IJMMS 25:5 (2001) 305–310
PII. S0161171201004781
http://ijmms.hindawi.com

ON CERTAIN ANALYTIC UNIVALENT FUNCTIONS

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(Received 10 March 2000)

ABSTRACT. We consider the class of analytic functions $B(\alpha)$ to investigate some properties for this class. The angular estimates of functions in the class $B(\alpha)$ are obtained. Finally, we derive some interesting conditions for the class of strongly starlike and strongly convex of order α in the open unit disk.

2000 Mathematics Subject Classification. Primary 30C45.

1. Introduction. Let $\mathbb{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. A function $f(z)$ belonging to $\mathbb{A}$ is said to be starlike of order α if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in U) \tag{1.2}$$

for some α $(0 \le \alpha < 1)$. We denote by S_α^* the subclass of $\mathbb{A}$ consisting of functions which are starlike of order α in U. Also, a function $f(z)$ belonging to $\mathbb{A}$ is said to be convex of order α if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in U) \tag{1.3}$$

for some α $(0 \le \alpha < 1)$. We denote by C_α the subclass of $\mathbb{A}$ consisting of functions which are convex of order α in U.

If $f(z) \in \mathbb{A}$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U) \tag{1.4}$$

for some α $(0 \le \alpha < 1)$, then $f(z)$ said to be strongly starlike of order α in U, and this class denoted by $\bar{S}_\alpha^*$.

If $f(z) \in \mathbb{A}$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U) \tag{1.5}$$

for some α $(0 \le \alpha < 1)$, then we say that $f(z)$ is strongly convex of order α in U, and we denote by $\bar{C}_\alpha$ the class of all such functions.

The object of the present paper is to investigate various properties of the following class of analytic functions defined as follows.

Definition 1.1. A function $f(z) \in \mathbb{A}$ is said to be a member of the class $B(\alpha)$ if and only if

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha \tag{1.6}$$

for some α $(0 \le \alpha < 1)$ and for all $z \in U$.

Note that condition (1.6) implies

$$\operatorname{Re}\left(\frac{z^2 f'(z)}{f^2(z)} \right) > \alpha. \tag{1.7}$$

2. Main results. In order to derive our main results, we have to recall here the following lemmas.

Lemma 2.1 (see [2]). *Let $f(z) \in \mathbb{A}$ satisfy the condition*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 \quad (z \in U), \tag{2.1}$$

then f is univalent in U.

Lemma 2.2 (see [1]). *Let $w(z)$ be analytic in U and such that $w(0) = 0$. Then if $|w(z)|$ attains its maximum value on circle $|z| = r < 1$ at a point $z_0 \in U$, we have*

$$z_0 w'(z_0) = k w(z_0), \tag{2.2}$$

where $k \ge 1$ is a real number.

Lemma 2.3 (see [3]). *Let a function $p(z)$ be analytic in U, $p(0) = 1$, and $p(z) \ne 0$ $(z \in U)$. If there exists a point $z_0 \in U$ such that*

$$|\arg(p(z))| < \frac{\pi}{2}\alpha, \quad \text{for } |z| < |z_0|, \qquad |\arg(p(z_0))| = \frac{\pi}{2}\alpha, \tag{2.3}$$

with $0 < \alpha \le 1$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha, \tag{2.4}$$

where

$$\begin{gathered} k \ge \frac{1}{2}\left(a + \frac{1}{a}\right) \ge 1 \quad \text{when } \arg(p(z_0)) = \frac{\pi}{2}\alpha, \\ k \le -\frac{1}{2}\left(a + \frac{1}{a}\right) \le -1 \quad \text{when } \arg(p(z_0)) = -\frac{\pi}{2}\alpha, \\ p(z_0)^{1/\alpha} = \pm a i, \quad (a > 0). \end{gathered} \tag{2.5}$$

We begin with the statement and the proof of the following result.

Theorem 2.4. *If $f(z) \in \mathbb{A}$ satisfies*

$$\left| \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right| < \frac{1-\alpha}{2-\alpha} \quad (z \in U), \tag{2.6}$$

for some α $(0 \le \alpha < 1)$, then $f(z) \in B(\alpha)$.

Proof. We define the function $w(z)$ by

$$\frac{z^2 f'(z)}{f^2(z)} = 1 + (1-\alpha)w(z). \tag{2.7}$$

Then $w(z)$ is analytic in U and $w(0) = 0$. By the logarithmic differentiations, we get from (2.7) that

$$\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} = \frac{(1-\alpha)zw'(z)}{1+(1-\alpha)w(z)}. \tag{2.8}$$

Suppose there exists $z_0 \in U$ such that

$$\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1, \tag{2.9}$$

then from Lemma 2.2, we have (2.2). □

Letting $w(z_0) = e^{i\theta}$, from (2.8), we have

$$\left| \frac{(z_0 f(z_0))''}{f'(z_0)} - \frac{2z_0 f'(z_0)}{f(z_0)} \right| = \left| \frac{(1-\alpha)ke^{i\theta}}{1+(1-\alpha)e^{i\theta}} \right| \ge \frac{1-\alpha}{2-\alpha}, \tag{2.10}$$

which contradicts our assumption (2.6). Therefore $|w(z)| < 1$ holds for all $z \in U$. We finally have

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| = (1-\alpha)|w(z)| < 1-\alpha \quad (z \in U), \tag{2.11}$$

that is, $f(z) \in B(\alpha)$.

Taking $\alpha = 0$ in Theorem 2.4 and using Lemma 2.1 we have the following corollary.

Corollary 2.5. *If $f(z) \in \mathbb{A}$ satisfies*

$$\left| \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right| < \frac{1}{2} \quad (z \in U), \tag{2.12}$$

then f is univalent in U.

Next, we prove the following theorem.

Theorem 2.6. *Let $f(z) \in \mathbb{A}$. If $f(z) \in B(\alpha)$, then*

$$\left| \arg\left(\frac{f(z)}{z} \right) \right| < \frac{\pi}{2}\alpha \quad (z \in U), \tag{2.13}$$

for some α $(0 < \alpha < 1)$ and $(2/\pi)\tan^{-1}\alpha - \alpha = 1$.

Proof. We define the function $p(z)$ by

$$\frac{f(z)}{z} = p(z) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1}. \tag{2.14}$$

Then we see that $p(z)$ is analytic in U, $p(0) = 1$, and $p(z) \neq 0$ $(z \in U)$. It follows from (2.14) that

$$\frac{z^2 f'(z)}{f^2(z)} = \frac{1}{p(z)}\left(1 + \frac{zp'(z)}{p(z)}\right). \tag{2.15}$$

Suppose there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha, \quad \text{for } |z| < |z_0|, \qquad |\arg(p(z_0))| = \frac{\pi}{2}\alpha. \tag{2.16}$$

Then, applying Lemma 2.3, we can write that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha, \tag{2.17}$$

where

$$\begin{gathered} k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \quad \text{when } \arg(p(z_0)) = \frac{\pi}{2}\alpha, \\ k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \quad \text{when } \arg(p(z_0)) = -\frac{\pi}{2}\alpha, \\ p(z_0)^{1/\alpha} = \pm ai, \quad (a > 0). \end{gathered} \tag{2.18}$$

Therefore, if $\arg(p(z_0)) = \pi\alpha/2$, then

$$\frac{z_0^2 f'(z_0)}{f^2(z_0)} = \frac{1}{p(z_0)}\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right) = a^{-\alpha} e^{-i\pi\alpha/2}(1 + ik\alpha). \tag{2.19}$$

This implies that

$$\begin{aligned} \arg\left(\frac{z_0^2 f'(z_0)}{f^2(z_0)}\right) &= \arg\left(\frac{1}{p(z_0)}\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right) \\ &= -\frac{\pi}{2}\alpha + \arg(1 + i\alpha k) \geq -\frac{\pi}{2}\alpha + \tan^{-1}\alpha \\ &= \frac{\pi}{2}\left(\frac{2}{\pi}\tan^{-1}\alpha - \alpha\right) = \frac{\pi}{2} \end{aligned} \tag{2.20}$$

if

$$\frac{2}{\pi}\tan^{-1}\alpha - \alpha = 1. \tag{2.21}$$

Also, if $\arg(p(z_0)) = -\pi\alpha/2$, we have

$$\arg\left(\frac{z_0^2 f'(z_0)}{f^2(z_0)}\right) \leq -\frac{\pi}{2} \tag{2.22}$$

if

$$\frac{2}{\pi}\tan^{-1}\alpha - \alpha = 1. \tag{2.23}$$

These contradict the assumption of the theorem.

Thus, the function $p(z)$ has to satisfy

$$|\arg(p(z))| < \frac{\pi}{2}\alpha \quad (z \in U) \tag{2.24}$$

or

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U). \tag{2.25}$$

This completes the proof. □

Now, we prove the following theorem.

Theorem 2.7. *Let $p(z)$ be analytic in U, $p(z) \neq 0$ in U and suppose that*

$$\left|\arg\left(p(z) + \frac{z^3 f'(z)}{f^2(z)} p'(z)\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U), \tag{2.26}$$

where $0 < \alpha < 1$ and $f(z) \in B(\alpha)$, then we have

$$|\arg(p(z))| < \frac{\pi}{2}\alpha \quad (z \in U). \tag{2.27}$$

Proof. Suppose there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha, \quad \text{for } |z| < |z_0|, \qquad |\arg(p(z_0))| = \frac{\pi}{2}\alpha. \tag{2.28}$$

Then, applying Lemma 2.3, we can write that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha, \tag{2.29}$$

where

$$\begin{aligned} k &\geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when } \arg(p(z_0)) = \frac{\pi}{2}\alpha, \\ k &\leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when } \arg(p(z_0)) = -\frac{\pi}{2}\alpha, \\ &p(z_0)^{1/\alpha} = \pm ai, \quad (a > 0). \end{aligned} \tag{2.30}$$

Then it follows that

$$\begin{aligned} \arg\left(p(z_0) + \frac{z_0^3 f'(z_0)}{f^2(z_0)} p'(z_0)\right) &= \arg\left(p(z_0)\left(1 + \frac{z_0^2 f'(z_0)}{f^2(z_0)} \frac{z p'(z_0)}{p(z_0)}\right)\right) \\ &= \arg\left(p(z_0)\left(1 + i\frac{z_0^2 f'(z_0)}{f^2(z_0)}\alpha k\right)\right). \end{aligned} \tag{2.31}$$

When $\arg(p(z_0)) = \pi\alpha/2$, we have

$$\arg\left(p(z_0) + \frac{z_0^3 f'(z_0)}{f^2(z_0)} p'(z_0)\right) = \arg(p(z_0)) + \arg\left(1 + i\frac{z_0^2 f'(z_0)}{f^2(z_0)}\alpha k\right) > \frac{\pi}{2}\alpha, \tag{2.32}$$

because

$$\operatorname{Re}\frac{z_0^2 f'(z_0)}{f^2(z_0)}\alpha k > 0 \text{ and therefore } \arg\left(1 + i\frac{z_0^2 f'(z_0)}{f^2(z_0)}\alpha k\right) > 0. \tag{2.33}$$

Similarly, if $\arg(p(z_0)) = -\pi\alpha/2$, then we obtain that

$$\arg\left(p(z_0)+\frac{z_0^3 f'(z_0)}{f^2(z_0)}p'(z_0)\right) = \arg(p(z_0))+\arg\left(1+i\frac{z_0^2 f'(z_0)}{f^2(z_0)}\alpha k\right) < -\frac{\pi}{2}\alpha, \quad (2.34)$$

because

$$\operatorname{Re}\frac{z_0^2 f'(z_0)}{f^2(z_0)}\alpha k < 0 \text{ and therefore } \arg\left(1+i\frac{z_0^2 f'(z_0)}{f^2(z_0)}\alpha k\right) < 0. \quad (2.35)$$

Thus we see that (2.32) and (2.34) contradict our condition (2.26). Consequently, we conclude that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha \quad (z \in U). \quad (2.36)$$

□

Taking $p(z) = zf'(z)/f(z)$ in Theorem 2.7, we have the following corollary.

Corollary 2.8. *If $f(z) \in \mathbb{A}$ satisfying*

$$\left|\arg\left(\frac{zf'(z)}{f(z)}+\frac{z^3 f'(z)}{f^3(z)}\left((zf'(z))'-\frac{z(f'(z))^2}{f(z)}\right)\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U), \quad (2.37)$$

where $0 < \alpha < 1$ and $f(z) \in B(\alpha)$, then $f(z) \in \bar{S}_\alpha^*$.

Taking $p(z) = 1 + zf''(z)/f'(z)$ in Theorem 2.7, we have the following corollary.

Corollary 2.9. *If $f(z) \in \mathbb{A}$ satisfying*

$$\left|\arg\left(\frac{(zf'(z))'}{f'(z)}+\frac{z^3}{f^3(z)}\left((zf''(z))'-\frac{z(f''(z))^2}{f'(z)}\right)\right)\right| < \frac{\pi}{2}\alpha, \quad (2.38)$$

where $0 < \alpha < 1$ and $f(z) \in B(\alpha)$, then $f(z) \in \bar{C}_\alpha$.

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