On certain abelian-by-nilpotent varieties

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We show that, whenever m, n **are** coprime, each subvariety of the abelian-by-nilpotent variety $\underline{\underline{A}}_m\left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n\right)$ has a finite basis for its laws. We further show that the just non-Cross subvarieties of $\underline{\underline{A}}_m\left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n\right)$ are precisely those already known.

1. Introduction

In his papers [2, 3] Graham Higman has described techniques for answering a range of questions regarding certain varieties of groups. Here we work out the details which enable us to prove the following two results.

THEOREM 1.1. *The subvarieties of* $\underline{\underline{A}}_m\left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n\right)$ *are finitely based whenever* m, n *are coprime.*

(As usual $\underline{\underline{A}}_m$, $\underline{\underline{N}}_2$, $\underline{\underline{B}}_n$ are respectively the varieties of: abelian groups of exponent dividing m, groups of class at most 2 and groups of exponent dividing n. We shall follow the terminology of Hanna Neumann [5]).

THEOREM 1.2. *Let* $\underline{\underline{T}}_q$ *be the variety generated by the non-abelian group of order* q^3 *and exponent* q *when* q *is an odd prime, and by the dihedral group of order* 8 *when* q *is* 2 . *Then, whenever* m, n *are coprime, the just non-Cross subvarieties of* $\underline{\underline{A}}_m\left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n\right)$ *are precisely* $\underline{\underline{A}}_p\underline{\underline{T}}_q$ *($p|m$, $q|n$)* .

Recall that a just non-Cross variety is one not generated by a finite group every proper subvariety of which is generated by a finite group. By now Theorem 1.1 requires little motivation; and we refer the reader to

Kovács and Newman [4] for the motivation behind Theorem 1.2. In [3] Higman
asks another question relevant to the above varieties: if $\underline{V}$ is a
subvariety of $\underline{\underline{A}}_m\left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n\right)$ where m, n are coprime, and $f(n) = \left|F_n(\underline{V})\right|$,
is $f : Z^+ \to Z^+$ so defined CREAM ? We believe we can give an affirmative
answer to this question; as the proof is long and tedious we shall, however,
not reproduce it here.

Since $\underline{\underline{A}}_m\left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n\right)$ has a finite basis for its laws [2, Theorem 3.1],
Theorem 1.1 is proved if the subvarieties of $\underline{\underline{A}}_m\left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n\right)$ satisfy the
descending chain condition, and to show this it is enough to prove

LEMMA 1.3. *If $\{G_i : i \in Z^+\}$ is an infinite sequence of critical
groups in $\underline{\underline{A}}_m\left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n\right)$, with m, n coprime, then for some i ,
$G_i \in \text{var} \{G_{i+1}, G_{i+2}, \ldots\}$.*

The description of the critical groups of $\underline{\underline{A}}_m\left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n\right)$ given in §2.7
of [3] will be taken as read. There Higman uses the description to show
that to prove CREAM for $\underline{\underline{A}}_m\underline{\underline{B}}$, where $\underline{\underline{B}}$ is locally finite of exponent
prime to m , it suffices to show it for $\underline{\underline{A}}_p\underline{\underline{B}}$ for each prime divisor p of
m . The same argument shows that it is enough to prove Theorems 1.1 and 1.2
for the case when m is prime: henceforth we assume m is a prime p .

To each non-nilpotent critical group in $\underline{\underline{A}}_p\left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n\right)$, Higman
associates a unique irreducible linear group over $GF(p)$, the field of p
elements. The proof of the theorems now depends on a complete
classification of such irreducible linear groups; it is well known that
they have cyclic centre, so the classification is given by the following two
lemmas whose proofs are given in §2 (the second is a corollary of 2.4).

LEMMA 1.4. *If $\{K_i : i \in Z^+\}$ is an infinite sequence of groups of
class 2 and exponent dividing n with cyclic centres, then for some i, j
with i less than j , K_i is isomorphic to a central factor of K_j (i.e.
K_i is isomorphic to a subgroup of K_j which together with its centralizer
in K_j , this subgroup generates K_j) .*

LEMMA 1.5. *If K is a group of class 2 with cyclic centre and of order prime to p , then all faithful irreducible representations of K over GF(p) are linearly isomorphic.*

Proof of Lemma 1.3. Let K_i be the irreducible linear group associated with G_i for each i in Z^+ . By Lemma 1.4 there exist i, j with i less than j such that, as abstract group, K_i is a subgroup of K_j with the centre $Z(K_i)$ contained in $Z(K_j)$. Let M be the space over GF(p) on which K_j acts, and consider an irreducible component M_0 of M_{K_i} . Since each element of $Z(K_j)$ acts fixed-point-free on M , M_0 is faithful. Consequently even as linear group K_i is contained in K_j , by Lemma 1.5, and thus G_i is a subgroup of G_j . Lemma 1.3, and with it Theorem 1.1, is therefore proved.

2. Proofs of Lemmas 1.4 and 1.5

We begin by determining the structure of groups of class 2 with cyclic centre. Since a nilpotent group is a direct product of groups of prime power order, it suffices to consider q-groups for some prime q .

THEOREM 2.1. (cf. M.F. Newman [6]). *A finite class 2 q-group with cyclic centre is a central product either of two-generator subgroups with cyclic centre or of two-generator subgroups with cyclic centre and a cyclic subgroup.*

Proof. Suppose that K satisfies the hypotheses. The proof is by induction on the minimal number of generators for K . If K requires either one or two generators, there is nothing to prove. Hence suppose that K cannot be generated by two elements and that the result is true for such groups with fewer generators: note that K is then not abelian. Choose an arbitrary set of generators for K , say $K = \langle x, y, u, \ldots \rangle$. Since K' , the derived group of K , is a cyclic q-group and K' is generated by all the commutators $[x, y], [x, u], \ldots$ it follows that K' is generated by one of them, say by $[x, y]$. We will show that K is generated by $\{x, y, u', \ldots\}$ where $u', \ldots$ all centralize x and y : for there is an integer r such that

$$[x, u] = [x, y]^r = [x, y^r]$$

whence $[x, uy^{-r}] = 1$, and an integer s such that

$$[y, uy^{-r}] = [y, x]^s = [y, x^s]$$

whence $[y, uy^{-r}x^{-s}] = 1$. Then $u' = uy^{-r}x^{-s}$ commutes with x, y and $\langle x, y, u \rangle = \langle x, y, u' \rangle$. Similarly one replaces the other elements of the given generating set.

In this way we have K expressed as a central product of $\langle x, y \rangle$ and a subgroup K_1 on fewer generators; $\langle x, y \rangle$ and K_1 both have cyclic centres, and so induction concludes the proof. Note that the amalgamations of the centres of the central factors must be as large as possible.

Our need now is to determine all two-generator groups of class 2 with cyclic centre.

THEOREM 2.2. *The q-groups of class 2 on two generators with cyclic centre comprise the following list:*

$$Q(\alpha, \beta) \quad (2\beta \le \alpha) \qquad : \left\langle a, b : a^{q^\alpha} = b^{q^\beta} = 1 , \quad a^{q^{\alpha-\beta}} = [a, b] \right\rangle ;$$

$$Q(\alpha, \beta) \quad (\beta \le \alpha < 2\beta) : \left\langle a, b : a^{q^\alpha} = b^{q^\beta} = 1 , \quad a^{q^\beta} = [a, b]^{q^{2\beta-\alpha}} , \right.$$
$$\left. [a, b, a] = [a, b, b] = 1 \right\rangle ;$$

and if $q = 2$ we have as well

$$R(\beta) \quad (1 \le \beta) \qquad : \left\langle a, b : a^{2^{\beta+1}} = b^{2^{\beta+1}} = 1 , \quad a^{2^\beta} = [a, b]^{2^{\beta-1}} \right.$$
$$\left. = b^{2^\beta} , \quad [a, b, a] = [a, b, b] = 1 \right\rangle .$$

Proof. Let X be a two generator group of class 2 with cyclic centre. Choose a generating pair $\{a, b\}$ for X where the order of a is q^α , that of b , $q^\delta (\alpha \ge \delta)$ and that of $[a, b]$, q^β .

If $a^{q^\lambda} \in Z(X)$ then $1 = \left[a^{q^\lambda}, b\right] = [a, b]^{q^\lambda}$ whence $\lambda \ge \beta$. Treat b similarly, so that

$$Z(X) = \left\langle a^{q^\beta}, b^{q^\beta}, [a, b] \right\rangle .$$

Since $Z(X)$ is cyclic, two cases arise.

Case 1. $Z(X) = \left\langle a^{q^\beta} \right\rangle$. There exists an integer r such that

$$[a,\, b] = a^{rq^\beta}$$

whence $[a,\, b]^{q^\lambda} = a^{rq^{\beta+\lambda}}$; consequently $1 = a^{rq^{2\beta}}$ and $q^\alpha \| rq^{2\beta}$ so that $\alpha \gneqq 2\beta$ and $q^{\alpha-2\beta} \| r$. Write $r = q^{\alpha-2\beta} s$ $(q \nmid s)$ and choose t so that $st \equiv 1 \pmod{q^\alpha}$; then

$$[a,\, b^t] = [a,\, b]^t = a^{q^{\alpha-\beta}} .$$

Since $\{a,\, b^t\}$ is a generating set for X we may as well assume that a, b as given satisfy

(*) $$[a,\, b] = a^{q^{\alpha-\beta}} .$$

Now there is an integer u such that $b^{q^\beta} = a^{uq^\beta}$ so that, for all integers v ,

$$\left(ba^v\right)^{q^\beta} = b^{q^\beta} a^{vq^\beta} [a,\, b]^{v\binom{q^\beta}{2}} = a^{uq^\beta + v\left(q^\beta + \binom{q^\beta}{2}q^{\alpha-\beta}\right)} .$$

We aim to choose v so that $\left(ba^v\right)^{q^\beta} = 1$, and this can certainly be done if the congruence

$$uq^\beta + v\left(q^\beta + q^{\alpha-\beta}\binom{q^\beta}{2}\right) \equiv 0 \pmod{q^\alpha}$$

can be solved for v . In case q is odd this can always be done; and when $q = 2$ it can be done in all cases except $\alpha = 2$, $\beta = 1$, $u \not\equiv 0$ (mod 2) (this exception corresponding to the quaternion group Q_8). Notice that a, ba^v satisfy the relation (*) . Hence X is a homomorphic image of $Q(\alpha,\, \beta)$ and since $Q(\alpha,\, \beta)$ has cyclic centre and its derived group has the same order as that of X , the kernel of the homomorphism is 1 .

Case 2. $Z(X) = \langle [a,\, b] \rangle$. By copying the earlier argument one deduces that $\alpha \leq 2\beta$ and that b may be changed so that

$$[a, b]^{q^{2\beta-\alpha}} = a^{q^\beta} .$$

Again one looks for v so that $(ba^v)^{q^\beta} = 1$, and if $[a, b]^u = b^{q^\beta}$, v must satisfy

$$u + v\left[q^{2\beta-\alpha} + \binom{q^\beta}{2}\right] \equiv 0 \pmod{q^\beta} .$$

When q is odd such a v always exists; and if $q = 2$ one may be found in all cases except $\alpha = \beta + 1$. Hence one gets either $Q(\alpha, \beta)$ or $R(\beta)$.

COROLLARY 2.3. *Define* $Q(\beta) = Q(\beta, \beta)$ *and let* $C = \left\langle c : c^{q^\alpha} = 1\right\rangle$ *($\alpha \geq \beta$) . Form the central product* $D(\alpha, \beta)$ *of* $Q(\beta)$ *and* C *with the amalgamation*

$$c^{q^{\alpha-\beta}} = [a, b] .$$

Then $Q(\alpha, \beta) \cong \langle ac, b\rangle$. *Also when* $q = 2$ *and* $\alpha = \beta + 1$ $R(\beta) \cong \langle ba, bc\rangle$ *in* $D(\beta+1, \beta)$ *and this is the unique copy of* $R(\beta)$ *in* $D(\beta+1, \beta)$.

Proof. This first statement requires showing that ac, b satisfy the appropriate relations, and that $|\langle ac, b\rangle'| = q^\beta$; we omit the details.

For the second case write $\binom{2^\beta}{2} = 2^{\beta-1}u$, so that u is odd, and choose v so that $uv \equiv 1 \pmod{2^\alpha}$. Put $a_1 = ba^v$, $b_1 = b^u c : a_1, b_1$ each have order 2^α , and

$$a_1^{2^\beta} = [a, b]^{v\binom{2^\beta}{2}} = [a, b]^{2^{\beta-1}} = c^{2^\beta} = b_1^{2^\beta} .$$

Hence $\langle a_1, b_1\rangle \cong R(\beta)$.

Finally note that if D_0 is a subgroup of $D = D(\beta+1, \beta)$ isomorphic to $R(\beta)$, then D_0 has index 2 in D . Now $|D_0 : \Phi(D_0)| = 4$, $|D : \Phi(D)| = 8$ (where Φ as usual denotes the Frattini subgroup); and since $\Phi(D) \leq D_0$, $\Phi(D_0) \leq \Phi(D)$ whence $\Phi(D) = \Phi(D_0)$. It follows that there are 7 *a priori* choices for D_0 , and noting that the canonic generators a', b' for D_0 have order 2^α and $a'b'$ does also, one

discards 6 of these. (Since u, v are odd and a^2, b^2, $c^2 \in \Phi(D_0)$,
$\langle a_1, b_1 \rangle = \langle a, b \rangle$.)

REMARK. If K , a group of class 2 with cyclic centre, is written as
a central product of $Q(\alpha, \beta)$'s , $R(\beta)$'s and perhaps one cycle, it is, up
to isomorphism, uniquely determined by the number of times each central
factor in this decomposition is repeated. For, each automorphism of $Z(Q)$,
where Q is a $Q(\alpha, \beta)$ or an $R(\beta)$ or cyclic, is the restriction of an
automorphism of Q ; and hence no matter how the amalgamations are
performed one gets the same group.

The crucial result of this section is:

THEOREM 2.4. *Let K be a group of class 2 with cyclic centre and
E a field whose characteristic does not divide the order of K . Then the
faithful irreducible representations of K over E are linearly
isomorphic.*

The proof of this fact requires a number of steps: we first show that
it is sufficient to establish the result for algebraically closed fields,
and then that only the groups of Theorem 2.2 really matter.

THEOREM 2.5. *Let X be a finite group, E a field and T_1, T_2
faithful irreducible representations of X over E . T_1, T_2 are linearly
isomorphic if and only if some composition factor of $T_1^{E^*}$ is linearly
isomorphic to a composition factor of $T_2^{E^*}$ (where E^* denotes the
algebraic closure of E) .*

Proof. The 'only if' part is trivial, so suppose $T_1^{E^*}$ has a
composition factor T_{11} linearly isomorphic to a composition factor T_{21}
of $T_2^{E^*}$. Since X is finite, T_{11} and T_{21} may be written in some
finite extension field F of E . Choose F so that $|F : E| > \dim T_{11}$;
the first part or the proof of (29.7) in Curtis and Reiner [1] can then be
routinely amended to show that T_{11} and T_{21} are linearly isomorphic
F-representations. That is T_1^F, T_2^F have a linearly isomorphic
composition factor.

Now F may be chosen as a splitting field for X and also as a
finite, normal extension of E , so that (70.15) of [1] tells us that
T_1^F, T_2^F are completely reducible, say

$$T_1{}^F = T_{11} \oplus \ldots \oplus T_{1m} \ , \quad T_2{}^F = T_{21} \oplus \ldots \oplus T_{2n} \ ,$$

and that T_{1i}, T_{2i} are all faithful representations of X . Now there exists $U \in GL(\dim T_{11}, F)$ such that

$$T_{11}(X)^U = T_{21}(X) \ .$$

Define the automorphism α of X by

$$T_{21}(x\alpha) = T_{11}(x)^U \ , \quad x \in X \ ,$$

and the representation T_3 of X over F by

$$T_3(x) = T_1\!\left(x\alpha^{-1}\right) \ , \quad x \in X \ .$$

Clearly T_3 is faithful and irreducible. It is easy to calculate that $T_2{}^F$, $T_3{}^F$ have the composition factor T_{21} in common. Consequently T_2, T_3 are equivalent by (29.6) of [1] (being irreducible). But $T_3(X) = T_1(X)$ so that T_1, T_2 are linearly isomorphic as required.

In order to reduce the problem to the case when K is one of the groups of Theorem 2.2 we need the following manifestation of a well known result (adapt the proof of [1, (51.3)]).

THEOREM 2.6. *Let X, Y be finite groups and $H = X \times Y$; E an arbitrary field and M an EH-module. Let M_1 be an absolutely irreducible submodule of M_X . Then M is irreducible if and only if $M \cong M_1 \# M_2$ for any irreducible submodule M_2 of M_Y .*

COROLLARY 2.7. *If X, Y are groups each of which has just one linear isomorphism class of faithful irreducible representations over an algebraically closed field E , then every central product of X and Y with cyclic centre has the same property. (Note that this includes the case $X \times Y$, X, Y of coprime orders.)*

Proof. If H_0 is a central product of X and Y then there is an epimorphism $H = X \times Y \twoheadrightarrow H_0$. Let M, N be faithful irreducible modules over E for H_0 ; M, N therefore can be regarded as irreducibles of H . Consequently, by the theorem, $M = M_1 \# M_2$, $N = N_1 \# N_2$ where M_1, N_1 are faithful irreducibles of X and M_2, N_2 faithful irreducibles of Y .

Now if T_i, U_i are the representations for X, Y afforded by M_i, N_i $(i = 1, 2)$ respectively, then there exist non-singular linear transformations V, W such that

$$T_1(X)^V = T_2(X) \ , \quad U_1(Y)^W = U_2(Y) \ .$$

Put $V_0 = V \otimes W$ and one easily checks that

$$\left(T_1(X)U_1(Y)\right)^{V_0} = T_2(X)U_2(Y)$$

as required.

Theorem 2.5 and Corollary 2.7 have therefore reduced the proof of Theorem 2.4 to the case when K is either $Q(\alpha, \beta)$, $R(\beta)$ or cyclic, and E is algebraically closed; the result is well known for cyclic groups and we treat the remainder in a number of steps starting with $K = Q(\beta)\left(= Q(\beta, \beta)\right)$.

LEMMA 2.8. *All faithful irreducibles of the groups $Q(\beta)$ over a field whose characteristic is not q are linearly isomorphic.*

Proof. As we have observed we assume the ground field E to be closed. Construct representations T_i of the subgroup $A = \langle a, [a, b] \rangle$ of $Q(\beta)$ as follows: the kernel of each T_i is $\langle a \rangle$, and

$$T_i\left([b, a]\right) = \xi^i \ , \quad i \in \{1, \ldots, q^{\beta-1}\} \ , \quad q \nmid i \ ,$$

where ξ is a primitive q^βth root of 1 . For each i and each $x \in Q(\beta) - A$, the representations T_i and $T_i^{(x)} : h \to T_i\left(x^{-1}hx\right)$ of A are distinct, therefore inequivalent, and hence each induced representation $T_i^{Q(\beta)}$ is irreducible ([1, (45.2)']). Moreover if $i \neq j$, the representations T_i and $T_j^{(x)}$ are distinct, therefore inequivalent, for each $x \in Q(\beta)$, and so $T_i^{Q(\beta)}$ and $T_j^{Q(\beta)}$ are inequivalent ([1, (45.6)]). The number of these $T_i^{Q(\beta)}$'s is $\left(q^\beta - 1\right) - \left(q^{\beta-1} - 1\right) = q^{\beta-1}(q-1)$. We show now that this accounts for all faithful irreducibles of $Q(\beta)$ over E .

LEMMA 2.9. *Let X be a q-group of class 2 with cyclic centre of*

order q^β . *The number of faithful absolutely irreducible representations*
of X *is* $q^{\beta-1}(q-1)$.

     Proof. We may assume X non-abelian as otherwise the result is well
known. Notice that the unique minimal normal subgroup (the monolith) σX
of X is the socle of X' . If elements x, y of X are conjugate
modulo σX , then there exists $t \in X$ and $z \in \sigma X$ such that

$$xz = y^t .$$

Now there exists $u \in X$ such that $z = [x, u]$ whence $x = y^{tu^{-1}}$, and
x, y are conjugate. Moreover a central element cannot be conjugate
modulo σX to a non-central element. It follows that if N is the number
of non-central conjugacy classes of X , the number of conjugacy classes in
$X/\sigma X$ is $N + |Z(X)/\sigma X| = N + q^{\beta-1}$. Hence the number of faithful
absolutely irreducible representations of X is

$(N+q^\beta) - (N+q^{\beta-1}) = q^{\beta-1}(q-1)$ as required.

     Continuing with the proof of Lemma 2.8 we find that a basis for the
modules underlying the $T_i^{Q(\beta)}$ may be constructed as follows: if $\{l_i\}$ is
a basis for the A-module T_i , $\{l_i \otimes b^j : 0 \le j \le q^\beta - 1\}$ is a basis for
$T_i^{Q(\beta)}$. The matrices representing a, b are then easily determined:

$$T_i^{Q(\beta)} : b \to \begin{pmatrix} & 1_1 & & \\ & & \ddots & \\ & & & \ddots 1 \\ 1 & & & \end{pmatrix},$$

$$T_i^{Q(\beta)} : a \to \begin{pmatrix} 1 & & & & \\ & \xi^i & & & \\ & & \xi^{2i} & & \\ & & & \ddots & \\ & & & & \xi^{(q^\beta-1)i} \end{pmatrix}.$$

It is immediately clear that the $T_i^{Q(\beta)}$ are faithful and linearly
isomorphic.

     We now use Corollary 2.3 to complete the proof of Theorem 2.4. Notice
that $Q(\alpha, \beta)$ is a subgroup of $D(\alpha, \beta)$ in such a way that the monoliths

coincide. The regular representation of $D(\alpha, \beta)$ restricted to $Q(\alpha, \beta)$
contains a copy of the regular of $Q(\alpha, \beta)$. Consequently every faithful
irreducible of $Q(\alpha, \beta)$ arises by restricting a faithful irreducible of
$D(\alpha, \beta)$; moreover it follows from Theorem 2.6 that if M is a faithful
irreducible $D(\alpha, \beta)$-module then $M_{Q(\beta)}$ is irreducible and, since all
subspaces of M admit C , $M_{Q(\alpha,\beta)}$ is irreducible. It is therefore easy
to write down the matrices representing the irreducibles of $Q(\alpha, \beta)$ and
see that they form a single linear group.

The case of $R(\beta)$ is similar: note that by Corollary 2.7 all
faithful irreducibles of $D(\beta+1, \beta)$ are linearly isomorphic; and since
there is precisely one copy of $R(\beta)$ in $D(\beta+1, \beta)$, by Corollary 2.3, all
faithful irreducibles of $R(\beta)$ are linearly isomorphic. The proof of
Theorem 2.4 is therefore complete.

It remains to prove Lemma 1.4.

Proof of Lemma 1.4. The two-generator groups with cyclic centre in
$\underline{N}_2 \wedge \underline{B}_n$ are finite in number; call them $L_1, \ldots, L_N$ say. By Theorem 2.1
each K_i is a central product of L_j's , possibly not uniquely;
nevertheless with a fixed decomposition of K_i as such a central product
containing r_{ij} copies of L_j associate the N-tuple $\left(r_{i1}, \ldots, r_{iN}\right)$.
It is well known that the set of all N-tuples with ordering by components
is partially well ordered. Consequently for some $i < j$, $r_{il} \leq r_{jl}$
$(1 \leq l \leq N)$. Thus K_i is a central factor of K_j , proving Lemma 1.4
(see the Remark following the proof of Corollary 2.3).

3. The proof of Theorem 1.2

Let $\underline{V} \leq \underline{A}_m \left(\underline{N}_2 \wedge \underline{B}_n \right)$ be just non-Cross and let $\underline{X}$ be the closed class
of linear groups associated with $\underline{V}$ à la Higman [2, §4] : $\underline{U}(\underline{X}) = \underline{V}$. Now
$\underline{X}$ is infinite and every proper closed subclass is finite; and each group
in $\underline{X}$ is a direct product of groups described in Theorem 2.1. There are
but finitely many cyclic groups and two generator groups with cyclic centre
in $\underline{N}_2 \wedge \underline{B}_n$ and consequently arbitrarily large central powers of one of
them, say Q , occur in $\underline{X}$. Since each group in $\underline{X}$ has at most one cyclic

central factor, Q is not cyclic.

Write

$$\underline{\underline{X}}_0 = \left\{ C_n : n \in Z^+ \right\}$$

where C_n is a central product with cyclic centre of n copies $Q_1, \ldots, Q_n$ of Q ; then $\underline{\underline{X}}$ is the closure of $\underline{\underline{X}}_0$.

LEMMA 3.1. *For each* $n > 1$, C_n *contains, as linear factor, the central product of* $n - 1$ *copies of* $Q(1)$.

Proof. We show first that C_n contains, as linear factor, the central product with cyclic centre of groups $S_1, \ldots, S_n$ each isomorphic to a $Q(\alpha, 1)$ or to $R(1)$. For, if $\{a_i, b_i\}$ is a canonic generating set for Q_i consider $H_i = \left\langle a_i, b_i^{q^{\beta-1}} \right\rangle$ and set $K_n = H_1 H_2 \ldots H_n$. One easily checks that the centre $Z(H_i)$ of H_i is $\left\langle a_i^q, [a_i, b_i]^{q^{\beta-1}} \right\rangle$ and that

$$Z(K_n) = Z(H_1)\ldots Z(H_n) , \quad K_n' = \left\langle [a_i, b_i]^{q^{\beta-1}} \right\rangle \quad (1 \le i \le n) .$$

Notice also that $Z(K_n)$ is the Frattini subgroup $\Phi(K_n)$ of K_n .

Now let M be a faithful irreducible module for C_n over $GF(p)$ and consider an irreducible component M_0 of M_{K_n} . Since the kernel of M_0 must avoid K_n' it is easy to see that

$$\ker M_0 \le Z(K_n) = \Phi(K_n) .$$

M_0 is a faithful and irreducible module for $P_n = K_n/\ker M_0$. P_n has cyclic centre, derived group exponent q , and it requires $2n$ generators since $\Phi(P_n) = \Phi(K_n)/\ker M_0$. Hence P_n is a central product as asserted in the first sentence of the proof.

Suppose now that $S_1, \ldots, S_n$ have as canonic generating sets $\{a_i', b_i'\}$; we write α_i for the order of a_i' and suppose

$\alpha_1 \geq \alpha_i$ $(i \geq 1)$. If $\alpha_i > 1$ $(i > 1$ we have that the subgroup

$\left\langle a_1 '^{q^{\alpha_1 - \alpha}i} , a_i ', b_i ' \right\rangle$ has cyclic centre $\left\langle a_1 '^{q^{\alpha_1 - \alpha}i} \right\rangle$ and is a central

product of $\left\langle a_1 '^{q^{\alpha_1 - \alpha}i} \right\rangle$ and $\langle a_i ', b_i ' \rangle$. Corollary 2.3 then yields that

$\left\langle a_1 '^{q^{\alpha_1 - \alpha}i} , a_i ', b_i ' \right\rangle$ is a central product of $\left\langle a_1 '^{q^{\alpha_1 - \alpha}i} \right\rangle$ and a subgroup

J_i isomorphic to $Q(1)$. Clearly $I_n = J_2 \ldots J_n$ (where $J_i = S_i$ if

$\alpha_i = 1$) is a central product with cyclic centre of $n - 1$ copies of $Q(1)$.

Since $Z\left(I_n\right) \leq Z\left(P_n\right)$, I_n is a component of P_n , and the lemma is proved.

Finally note that Higman [2, Ex. 4.10] has shown that if $Q \cong Q(1)$,

$\underline{\underline{Y}}_0 = \{C_n : n \in Z^+\}$, where C_n is a central product with cyclic centre of

n copies of Q and $\underline{Y}$ is the closure of $\underline{\underline{Y}}_0$, then $\underline{U}(\underline{Y}) = \underline{\underline{A}}_p\underline{\underline{T}}_q$ and that

this is just non-Cross. Hence, under the assumptions on $\underline{\underline{X}}$, $\underline{U}(\underline{\underline{X}}) = \underline{\underline{A}}_p\underline{\underline{T}}_q$

as required.

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