ON CENTRALITY FUNCTIONS OF A GRAPH

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Abstract: For a connected nondirected graph, a centrality function is a real valued function of the vertices defined as a linear combination of the numbers of the vertices classified according to the distance from a given vertex. Some fundamental properties of the centrality functions and the set of central vertices are summarized. Inserting an edge between a center and a vertex, the stability of the set of central vertices are investigated.

For a weakly connected directed graph, we can prove similar theorems with respect to a generalized centrality function based on a new definition of the modified distance from a vertex to another vertex.

1. Introduction

In many practical applications, it is often necessary to find the best location of facilities in networks or graphs. In this context, a real number $f(G,v)$ is associated with every vertex v of the graph G for the criterion of deciding what vertex is best. The criterion of optimality may be taken to be the minimization of the function $f(G,v)$ with respect to v.

One of the most important problems is to determine what kind of functions is suitable for the measure of centrality of vertices in a graph. It is well-known that the transmission number is an example of such functions. In this survey, the centrality function, a generalized form of the transmission number, is defined as a linear combination with real coefficients of the numbers of vertices classified according to the distance from a given vertex in a connected nondirected graph.

As a fundamental theorem, a necessary and sufficient condition for the function to satisfy the centrality axioms is stated in terms of the coefficients.

Inserting an edge between a center and a vertex, the sets of central vertices settled before and after the edge inserting are generally different. Some stability theorems of the sets of central vertices are presented for a connected nondirected graph.

However the situation often arises where a nondirected graph will not be able to meet various requirements and what is then needed is to introduce a centrality function for a directed graph. For a weakly connected directed graph, a modified distance from a vertex to another vertex is defined as a two-dimensional vector of integer components showing the numbers of forward and backward edges contained in the shortest path with respect to a newly defined order relation. It is shown that the major results for a nondirected graph can be extended similarly to a directed graph with respect to a generalized centrality function based on the modified distance.

2. Transmission Number

Let G be a connected nondirected graph with the set of vertices V. A distance $d(u,v)$ between a pair of vertices u and v in G is defined as the minimum number of edges in a path connecting u and v. We now define $c_0(G,v)$ for every vertex v in G as follows :

$$c_0(G,v) = \sum_{w \in V} d(v,w) \tag{1}$$

The number $c_0(G,v)$ is often refered to as the transmission number[1]. A central vertex v_0 for which

$$c_0(G,v_0) = \operatorname*{Min}_{v \in V} c_0(G,v) \tag{2}$$

is called a median[1] of the graph G.

3. Centrality Function

Let $c(G,v)$ be a real valued function of vertices of G. Then the function is said to be a centrality function if $c(G,v)$ satisfies the following centrality axioms[2].

Centrality Axioms : If there exist no edges between a pair of vertices p and q in a connected nondirected graph G, the insertion of an edge between p and q yields the graph G_{pq} and the difference

$$\Delta_{pq}(v) = c(G,v) - c(G_{pq},v) \tag{3}$$

for any vertex v in G.

Now the function $c(G,v)$ is called a centrality function if and only if

(i) $\Delta_{pq}(p) > 0$ (4)

(ii) $\Delta_{pq}(p) \geqq \Delta_{pq}(v)$ for any v satisfying

$d(v,p) \leqq d(v,q)$ (5)

for any pair of vertices p and q which are not adjacent. (End)

As a generalized form of the transmission number, we deal with a real valued function $c(G,v)$ as follows :

$$c(G,v) = \sum_{k=1}^{\infty} a_k n_k(v) \tag{6}$$

where $n_k(v)$ stands for the number of vertices whose distances from v are k, and a_k's are real constants.

For the function defined by (6), the following theorem can be proved[3].

Theorem 1 : The function $c(G,v)$ defined by (6) is a centrality function for any graph G if and only if a_k's satisfy

(i) $a_1 < a_2 \leqq a_3 \leqq a_4 \leqq \ldots$ (7)

(ii) $2a_k \geqq a_{k-1} + a_{k+1}$, $(k \geqq 2)$ (8)

(End)

As an illustrative example, suppose

$$a_k = k, \quad (k = 1,2,3, \ldots). \tag{9}$$

It is easily shown that

$$\sum_{k=1}^{\infty} k\, n_k(v) = \sum_{w \in V} d(v,w) = c_0(G,v) \tag{10}$$

and a_k's given by (9) satisfy (7) and (8). Thus we can conclude that the transmission number is a centrality function.

Let $c(G,v)$ defined by (6) be a centrality function for any connected nondirected graph G. A vertex v_0 for which

$$c(G,v_0) = \min_{v \in V} c(G,v) \tag{11}$$

is called a center of G with respect to $c(G,v)$ or shortly a c-center. Let $S_c(G)$ be the set of all the c-centers of G.

4. Stability Theorems

If a c-center p and a vertex q in G are not adjacent, the insertion of an edge between p and q yields the graph G_{pq} with its set of all the c-centers $S_c(G_{pq})$. Then two cases can occur, either

$$\text{Case A : } S_c(G_{pq}) \subseteqq S_c(G) \cup \{q\} \tag{12}$$

or

$$\text{Case B : } S_c(G_{pq}) \not\subseteqq S_c(G) \cup \{q\} \tag{13}$$

for any vertex p in $S_c(G)$ and q in V. A graph for which case B occurs is said to be unstable with respect to c(G,v).

Case A can be classified into two cases,

$$\text{Case A-1 : } S_c(G_{pq}) \subseteqq S_c(G) \text{ and } p \in S_c(G_{pq}) \tag{14}$$

and

$$\text{Case A-2 : } S_c(G_{pq}) \not\subseteqq S_c(G) \text{ or } p \notin S_c(G_{pq}) \tag{15}$$

for any vertex p in $S_c(G)$ and q in V.

A graph G is said to be stable if case A-1 occurs. A quasi-stable graph is a graph for which case A-2 occurs.

We can then prove the following theorem[4].

Theorem 2 : For any centrality function c(G,v) satisfying $a_2 < a_3$, there exist a quasi-stable graph. (End)

A quasi-stable graph with respect to the transmission number is shown in Fig. 1[4].

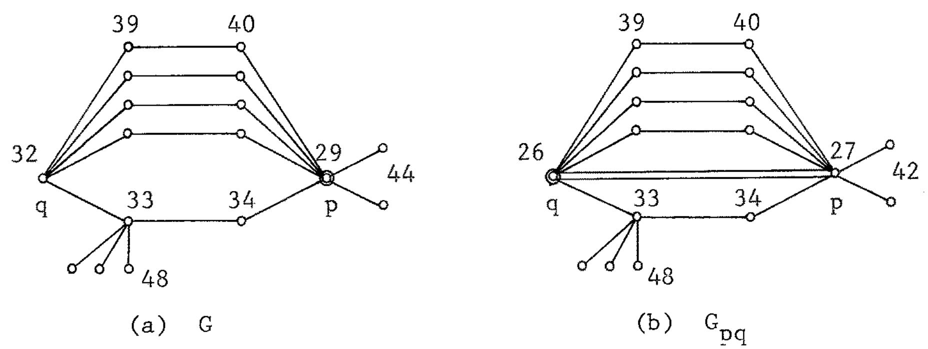

Fig. 1. Quasi-stable graph.

Theorem 3 : For any centrality function c(G,v) satisfying $a_2 = a_3$, all the connected nondirected graphs are stable. (End)

Theorem 4 : Any connected nondirected graph is stable if and only if the centrality function c(G,v) given by (6) satisfies $a_2 = a_3$. (End)

Theorem 5 : For any centrality function c(G,v) satisfying $a_3 < a_4$, there exists an unstable graph. (End)

An unstable graph with respect to the transmission number is shown in Fig. 2[3].

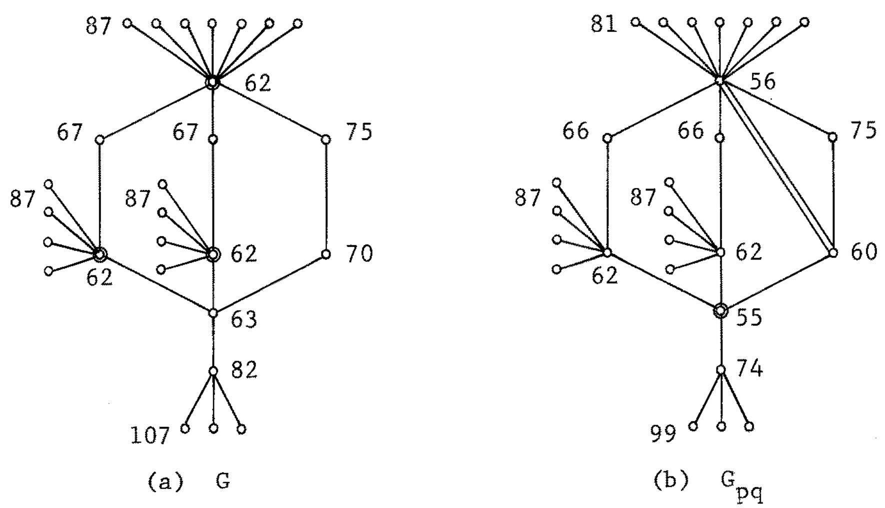

Fig. 2. Unstable graph.

Theorem 6 : For any centrality function satisfying $a_3 = a_4$ all the connected nondirected graphs are quasi-stable or stable. (End)

Theorem 7 : Any connected nondirected graph is not unstable if and only if the centrality function given by (6) satisfies $a_3 = a_4$. (End)

5. Stable Graphs

The theorems in the preceding section show that a centrality function with which all the graphs are stable or quasi-stable is rather trivial one. Characterizing stable or quasi-stable graphs with respect to a given centrality function is an important problem to be solved. The following theorem[2] is basic with respect to the centrality function specified as the transmission number.

Theorem 8 : If a graph G forms a tree, then G is stable with respect to the transmission number. (End)

Let H_k $(k = 0,1,2, \ldots)$ be the collection of all the connected graphs of nullity k. Then Theorem 8 shows that any graph of H_0 is stable. Since H_2 contains an unstable graph shown in Fig. 2, we may ask if there exists an unstable or a quasi-stable graphs in H_1. Counting the number m of edges in the only loop contained in any graph of H_1, we can define a subset $H_1(m)$ as the collection of graphs containing the single loop of length m.

Recent results with respect to the transmission number include the following two theorems[5].

Theorem 9 : For any $m \leqq 4$, all the graphs of $H_1(m)$ are stable. For any $m \geqq 5$, $H_1(m)$ contains a quasi-stable graph. (End)

Theorem 10 : For $m = 7$, $H_1(m)$ contains an unstable graph. For $m \leqq 6$, $H_1(m)$ contains no unstable graphs. (End)

The graph shown in Fig. 3 is an example of unstable graph of m=7.

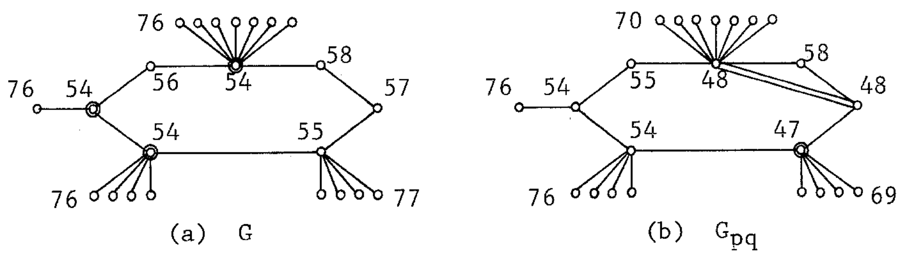

Fig. 3. Unstable graph.

6. Centrality Functions for A Directed Graph

The definitions and the theorems discussed so far can be extended for a directed graph[6]. Let us begin with some preliminary definitions.

Let R^2 be the two dimensional real space defined by

$$R^2 = \{(x,y) \mid x,y \in R\} \tag{16}$$

where R is the set of real numbers. For the simplicity, a vector $(x,y) \in R^2$ is expressed by $x+y\omega \in R^2$, where ω is the symbol specifying the second component.

A natural order and the vector addition can be defined in R^2 as follows.

$$\text{(i)} \quad x+y\omega > 0 \quad \text{if and only if} \quad y > 0 \quad \text{or} \quad y = 0 \ \text{and} \ x > 0 \tag{17}$$

$$\text{(ii)} \quad (x+y\omega)+(x'+y'\omega) = (x+x')+(y+y')\omega \quad \text{where} \quad 0 = 0 + 0\omega \tag{18}$$

Let N^2 be the subset of R^2 similarly defined with the set of non-negative integer N. It is obvious that R^2 is an ordered abelian group, while N^2 is an ordered semigroup contained in R^2.

Let a directed graph G be weakly connected. A path P between two vertices u and v may be oriented as from u to v. We can then define a vector (a_p, b_p) of integer component associated with the path P where

a_p and b_p are the number of coincide and opposite edges in the path P, respectively. Since (a_p, b_p) can be interpreted as an element $a_p+b_p\omega$ in N^2, we can define a generalized length of the path P such that

$$L_{uv}(P) = a_p + b_p\omega \tag{19}$$

The modified distance from vertex u to vertex v in a weakly connected graph is given by

$$D(u,v) = \operatorname*{Min}_{P} L_{uv}(p) \tag{20}$$

where P is an arbitrary path connecting u and v.

Naturally D(u,v) does not fulfil the reflective law, but still satisfies

$$D(u,v) \leqq D(u,w) + D(w,v) \tag{21}$$

Similar to the centrality axioms for a nondirected graph, a centrality function C(G,v) whose values are in R^2 can be defined in terms of the modified distance.

Centrality Axioms : If there exist no edges between a pair of vertices p and q in a weakly connected directed graph G, the insertion of edges from p to q and from q to p yields two graphs G'_{pq} and G''_{pq}, respectively. Let us define

$$\left.\begin{aligned} \Delta'_{pq}(v) &= C(G,v)-C(G'_{pq},v) \\ \Delta''_{pq}(v) &= C(G,v)-C(G''_{pq},v) \end{aligned}\right\} \tag{22}$$

for any vertex v in G.

Now the function C(G,v) is called a centrality function if and only if

$$\text{(i)}\quad \Delta'_{pq}(p) > 0, \quad \Delta''_{pq}(p) \geqq 0 \tag{23}$$

$$\text{(ii)}\quad \Delta'_{pq}(p) \geqq \Delta'_{pq}(v) \quad \text{and} \quad \Delta''_{pq}(p) \geqq \Delta''_{pq}(v)$$

for any v satisfying

$$D(v,p) \leqq D(v,p) \tag{24}$$

for any pair of vertices p and q which are not adjacent. (End)

We will deal with the function defined by

$$C(G,v) = \sum_{1<\mu\varepsilon N^2} \alpha_\mu n_\mu(v) \tag{25}$$

where $\alpha_\mu(\varepsilon\ R^2)$ does not depend on G and $n_\mu(v)$ denotes the number of vertices whose modified distance from v are $\mu(\varepsilon\ N^2)$.

Corresponding to Theorem 1, we now obtain the following theorem.

Theorem 11 : The function defined by (25) is a centrality function if α_μ's satisfy

$$\text{(i)}\quad \alpha_1 < \alpha_2, \quad \alpha_{\mu_1} \leqq \alpha_{\mu_2} \tag{26}$$

$$\text{(ii)}\quad \alpha_{\mu_2} - \alpha_{\mu_1} \geqq \alpha_{\mu_2+\delta} - \alpha_{\mu_1+\delta} \tag{27}$$

where $1 \leqq \mu_1 < \mu_2$ and $1 \leqq \delta$. (End)

For a directed graph, we can also prove some stability theorems corresponding to those for a nondirected graph.

7. Conclusion

It has been supposed to be true that any connected nondirected graph is stable with respect to the transmission number [2]. The theorems given here show that the conjecture is false.

Theorem 4 and 6 show that centrality functions with which all the nondirected graphs are stable or quasi-stable are rather trivial. Characterizing stable or quasi-stable graphs with respect to a given centrality function is an interesting problem.

The definitions and theorems of centrality functions for a nondirected graph can be extended for a directed graph, employing the concept of modified distance which seems to be useful in the theory of directed graphs.

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