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ON CERTAIN CLASS OF MEROMORPHIC HARMONIC CONCAVE FUNCTIONS

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Abstract. In this paper, a class of meromorphic harmonic functions concave in the unit disc is introduced. Coefficient bounds, distortion inequalities, extreme points, geometric convolution, integral convolution for the functions belonging to this class are obtained.

1. Introduction

Conformal maps of the unit disc onto convex domain are a classical topic and many results are found related to this field. Recently, Avkhadiev and Wirths [4] discovered the conformal mapping of a unit disc onto concave domains (the complements of convex closed sets). This is interesting as not many problems are discussed thoroughly towards this approach.

Let $\mathbb{U}$ denote the open unit disc, where f has the form given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that maps $\mathbb{U}$ conformally onto a domain whose complement with respect to C is convex and that satisfies the normalization $f(1) = \infty$. Furthermore, they imposed on these functions the condition that the opening angle of $f(\mathbb{U})$ at infinity is less than or equal to $\alpha\pi$, $\alpha \in (1,2]$. These families of functions are denoted by $C_0(\alpha)$. The class $C_0(\alpha)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions we refer to [3],[4],[5] and [8]. We observe that $C_0(2)$ contains the classes $C_0(\alpha)$, $\alpha \in (1,2]$ and the class $C_0(1)$ consists of all concave univalent functions normalized such that $f(1) = \infty$ and f is given by (1).

Recently, Chuaqui et al. [6] introduced the concept of meromorphic concave mappings. A conformal mapping of meromorphic function on the unit disc $\mathbb{U}$ is said to be a concave mapping if its image is the complement of a compact, convex set.

If f has the form

$$f(z) = \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \cdots,$$

then a necessary and sufficient condition for f to be a concave mapping is

$$1 + Re\left\{z\frac{f''(z)}{f'(z)}\right\} < 0, \quad |z| < 1.$$

The subject of harmonic univalent functions is not that recent as it has been around since 1984 and perhaps was first established by Clunie and Sheil-Small [7]. In [14] there is more comprehensive study on harmonic univalent functions.

The importance of these functions is due to their usage in the study of minimal surfaces as well as in various problems related to engineering, operations research, applied Mathematics and perhaps of other areas of sciences [1]. A continuous function $f = u + iv$ is a complex valued harmonic function in a domain $\mathbb{U} \subset C$ if both u and v are real harmonic in $\mathbb{U}$. In any simply connected domain, we write $f = h + \bar{g}$ where h and g are analytic in $\mathbb{U}$. A necessary and sufficient condition for f to be locally univalent and orientation preserving in $\mathbb{U}$ is that $|h'| > |g'|$ in $\mathbb{U}$ (see [7]). Hengartner and Schober [10] investigated functions harmonic in the exterior of the unit disc $\widetilde{\mathbb{U}} = \{z : |z| > 1\}$. They showed that complex valued, harmonic, sense preserving, univalent mapping f must admit the representation

$$f(z) = h(z) + \overline{g(z)} + A log|z|,$$

where $h(z)$ and $g(z)$ are defined by

$$h(z) = \alpha z + \sum_{n=1}^{\infty} a_n z^{-n}, \quad g(z) = \beta \bar{z} + \sum_{n=1}^{\infty} b_n z^{-n}$$

for $0 \le |\beta| < |\alpha|$, $A \in C$ and $z \in \widetilde{\mathbb{U}}$.

For $z \in \mathbb{U}\backslash\{0\}$, let M_H be the class of functions:

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \tag{2}$$

which are harmonic in the punctured unit disc $\mathbb{U}\backslash\{0\}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{U}\backslash\{0\}$ and $\mathbb{U}$, respectively, and $h(z)$ has a simple pole at the origin with residue 1 here (see [2]).

A function $f \in M_H$ is said to be in the subclass MS_H^* of meromorphically harmonic starlike functions in $\mathbb{U}\backslash\{0\}$ if it satisfies the condition

$$Re\left\{-\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} > 0, \quad (z \in \mathbb{U}\backslash\{0\}).$$

Also, a function $f \in M_H$ is said to be in the subclass MC_H of meromorphically harmonic convex functions in $\mathbb{U}\backslash\{0\}$ if it satisfies the condition

$$Re\left\{-\frac{zh''(z) + h'(z) - \overline{zg''(z) + g'(z)}}{h'(z) - \overline{g'(z)}}\right\} > 0, \quad (z \in \mathbb{U}\backslash\{0\}).$$

Note that the classes of harmonic meromorphic starlike functions and harmonic meromorphic convex functions have been studied by Jahangiri and Silverman [11], and Jahangiri [12, 13].

This work is an attempt to give a connection between harmonic function and meromorphic concave functions by introducing a class MHC_0 of meromorphic harmonic concave functions.

Definition 1.1. Let MHC_0 denote the class of meromorphic harmonic concave functions f of the form (2) such that

$$1 + Re\left\{\frac{zf''(z)}{f'(z)}\right\} < 0, \quad |z| < 1. \tag{3}$$

The article is organized as follows: In section 2, we study a sufficient condition for functions $f = h + \overline{g}$, where h and g given by (2) to be in the class MHC_0. In section 3, we obtain distortion bounds, characterize the extreme points for functions in MHC_0. In section 4, we define convolution properties for functions belonging to the class MHC_0.

2. Coefficient Conditions

In this section, sufficient coefficient condition for a function $f \in M_H$ to be in MHC_0 is derived.

Theorem 2.1. Let $f = h + \overline{g}$ be of the form (2). If

$$\sum_{n=1}^{\infty} n^2 (|a_n| + |b_n|) \le 1, \tag{4}$$

then, f is harmonic univalent, sense preserving in $\mathbb{U}\backslash\{0\}$.

Proof. First, for $0 < |z_1| \le |z_2| < 1$, we have

$$\begin{aligned}
\left|f(z_1) - f(z_2)\right| &\ge |h(z_1) - h(z_2)| - \left|g(z_1) - g(z_2)\right| \\
&\ge \frac{|z_1 - z_2|}{|z_1||z_2|} - |z_1 - z_2| \sum_{n=1}^{\infty} (|a_n| + |b_n|) |z_1^{n-1} + \cdots + z_2^{n-1}| \\
&> \frac{|z_1 - z_2|}{|z_1||z_2|} \left[1 - |z_2|^2 \sum_{n=1}^{\infty} n (|a_n| + |b_n|)\right] \\
&> \frac{|z_1 - z_2|}{|z_1||z_2|} \left[1 - \sum_{n=1}^{\infty} n^2 (|a_n| + |b_n|)\right].
\end{aligned}$$

The last expression is non negative by (4) and f is univalent in $\mathbb{U} \backslash \{0\}$.

To show that f is sense preserving in $\mathbb{U} \backslash \{0\}$, we need to show that $|h'(z)| \geq |g'(z)|$ in $\mathbb{U} \backslash \{0\}$.

$$
\begin{aligned}
|h'(z)| &\geq \frac{1}{|z|^2} - \sum_{n=1}^{\infty} n|a_n||z|^{n-1} \\
&= \frac{1}{r^2} - \sum_{n=1}^{\infty} n|a_n|r^{n-1} > 1 - \sum_{n=1}^{\infty} n|a_n| \\
&\geq 1 - \sum_{n=1}^{\infty} n^2|a_n| \\
&\geq \sum_{n=1}^{\infty} n^2|b_n| > \sum_{n=1}^{\infty} n|b_n|r^{n-1} \\
&= \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \geq |g'(z)|.
\end{aligned}
$$

Thus, this completes the proof of the theorem.

Theorem 2.2. Let $f = h + \overline{g}$ be of the form (2). Then $f \in MHC_0$ if and only if the inequality (4) holds for the coefficient $f = h + \overline{g}$.

Proof. Suppose that inequality (4) holds. By using the fact $Rew < 0 \leftrightarrow \left| \frac{w+1}{w-1} \right| < 1$. So it suffices to show that $\left| \frac{w+1}{w-1} \right| < 1$. We have

$$
\begin{aligned}
&= \left| \frac{1 + \frac{zh''(z) - z\overline{g''(z)}}{h'(z) - \overline{g'(z)}} + 1}{1 + \frac{zh''(z) - z\overline{g''(z)}}{h'(z) - \overline{g'(z)}} - 1} \right| \\
&= \left| \frac{2(h'(z) - \overline{g'(z)}) + zh''(z) - z\overline{g''(z)}}{zh''(z) - z\overline{g''(z)}} \right| \\
&= \left| \frac{\frac{2}{z^2} + \sum_{n=1}^{\infty} n(n-1)a_n z^{n-1} - \frac{2}{z^2} + 2\sum_{n=1}^{\infty} na_n z^{n-1} - \sum_{n=1}^{\infty} n(n-1)\overline{b_n z^{n-1}} - 2\sum_{n=1}^{\infty} n\overline{b_n z^{n-1}}}{\frac{2}{z^2} + \sum_{n=1}^{\infty} n(n-1)a_n z^{n-1} - \sum_{n=1}^{\infty} n(n-1)\overline{b_n z^{n-1}}} \right| \\
&= \left| \frac{\sum_{n=1}^{\infty} n(n+1)a_n z^{n-1} - \sum_{n=1}^{\infty} n(n+1)\overline{b_n z^{n-1}}}{\frac{2}{z^2} + \sum_{n=1}^{\infty} n(n-1)a_n z^{n-1} - \sum_{n=1}^{\infty} n(n-1)\overline{b_n z^{n-1}}} \right| \\
&< \frac{\sum_{n=1}^{\infty} n(n+1)|a_n| + \sum_{n=1}^{\infty} n(n+1)|b_n|}{2 - \sum_{n=1}^{\infty} n(n-1)|a_n| - \sum_{n=1}^{\infty} n(n-1)|b_n|}.
\end{aligned}
$$

The last expression is bounded above by 1 if

$$
\sum_{n=1}^{\infty} n(n+1)|a_n| + \sum_{n=1}^{\infty} n(n+1)|b_n| \leq 2 - \sum_{n=1}^{\infty} n(n-1)|a_n| - \sum_{n=1}^{\infty} n(n-1)|b_n|,
$$

which is equivalent to our condition

$$
\sum_{n=1}^{\infty} n^2 \left(|a_n| + |b_n| \right) \leq 1.
$$

of the theorem.

Conversely, assume $f \in MHC_0$, then we have

$$\left| \frac{1 + \frac{zh''(z) - z\overline{g''(z)}}{h'(z) - \overline{g'(z)}} + 1}{1 + \frac{zh''(z) - z\overline{g''(z)}}{h'(z) - \overline{g'(z)}} - 1} \right| < 1$$

$$= \left| \frac{\sum_{n=1}^{\infty} n(n+1) a_n z^{n-1} - \sum_{n=1}^{\infty} n(n+1) \overline{b_n z^{n-1}}}{\frac{2}{z^2} + \sum_{n=1}^{\infty} n(n-1) a_n z^{n-1} - \sum_{n=1}^{\infty} n(n-1) \overline{b_n z^{n-1}}} \right| < 1.$$

Letting $|z| \to 1$, we obtain the required condition (4).

3. Distortion bounds and extreme points

In this section, bounds and extreme points for functions belonging to the class MHC_0 are estimated.

Theorem 3.1. If $f_k = h_k + \overline{g_k} \in MHC_0$ and $0 < |z| = r < 1$, then

$$|f_k(z)| \leq \frac{1 + r^2}{r}$$

and

$$|f_k(z)| \geq \frac{1 - r^2}{r}.$$

Proof. Let $f_k = h_k + \overline{g_k} \in MHC_0$. Taking the absolute value of f we obtain

$$|f_k| = \left| \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n} \right|$$

$$\geq \frac{1}{r} - \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \geq \frac{1}{r} - \sum_{n=1}^{\infty} (|a_n| + |b_n|) r$$

$$\geq \frac{1}{r} - \sum_{n=1}^{\infty} n^2 (|a_n| + |b_n|) r$$

$$\geq \frac{1}{r} - r = \frac{1 - r^2}{r}$$

$$|f_k| = \left| \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n} \right|$$

$$|f_k| \leq \frac{1}{r} + \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n$$

$$\leq \frac{1}{r} + \sum_{n=1}^{\infty} n^2 (|a_n| + |b_n|) r^n$$

$$\leq \frac{1}{r} + \sum_{n=1}^{\infty} n^2 (|a_n| + |b_n|) r$$

$$\leq \frac{1 + r^2}{r}.$$

Now the theorem follows by applying (4).

Theorem 3.2. Let $f_k = h_k + \overline{g_k}$ where h_k and g_k are given by (2). Set

$$h_{k,0} = g_{k,0} = \frac{1}{z}$$

$$h_{k,n}(z) = \frac{1}{z} + \frac{1}{n^2} z^n,$$

for $n = 1,2,3,\dots$ and

$$g_{k,n} = \frac{1}{z} + \frac{1}{n^2} \overline{z^n}$$

for $n = 1,2,3,\dots$.

Then, $f_k \in MHC_0$ if and only if f_k can be expressed as

$$f_{k,n} = \sum_{n=0}^{\infty} \left(\lambda_n h_{k,n}(z) + \gamma_n g_{k,n}(z) \right),$$

where $\lambda_n \geq 0, \gamma_n \geq 0$. and $\Sigma_{n=0}^{\infty}(\lambda_n + \gamma_n) = 1$. In particular, the extreme points of MHC_0 are $\{h_{k,n}\}$ and $\{g_{k,n}\}$.

Proof. For functions $f_k = h_k + \overline{g_k}$, where h_k and g_k are given by (2), we have

$$\begin{aligned}
f_{k,n}(z) &= \sum_{n=0}^{\infty} \left(\lambda_n h_{k,n}(z) + \gamma_n g_{k,n}(z) \right) \\
&= \lambda_0 h_{k,0} + \gamma_0 g_{k,0} + \sum_{n=1}^{\infty} \lambda_n \left(\frac{1}{z} + \frac{1}{n^2} z^n \right) + \sum_{n=1}^{\infty} \gamma_n \left(\frac{1}{z} + \frac{1}{n^2} \overline{z^n} \right) \\
&= \sum_{n=0}^{\infty} (\lambda_n + \gamma_n) \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\lambda_n z^n + \overline{\gamma_n z^n} \right).
\end{aligned}$$

Now by Theorem 2.1,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} n^2 \lambda_n + \frac{1}{n^2} n^2 \gamma_n$$

$$= \sum_{n=1}^{\infty} (\lambda_n + \gamma_n) = 1 - \lambda_0 - \gamma_0 \leq 1.$$

So $f_k \in MHC_0$.

Conversely, suppose that $f_k \in MHC_0$. Setting

$$\lambda_n = n^2 |a_n|, \quad n \geq 1$$
$$\gamma_n = n^2 |b_n|, \quad n \geq 1.$$

We define

$$\lambda_0 + \gamma_0 = 1 - \sum_{n=1}^{\infty} \lambda_n - \sum_{n=1}^{\infty} \gamma_n.$$

Therefore, f can be written as

$$
\begin{aligned}
f_k(z) &= \frac{1}{z} + \sum_{n=1} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z^n} \\
&= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{n^2} \lambda_n z^n + \sum_{n=1}^{\infty} \frac{1}{n^2} \gamma_n \overline{z^n} \\
&= \frac{1}{z} + \sum_{n=1}^{\infty} \left(h_{k,n}(z) - \frac{1}{z} \right) \lambda_n + \sum_{n=1}^{\infty} \left(g_{k,n}(z) - \frac{1}{z} \right) \gamma_n \\
&= \sum_{n=1}^{\infty} h_{k,n} \lambda_n + \sum_{n=1}^{\infty} g_{k,n} \gamma_n + \frac{1}{z} \left(1 - \sum_{n=1}^{\infty} \lambda_n - \sum_{n=1}^{\infty} \gamma_n \right) \\
&= \lambda_0 h_{k,0} + \gamma_0 g_{k,o} + \sum_{n=1}^{\infty} h_{k,n} \lambda_n + \sum_{n=1}^{\infty} g_{k,n} \gamma_n \\
&= \sum_{n=0}^{\infty} \left(\lambda_n h_{k,n}(z) + \gamma_n g_{k,n}(z) \right).
\end{aligned}
$$

The proof is complete. Therefore $\{h_{k,n}\}$ and $\{g_{k,n}\}$ are extreme points.

4. Convolution properties

In this section, convolution, geometric convolution, integral convolution of the class MHC_0 are defined and studied.

For harmonic functions, f_k and F_k defined as follows:

$$
f_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z^n} \tag{5}
$$

and

$$
F_k = \frac{1}{z} + \sum_{n=1}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \overline{z^n}, \tag{6}
$$

the convolution of f_k and F_k is given by

$$
\begin{aligned}
(f_k \star F_k)(z) &= f_k(z) \star F_k(z) \\
&= \frac{1}{z} + \sum_{n=1}^{\infty} |a_n||A_n| z^n + \sum_{n=1}^{\infty} |b_n||B_n| \overline{z^n}.
\end{aligned} \tag{7}
$$

The geometric convolution of f_k and F_k is given by

$$
\begin{aligned}
(f_k * F_k)(z) &= f_k(z) * F_k(z) \\
&= \frac{1}{z} + \sum_{n=1}^{\infty} \sqrt{|a_n A_n|} z^n + \sum_{n=1}^{\infty} \sqrt{|b_n B_n|} \overline{z^n}.
\end{aligned} \tag{8}
$$

The integral convolution of f_k and F_k is given by

$$
\begin{aligned}
(f_k \diamond F_k)(z) &= f_k(z) \diamond F_k(z) \\
&= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{|a_n A_n|}{n} z^n + \sum_{n=1}^{\infty} \frac{|b_n B_n|}{n} \overline{z^n}.
\end{aligned} \tag{9}
$$

Theorem 4.1. Let $f_k \in MHC_0$ and $F_k \in MHC_0$. Then the convolution $f_k \star F_k \in MHC_0$.

Proof. For f_k and F_k as given by (5) and (6), then the convolution $f_k \star F_k$ is given by (7). We wish to show that the coefficients of $f_k \star F_k$ satisfy the required condition given in Theorem 2.1. For $F_k \in MHC_0$ we note that $|A_n| \leq 1$ and $|B_n| \leq 1$. Now for convolution function $f_k \star F_k$ we obtain

$$\sum_{n=1}^{\infty} n^2 |a_n||A_n| + \sum_{n=1}^{\infty} n^2 |b_n||B_n|$$
$$\leq \sum_{n=1}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n|$$
$$\leq 1.$$

Therefore $f_k \star F_k \in MHC_0$, this proves the required result.

Theorem 4.2. If f_k and F_k of the form (5) and (6) belong to the class MHC_0, then the geometric convolution $f_k * F_k$ also belongs to the class MHC_0.

Proof. Since $f_k, F_k \in MHC_0$, it follows that

$$\sum_{n=1}^{\infty} n^2 (|a_n| + |b_n|) \leq 1$$
$$\sum_{n=1}^{\infty} n^2 (|A_n| + |B_n|) \leq 1.$$

Hence, by Cauchy-Schwartz's inequality, it is noted that,

$$\sum_{n=1}^{\infty} n^2 \left(\sqrt{|a_n A_n|} + \sqrt{|b_n B_n|} \right) \leq 1.$$

The proof is complete.

Theorem 4.3. If f_k and F_k of the form (5) and (6) belong to the class MHC_0, then the integral convolution $f_k \diamond F_k$ also belongs to the class MHC_0.

Proof. Since $f_k, F_k \in MHC_0$, it follows that $|A_n| \leq 1$ and $|B_n| \leq 1$. Then $f \diamond F \in MHC_0$, because

$$\sum_{n=1}^{\infty} n^2 \frac{|a_n A_n|}{n} + \sum_{n=1}^{\infty} n^2 \frac{|b_n B_n|}{n}$$
$$\leq \sum_{n=1}^{\infty} n^2 \frac{|a_n|}{n} + \sum_{n=1}^{\infty} n^2 \frac{|b_n|}{n}$$
$$\leq \sum_{n=1}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| \leq 1,$$

this proves the required result.

Acknowledgement

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