2\bigg(Mm \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\bigg)^{1/2} \\
&\qquad \times \bigg(Hh \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\bigg)^{1/2}.
\end{aligned} \tag{11}$$

Therefore, by using (11) and (7), we find that (8) is valid. The proof of Theorem 2.10 is completed. □

Remark 2.5 Obviously, inequality (8) extends the result in [33].

Remark 2.6 Under the assumptions of Theorem 2.4, and letting $mM > 0$, $hH > 0$, $0 < q \leq p < 1$, and $p + q = 1$, from the AG inequality (5) and (7) we have

$$\begin{aligned}
&\bigg(\frac{mM}{p} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\bigg)^p \\
&\quad \cdot \bigg(\frac{hH}{q} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\bigg)^q
\end{aligned}$$

$$\begin{aligned}
&\leq mM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n \\
&\quad + hH\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n \\
&\leq (mh+MH)\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^q \\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^p \\
&\leq p^p(1-p)^{1-p}\frac{mh+MH}{(mM)^p(hH)^{1-p}}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n. \qquad (12)
\end{aligned}$$

Letting $p\to 1^-$ on the both sides of inequality (12), we find

$$\begin{aligned}
&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n \\
&\leq\left(\frac{H}{m}+\frac{h}{M}\right)\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n \\
&=\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\frac{Ht(\mathbf{x})}{m}\diamond_\alpha x_1\cdots\diamond_\alpha x_n \\
&\quad+\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\frac{ht(\mathbf{x})}{M}\diamond_\alpha x_1\cdots\diamond_\alpha x_n.
\end{aligned}$$

Letting $p\to 0^+$ on the both sides of inequality (12), we find

$$\begin{aligned}
&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n \\
&\leq\left(\frac{m}{H}+\frac{M}{h}\right)\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n. \qquad (13)
\end{aligned}$$

Remark 2.7 Let $S(\mathbf{x}):[a_i,b_i]_{\mathbb{T}}^n\to(0,+\infty)$ be a $\diamond_\alpha$-integrable function. If $s(\mathbf{x})=S^{\frac{-1}{2}}(\mathbf{x})$, $t(\mathbf{x})=S^{\frac{1}{2}}(\mathbf{x})$, then inequality (7) reduces to

$$\begin{aligned}
&hH\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})S(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n+mM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{\lambda(\mathbf{x})}{S(\mathbf{x})}\diamond_\alpha x_1\cdots\diamond_\alpha x_n \\
&\leq(mh+MH)\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n,
\end{aligned}$$

which generalizes the result in [34].

Remark 2.8 Letting $\lambda(\mathbf{x}) = 1$ in (8), then inequality (8) reduces to

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\&\quad\le \frac{(hm+HM)^2}{4hmHM}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2,\end{aligned}$$

which extends Pólya and Szegö's result [35].

Remark 2.9 Let $S(\mathbf{x}) : [a_i,b_i]_{\mathbb{T}}^n \to (0,+\infty)$ be a $\diamond_\alpha$-integrable function. If $s(\mathbf{x}) = S^{\frac{-1}{2}}(\mathbf{x})$, $t(\mathbf{x}) = S^{\frac{1}{2}}(\mathbf{x})$, then inequality (8) reduces to

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \frac{\lambda(\mathbf{x})}{S(\mathbf{x})}\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})S(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\&\quad\le \frac{(hm+HM)^2}{4hmHM}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2,\end{aligned}$$

which extends some results in [36, 37].

Remark 2.10 Letting $h = H = 1$ in (8), inequality (8) reduces to

$$\begin{aligned}&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\quad + mM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\quad\le (m+M)\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\\&\quad\le |m+M|\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^{\frac{1}{2}}\\&\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^{\frac{1}{2}}. \qquad (14)\end{aligned}$$

Moreover, if $mM > 0$, then

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\&\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\&\quad\le \frac{(m+M)^2}{4mM}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2. \qquad (15)\end{aligned}$$

The above inequalities (14) and (15) extend some results in [31].

Theorem 2.11 *Let $\lambda(\mathbf{x}), s(\mathbf{x}), t(\mathbf{x}) : [a_i,b_i]_{\mathbb{T}}^n \to [0,+\infty)$ be $\diamond_\alpha$-integrable functions, let there be constants $h,H,m,M,p,q > 0$ such that $m \le t(\mathbf{x}) \le M$, $h \le s(\mathbf{x}) \le H$, $0 < q \le p < 1$, and*

$p+q=1$. *Then*

$$\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^p\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^q \\ \leq \frac{pHM+qhm}{(hH)^q(mM)^p}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right). \tag{16}$$

Proof As $(pMs(\mathbf{x})-hqt(\mathbf{x}))(ms(\mathbf{x})-Ht(\mathbf{x}))\leq 0$, we have

$$pmMs^2(\mathbf{x})-(pHM+qhm)s(\mathbf{x})t(\mathbf{x})+qHht^2(\mathbf{x})\leq 0.$$

Then

$$pmMs^2(\mathbf{x})+qHht^2(\mathbf{x})\leq (pHM+qhm)s(\mathbf{x})t(\mathbf{x}). \tag{17}$$

By the AG inequality (5) and (17), we find

$$\begin{aligned}
&\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\right)^p\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^q \\
&\quad=\frac{1}{(hH)^q(mM)^p}\left(mM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^p \\
&\qquad\times\left(hH\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^q \\
&\quad\leq\frac{1}{(hH)^q(mM)^p}\left(pmM\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right. \\
&\qquad\left.+qhH\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\
&\quad\leq\frac{pHM+qhm}{(hH)^q(mM)^p}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right),
\end{aligned}$$

which implies (16) holds. □

Theorem 2.12 *Let* $\lambda(\mathbf{x}),s(\mathbf{x}),t(\mathbf{x}):[a_i,b_i]_{\mathbb{T}}^n\to[0,+\infty)$ *be* $\diamond_\alpha$*-integrable functions.*

(i) *If there are constants* $h,H,m,M\in\mathbb{R}$ *such that* $[Ht(\mathbf{x})-ms(\mathbf{y})][Ms(\mathbf{y})-ht(\mathbf{x})]\geq 0$ *for all* $\mathbf{x},\mathbf{y}\in[a_i,b_i]_{\mathbb{T}}^n$, *then*

$$\begin{aligned}
&(hm+HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\
&\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\
&\quad\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\
&\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)
\end{aligned}$$

$$+ mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)$$
$$\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right). \tag{18}$$

(ii) *If* $hH > 0$, $mM > 0$ *such that* $[Ht(\mathbf{x}) - ms(\mathbf{y})][Ms(\mathbf{y}) - ht(\mathbf{x})] \geq 0$ *for all* $\mathbf{x}, \mathbf{y} \in [a_i, b_i]_{\mathbb{T}}^n$, *and if* $p, q > 0$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$(hm + HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)$$
$$\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)$$
$$\geq\left[\frac{hH}{p}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right.$$
$$\left.\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right]^p$$
$$\times\left[\frac{mM}{q}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right.$$
$$\left.\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\right]^q. \tag{19}$$

(iii) *If* $mM > 0$, $hH > 0$ *such that* $[Ht(\mathbf{x}) - ms(\mathbf{x})][Ms(\mathbf{x}) - ht(\mathbf{x})] \geq 0$, *then*

$$(hm + HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)$$
$$\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)$$
$$\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2$$
$$+ mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2.$$

(iv) *If* $mM > 0$, $hH > 0$ *such that* $[Ht(\mathbf{x}) - ms(\mathbf{y})][Ms(\mathbf{y}) - ht(\mathbf{x})] \geq 0$ *for all* $\mathbf{x}, \mathbf{y} \in [a_i, b_i]_{\mathbb{T}}^n$, *then*

$$(hm + HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)$$
$$\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)$$
$$\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2$$
$$+ mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2.$$

Proof Case (i). From the assumption we find that

$$\lambda(\mathbf{x})\lambda(\mathbf{y})\big(Ht(\mathbf{x}) - ms(\mathbf{y})\big)\big(Ms(\mathbf{y}) - ht(\mathbf{x})\big) \geq 0,$$

which means that

$$\begin{aligned} &HM\lambda(\mathbf{x})\lambda(\mathbf{y})s(\mathbf{y})t(\mathbf{x}) + hm\lambda(\mathbf{x})\lambda(\mathbf{y})s(\mathbf{y})t(\mathbf{x}) \\ &\quad\geq hH\lambda(\mathbf{x})\lambda(\mathbf{y})t^2(\mathbf{x}) + mM\lambda(\mathbf{x})\lambda(\mathbf{y})s^2(\mathbf{x}). \end{aligned}$$

Therefore

$$\begin{aligned} &(hm + HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{y})s(\mathbf{y})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\quad\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\qquad+ mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right). \end{aligned}$$

Case (ii). From AG inequality (5) and Case (i), it is easy to find that (19) holds.

Case (iii). From Cauchy–Schwarz inequality (4) we find that

$$\begin{aligned} &\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\quad\geq\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2, \end{aligned}\tag{20}$$

and

$$\begin{aligned} &\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\quad\geq\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2. \end{aligned}\tag{21}$$

Combining (7), (20), and (21), we have

$$\begin{aligned} &(hm + HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \\ &\qquad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right) \end{aligned}$$

$$
\begin{aligned}
&\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad+mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\quad+mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2.
\end{aligned}
$$

The proof of Case (iii) is completed.

Case (iv). Combining (18), (20), and (21), we have

$$
\begin{aligned}
&(hm+HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad+mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\geq hH\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\quad+mM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2,
\end{aligned}
$$

which implies that Case (iv) holds. □

Remark 2.13 From Case (i) of Theorem 2.12 we find that

$$
\begin{aligned}
&(hm+HM)^2\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\geq h^2H^2\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ m^2M^2\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&+ 2hmHM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
\geq{}& 4hmHM\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2\\
&\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)^2.
\end{aligned}
$$

Therefore, if $hmHM > 0$, we find

$$
\begin{aligned}
&(hm + HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\geq 2\sqrt{hmHM}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right).
\end{aligned}
$$

Similarly, from Case (iv) of Theorem 2.12 we have

$$
\begin{aligned}
&(hm + HM)\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\geq 2\sqrt{hmHM}\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right)\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\right).
\end{aligned}
$$

By using methods similar to that in [31], we can prove the following theorem.

Theorem 2.14 *Let* $\lambda(\mathbf{x}), s(\mathbf{x}), t(\mathbf{x}) : [a_i, b_i]_{\mathbb{T}}^n \to [0, +\infty)$ *be* $\diamond_\alpha$*-integrable functions.*

Case (1).

$$\begin{aligned}
&\biggl[\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)\\
&\quad+\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)^2\biggr]\\
&\quad\times\biggl[\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)\\
&\quad+\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)^2\biggr]\\
&\geq\biggl[\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\biggr)\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\biggr)\\
&\quad+\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\biggr)\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\biggr)\biggr]^2.
\end{aligned}$$

Case (2). *If there are constants* $h, H, m, M \in \mathbb{R}$ *such that* $[Ht(\mathbf{x}) - ms(\mathbf{x})][Ms(\mathbf{x}) - ht(\mathbf{x})] \geq 0$, *then*

$$\begin{aligned}
&(hm+HM)^2\biggl[\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)\\
&\quad\times\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)\\
&\quad+\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)\\
&\quad\times\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)\biggr]^2\\
&\geq 4hmHM\biggl[\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)\\
&\quad\times\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)\\
&\quad+\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})t(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)^2\biggr]\\
&\quad\times\biggl[\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)\\
&\quad\times\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s^2(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)\\
&\quad+\biggl(\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\lambda(\mathbf{x})s(\mathbf{x})\diamond_\alpha x_1\cdots\diamond_\alpha x_n\biggr)^2\biggr]^2.
\end{aligned}$$

Case (3). *If there are constants* $h, H, m, M \in \mathbb{R}$ *such that* $[Ht(\mathbf{x}) - ms(\mathbf{y})][Ms(\mathbf{y}) - ht(\mathbf{x})] \geq 0$ *for all* $\mathbf{x}, \mathbf{y} \in [a_i, b_i]_{\mathbb{T}}^n$, *then*

$$\begin{aligned} 1 \leq{} & \Bigg[\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) s^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ & + \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) s(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right)^2\Bigg] \\ & \times \Bigg[\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right)\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) t^2(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ & + \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) t(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right)^2\Bigg] \\ & \Big/ \Bigg[\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ & \times \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) s(\mathbf{x}) t(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ & + \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) s(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right) \\ & \times \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \lambda(\mathbf{x}) t(\mathbf{x}) \diamond_\alpha x_1 \cdots \diamond_\alpha x_n\right)\Bigg]^2 \\ \leq{} & \frac{(hm + HM)^2}{4hmHM}. \end{aligned}$$

Acknowledgements
The authors gratefully acknowledge the anonymous referees for their constructive comments and advices on the earlier version for this paper.

Funding
This work was supported by the Fundamental Research Funds for the Central Universities (No. 2015ZD29).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

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All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Publisher's Note

Received: 5 September 2019 Accepted: 6 January 2020 Published online: 13 January 2020

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