ON CERTAIN ANALYTIC UNIVALENT FUNCTIONS

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ABSTRACT. We consider the class of analytic functions $B(\alpha)$ to investigate some properties for this class. The angular estimates of functions in the class $B(\alpha)$ are obtained. Finally, we derive some interesting conditions for the class of strongly starlike and strongly convex of order α in the open unit disk.

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1. Introduction. Let \mathbb{A} denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$
 (1.1)

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. A function f(z) belonging to \mathbb{A} is said to be starlike of order α if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in U)$$
 (1.2)

for some α ($0 \le \alpha < 1$). We denote by S_{α}^* the subclass of $\mathbb A$ consisting of functions which are starlike of order α in U. Also, a function f(z) belonging to $\mathbb A$ is said to be convex of order α if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in U)$$
(1.3)

for some α ($0 \le \alpha < 1$). We denote by C_{α} the subclass of $\mathbb A$ consisting of functions which are convex of order α in U.

If $f(z) \in \mathbb{A}$ satisfies

$$\left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U)$$
 (1.4)

for some α ($0 \le \alpha < 1$), then f(z) said to be strongly starlike of order α in U, and this class denoted by \bar{S}_{α}^* .

If $f(z) \in \mathbb{A}$ satisfies

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U)$$
 (1.5)

for some α ($0 \le \alpha < 1$), then we say that f(z) is strongly convex of order α in U, and we denote by \bar{C}_{α} the class of all such functions.

The object of the present paper is to investigate various properties of the following class of analytic functions defined as follows.

DEFINITION 1.1. A function $f(z) \in \mathbb{A}$ is said to be a member of the class $B(\alpha)$ if and only if

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha \tag{1.6}$$

for some α ($0 \le \alpha < 1$) and for all $z \in U$.

Note that condition (1.6) implies

$$\operatorname{Re}\left(\frac{z^2 f'(z)}{f^2(z)}\right) > \alpha. \tag{1.7}$$

2. Main results. In order to derive our main results, we have to recall here the following lemmas.

LEMMA 2.1 (see [2]). Let $f(z) \in A$ satisfy the condition

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 \quad (z \in U), \tag{2.1}$$

then f is univalent in U.

LEMMA 2.2 (see [1]). Let w(z) be analytic in U and such that w(0) = 0. Then if |w(z)| attains its maximum value on circle |z| = r < 1 at a point $z_0 \in U$, we have

$$z_0 w'(z_0) = k w(z_0), (2.2)$$

where $k \ge 1$ is a real number.

LEMMA 2.3 (see [3]). Let a function p(z) be analytic in U, p(0) = 1, and $p(z) \neq 0$ ($z \in U$). If there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha$$
, for $|z| < |z_0|$, $|\arg(p(z_0))| = \frac{\pi}{2}\alpha$, (2.3)

with $0 < \alpha \le 1$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha, \tag{2.4}$$

where

$$k \ge \frac{1}{2} \left(a + \frac{1}{a} \right) \ge 1 \quad \text{when } \arg \left(p(z_0) \right) = \frac{\pi}{2} \alpha,$$

$$k \le -\frac{1}{2} \left(a + \frac{1}{a} \right) \le -1 \quad \text{when } \arg \left(p(z_0) \right) = -\frac{\pi}{2} \alpha,$$

$$p(z_0)^{1/\alpha} = \pm ai, \quad (a > 0).$$

$$(2.5)$$

We begin with the statement and the proof of the following result.

THEOREM 2.4. If $f(z) \in \mathbb{A}$ satisfies

$$\left| \frac{\left(zf(z) \right)^{\prime \prime}}{f^{\prime}(z)} - \frac{2zf^{\prime}(z)}{f(z)} \right| < \frac{1-\alpha}{2-\alpha} \quad (z \in U), \tag{2.6}$$

for some α (0 $\leq \alpha < 1$), then $f(z) \in B(\alpha)$.

PROOF. We define the function w(z) by

$$\frac{z^2 f'(z)}{f^2(z)} = 1 + (1 - \alpha) w(z). \tag{2.7}$$

Then w(z) is analytic in U and w(0) = 0. By the logarithmic differentiations, we get from (2.7) that

$$\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} = \frac{(1-\alpha)zw'(z)}{1+(1-\alpha)w(z)}.$$
 (2.8)

Suppose there exists $z_0 \in U$ such that

$$\max_{|z| < |z_0|} |w(z)| = |w(z_0)| = 1, \tag{2.9}$$

then from Lemma 2.2, we have (2.2).

Letting $w(z_0) = e^{i\theta}$, from (2.8), we have

$$\left| \frac{(z_0 f(z_0))^{"}}{f'(z_0)} - \frac{2z_0 f'(z_0)}{f(z_0)} \right| = \left| \frac{(1-\alpha)ke^{i\theta}}{1 + (1-\alpha)e^{i\theta}} \right| \ge \frac{1-\alpha}{2-\alpha},\tag{2.10}$$

which contradicts our assumption (2.6). Therefore |w(z)| < 1 holds for all $z \in U$. We finally have

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| = (1 - \alpha) |w(z)| < 1 - \alpha \quad (z \in U), \tag{2.11}$$

that is, $f(z) \in B(\alpha)$.

Taking $\alpha = 0$ in Theorem 2.4 and using Lemma 2.1 we have the following corollary.

COROLLARY 2.5. *If* $f(z) \in \mathbb{A}$ *satisfies*

$$\left| \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right| < \frac{1}{2} \quad (z \in U), \tag{2.12}$$

then f is univalent in U.

Next, we prove the following theorem.

THEOREM 2.6. Let $f(z) \in \mathbb{A}$. If $f(z) \in B(\alpha)$, then

$$\left| \arg \left(\frac{f(z)}{z} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U),$$
 (2.13)

for some α (0 < α < 1) and (2/ π) tan⁻¹ α - α = 1.

PROOF. We define the function p(z) by

$$\frac{f(z)}{z} = p(z) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1}.$$
 (2.14)

Then we see that p(z) is analytic in U, p(0) = 1, and $p(z) \neq 0$ ($z \in U$). It follows from (2.14) that

$$\frac{z^2 f'(z)}{f^2(z)} = \frac{1}{p(z)} \left(1 + \frac{z p'(z)}{p(z)} \right). \tag{2.15}$$

Suppose there exists a point $z_0 \in U$ such that

$$\left| \operatorname{arg} (p(z)) \right| < \frac{\pi}{2} \alpha, \quad \text{for } |z| < |z_0|, \qquad \left| \operatorname{arg} (p(z_0)) \right| = \frac{\pi}{2} \alpha.$$
 (2.16)

Then, applying Lemma 2.3, we can write that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha, \tag{2.17}$$

where

$$k \ge \frac{1}{2} \left(a + \frac{1}{a} \right) \ge 1 \quad \text{when } \arg \left(p(z_0) \right) = \frac{\pi}{2} \alpha,$$

$$k \le -\frac{1}{2} \left(a + \frac{1}{a} \right) \le -1 \quad \text{when } \arg \left(p(z_0) \right) = -\frac{\pi}{2} \alpha,$$

$$p(z_0)^{1/\alpha} = \pm ai, \quad (a > 0).$$

$$(2.18)$$

Therefore, if $arg(p(z_0)) = \pi \alpha/2$, then

$$\frac{z_0^2 f'(z_0)}{f^2(z_0)} = \frac{1}{p(z_0)} \left(1 + \frac{z_0 p'(z_0)}{p(z_0)} \right) = a^{-\alpha} e^{-i\pi\alpha/2} (1 + ik\alpha). \tag{2.19}$$

This implies that

$$\arg\left(\frac{z_0^2 f'(z_0)}{f^2(z_0)}\right) = \arg\left(\frac{1}{p(z_0)} \left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right)$$

$$= -\frac{\pi}{2} \alpha + \arg(1 + i\alpha k) \ge -\frac{\pi}{2} \alpha + \tan^{-1} \alpha$$

$$= \frac{\pi}{2} \left(\frac{2}{\pi} \tan^{-1} \alpha - \alpha\right) = \frac{\pi}{2}$$
(2.20)

if

$$\frac{2}{\pi}\tan^{-1}\alpha - \alpha = 1. \tag{2.21}$$

Also, if $arg(p(z_0)) = -\pi \alpha/2$, we have

$$\arg\left(\frac{z_0^2 f'(z_0)}{f^2(z_0)}\right) \le -\frac{\pi}{2} \tag{2.22}$$

if

$$\frac{2}{\pi}\tan^{-1}\alpha - \alpha = 1. \tag{2.23}$$

These contradict the assumption of the theorem.

Thus, the function p(z) has to satisfy

$$\left| \arg \left(p(z) \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U)$$
 (2.24)

or

$$\left| \arg \left(\frac{f(z)}{z} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U).$$
 (2.25)

This completes the proof.

Now, we prove the following theorem.

THEOREM 2.7. Let p(z) be analytic in U, $p(z) \neq 0$ in U and suppose that

$$\left| \arg \left(p(z) + \frac{z^3 f'(z)}{f^2(z)} p'(z) \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U), \tag{2.26}$$

where $0 < \alpha < 1$ and $f(z) \in B(\alpha)$, then we have

$$\left| \operatorname{arg} \left(p(z) \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U).$$
 (2.27)

PROOF. Suppose there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha$$
, for $|z| < |z_0|$, $|\arg(p(z_0))| = \frac{\pi}{2}\alpha$. (2.28)

Then, applying Lemma 2.3, we can write that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha, \tag{2.29}$$

where

$$k \ge \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when } \arg \left(p(z_0) \right) = \frac{\pi}{2} \alpha,$$

$$k \le -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when } \arg \left(p(z_0) \right) = -\frac{\pi}{2} \alpha,$$

$$p(z_0)^{1/\alpha} = \pm ai, \quad (a > 0).$$

$$(2.30)$$

Then it follows that

$$\arg\left(p(z_{0}) + \frac{z_{0}^{3}f'(z_{0})}{f^{2}(z_{0})}p'(z_{0})\right) = \arg\left(p(z_{0})\left(1 + \frac{z_{0}^{2}f'(z_{0})}{f^{2}(z_{0})}\frac{zp'(z_{0})}{p(z_{0})}\right)\right)$$

$$= \arg\left(p(z_{0})\left(1 + i\frac{z_{0}^{2}f'(z_{0})}{f^{2}(z_{0})}\alpha k\right)\right). \tag{2.31}$$

When $arg(p(z_0)) = \pi \alpha/2$, we have

$$\arg\left(p(z_0) + \frac{z_0^3 f'(z_0)}{f^2(z_0)}p'(z_0)\right) = \arg\left(p(z_0)\right) + \arg\left(1 + i\frac{z_0^2 f'(z_0)}{f^2(z_0)}\alpha k\right) > \frac{\pi}{2}\alpha, \quad (2.32)$$

because

Re
$$\frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k > 0$$
 and therefore $\arg\left(1 + i\frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k\right) > 0$. (2.33)

Similarly, if $arg(p(z_0)) = -\pi \alpha/2$, then we obtain that

$$\arg\left(p(z_0) + \frac{z_0^3 f'(z_0)}{f^2(z_0)}p'(z_0)\right) = \arg\left(p(z_0)\right) + \arg\left(1 + i\frac{z_0^2 f'(z_0)}{f^2(z_0)}\alpha k\right) < -\frac{\pi}{2}\alpha, (2.34)$$

because

$$\operatorname{Re} \frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k < 0 \text{ and therefore } \operatorname{arg} \left(1 + i \frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k \right) < 0. \tag{2.35}$$

Thus we see that (2.32) and (2.34) contradict our condition (2.26). Consequently, we conclude that

$$\left| \operatorname{arg} (p(z)) \right| < \frac{\pi}{2} \alpha \quad (z \in U).$$
 (2.36)

Taking p(z) = zf'(z)/f(z) in Theorem 2.7, we have the following corollary.

COROLLARY 2.8. *If* $f(z) \in \mathbb{A}$ *satisfying*

$$\left| \arg \left(\frac{zf'(z)}{f(z)} + \frac{z^3f'(z)}{f^3(z)} \left((zf'(z))' - \frac{z(f'(z))^2}{f(z)} \right) \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U), \tag{2.37}$$

where $0 < \alpha < 1$ and $f(z) \in B(\alpha)$, then $f(z) \in \bar{S}_{\alpha}^*$.

Taking p(z) = 1 + z f''(z) / f'(z) in Theorem 2.7, we have the following corollary.

COROLLARY 2.9. *If* $f(z) \in \mathbb{A}$ *satisfying*

$$\left| \arg \left(\frac{(zf'(z))'}{f'(z)} + \frac{z^3}{f^3(z)} \left((zf''(z))' - \frac{z(f''(z))^2}{f'(z)} \right) \right) \right| < \frac{\pi}{2} \alpha, \tag{2.38}$$

where $0 < \alpha < 1$ and $f(z) \in B(\alpha)$, then $f(z) \in \bar{C}_{\alpha}$.

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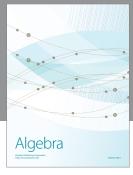
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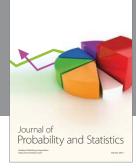
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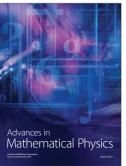






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