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On Certain Class of Multivalent Functions Involving the Cho-Kwon-Srivastava Operator

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Abstract. In this paper a new subclass of multivalent functions with negative coefficients defined by Cho-Kwon-Srivastava operator is introduced. Coefficient estimate and inclusion relationships involving the neighborhoods of p-valently analytic functions are investigated for this class. Further subordination result and results on partial sums for this class are also found.

1. Introduction

Let S_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N = \{1, 2, 3...\}), \tag{1.1}$$

which are analytic and p-valent in the unit disk $U = \{z : |z| < 1\}$. Also denote by T_p the class of functions of the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (a_{p+k} \geq 0;\ p \in N = \{1, 2, 3...\}). \tag{1.2}$$

For functions

$$f_j(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,j}\ z^{p+k}\ (a_{p+k,j} \geq 0; j = 1, 2), \tag{1.3}$$

in the class T_p, the modified Hadamard product $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,1}\ a_{p+k,2}\ z^{p+k}. \tag{1.4}$$

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Saitoh [9] introduced a linear operator:

$$L_p(a,c) : S_p \longrightarrow S_p$$

defined by

$$L_p(a,c)f(z) = \phi_p(a,c;z) * f(z) \qquad (z \in U),$$

where

$$\phi_p(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k}, \tag{1.5}$$

and $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1; & (\text{k=0}), \\ a(a+1)(a+2)...(a+k-1), & (k \in N). \end{cases}$$

In 2004, Cho, Kwon and Srivastava [3] introduced the following linear operator $I_p^\lambda(a,c)$ analogous to $L_p(a,c)$:

$$I_p^\lambda(a,c) : S_p \longrightarrow S_p$$

defined by

$$I_p^\lambda(a,c)f(z) = \phi_p^\star(a,c;z) * f(z) \qquad (z \in U;\ a,c \in R\backslash Z_0^-;\ \lambda > -p;\ f \in A_p), \tag{1.6}$$

where $\phi_p^\star$ is the function defined in terms of the Hadamard product (or convolution) by the following condition:

$$\phi_p(a,c;z) * \phi_p^\star(a,c;z) = \frac{z^p}{(1-z)^{\lambda+p}}. \tag{1.7}$$

We can easily find from (1.5), (1.6) and (1.7) and for the function $f(z) \in T_p$ that

$$I_p^\lambda(a,c)f(z) = z^p - \sum_{k=1}^{\infty} \frac{(\lambda+p)_k (c)_k}{k!(a)_k} z^{p+k} \qquad (z \in U;\ \lambda > -p). \tag{1.8}$$

It is easily verified from (1.8) that

$$z(I_p^\lambda(a+1,c)f)'(z) = aI_p^\lambda(a,c)f(z) - (a-p)I_p^\lambda(a+1,c)f(z) \tag{1.9}$$

and

$$z(I_p^\lambda(a,c)f)'(z) = (\lambda+p)I_p^{\lambda+p}(a,c)f(z) - \lambda I_p^\lambda(a,c)f(z). \tag{1.10}$$

Also by specializing the parameter $\lambda,\ a$ and c we obtain from (1.8)

$$I_p^1(p+1,1)f(z) = f(z), \qquad I_p^1(p,1)f(z) = \frac{zf'(z)}{p},$$

and

$$I_p^n(a,a)f(z) = D^{n+p-1}f(z) \qquad (n > -p),$$

where D^{n+p-1} is the well-known Ruscheweyh derivative of order $n+p-1$.

Now making use of Cho-Kwon-Srivastava operator $I_p^\lambda(a,c)$ defined by (1.8), we introduced the following subclass $H_p(a,b,c,\lambda,\beta)$ of p-valent analytic function.

Definition 1. A function $f(z) \in T_p$ is said to be in the class $H_p(a,b,c,\lambda,\beta)$ if it satisfies the following inequality:

$$\left|\frac{1}{b}\left(\frac{z(I_p^\lambda(a,c)f(z))'}{I_p^\lambda(a,c)f(z)} - p\right)\right| < \beta, \tag{1.11}$$

$$(z \in U;\ p \in N;\ \lambda > -p;\ b \in C\backslash\{0\};\ 0 < \beta \le 1).$$

It may be noted that for suitable choice of a, b, c and λ the class $H_p(a,b,c,\lambda,\beta)$ extends several classes of analytic and p-valent functions such as

(*i*) $H_p(p+1,b,1,1,\beta) = S_p(b,\beta) = \left\{f(z) \in A_p : \left|\frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p\right)\right| < \beta\right\}$

$$(z \in U;\ p \in N;\ 0 < \beta \le 1).$$

(*ii*) $H_p(p,b,1,1,\beta) = C_p(b,\beta) = \left\{f(z) \in A_p : \left|\frac{1}{b}\left(\frac{zf''(z)}{f'(z)} + 1 - p\right)\right| < \beta\right\}$

$$(z \in U;\ p \in N;\ 0 < \beta \le 1).$$

Where the classes $S_p(b,\beta)$ and $C_p(b,\beta)$ are the well know classes of starlike and convex p-valent functions of complex order. The classes $S_p(1,\beta) = S_p^*(\beta)$ and $C_p(1,\beta) = K_p^*(\beta)$ are the classes of starlike and convex p-valent functions introduced by Owa [5] and studied by Patil and Thakare [6].

Now following the earlier investigation by Goodman [4], Ruscheweyh [8], Altintas and Owa [1], Raina and Srivastava [7], Aouf [2] and others, we define the δ-neighborhood of a function $f(z) \in T_p$ by (see, for example, [5, p. 1668])

$$N_\delta(f) = \{g : g \in T_p, g(z) = z^p - \sum_{k=1}^{\infty} b_{p+k}z^{p+k} \text{ and } \sum_{k=1}^{\infty}(k+p)|a_{p+k} - b_{p+k}| \le \delta\} \tag{1.12}$$

In particular, if

$$h(z) = z^p \qquad (p \in N),$$

we immediately have

$$N_\delta(h) = \{g : g \in T_p,\ g(z) = z^p - \sum_{k=1}^{\infty} b_{p+k}z^{p+k} \text{ and } \sum_{k=1}^{\infty}(k+p)|b_{p+k}| \le \delta\}. \tag{1.13}$$

2. Coefficient estimates

Theorem 1. *Let the function* $f(z) \in T_p$ *be defined by (1.2). Then* $f(z) \in H_p(a,b,c,\lambda,\beta)$ *if and only if*

$$\sum_{k=1}^{\infty}\{k+\beta|b|\}\frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k}\ a_{p+k} \leq \beta|b|, \tag{2.1}$$

$$(z\in U;\ p\in N;\ a,c\in R\backslash Z_0^-;\ \lambda>-p;\ b\in C\backslash\{0\};\ 0<\beta\leq 1).$$

The result is sharp.

Proof. Let the function $f(z) \in T_p$ be defined by (1.2) and belongs to $H_p(a,b,c,\lambda,\beta)$. Then in view of (1.8) and (1.11) we have

$$Re\left\{\frac{z(I_p^\lambda(a,c)f(z))'}{I_p^\lambda(a,c)f(z)}-p\right\} > -\beta|b| \qquad (z\in U), \tag{2.2}$$

or, equivalently,

$$Re\left\{\frac{-\sum\limits_{k=1}^{\infty}\frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k}ka_{p+k}z^k}{1-\sum\limits_{k=1}^{\infty}\frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k}a_{p+k}z^k}\right\} > -\beta|b| \quad (z\in U). \tag{2.3}$$

Setting $z=r$ $(0\leq r<1)$ in (2.3), we have that the expression in the denominator of the left-hand side of (2.3) is positive for $r=0$ and also for all $r(0<r<1)$. Thus by letting $r\longrightarrow 1^-$ through real values, (2.3) leads us to the desired assertion (2.1) of Theorem 1.

Conversely, by applying the hypothesis (2.1) and letting $|z|=1$, we find from (1.11) that

$$\left|\frac{z(I_p^\lambda(a,c)f(z))'}{I_p^\lambda(a,c)f(z)}-p\right| = \left|\frac{\sum\limits_{k=1}^{\infty}\frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k}ka_{p+k}z^k}{1-\sum\limits_{k=1}^{\infty}\frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k}a_{p+k}z^k}\right|$$

$$\leq \frac{\sum\limits_{k=1}^{\infty}\frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k}ka_{p+k}}{1-\sum\limits_{k=1}^{\infty}\frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k}a_{p+k}}$$

$$\leq \frac{\beta|b|\left\{1-\sum\limits_{k=1}^{\infty}\frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k}a_{p+k}\right\}}{1-\sum\limits_{k=1}^{\infty}\frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k}a_{p+k}} = \beta|b|.$$

Hence by maximum modulus principle we have $f(z) \in H_p(a,b,c,\lambda,\beta)$, which evidently completes the proof of Theorem. □

Our first inclusion relation involving $N_\delta(h)$ is given in the following theorem.

3. Inclusion relationships involving δ-neighborhoods for the class $H_p(a,b,c,\lambda,\beta)$.

Theorem 2. *Let*

$$\delta = \frac{a(p+1)\beta|b|}{c(\lambda+p)(1+\beta|b|)} \qquad (p > |b|), \tag{3.1}$$

then

$$H_p(a,b,c,\lambda,\beta) \subset N_\delta(h). \tag{3.2}$$

Proof. Let $f(z) \in H_p(a,b,c,\lambda,\beta)$. Then, in view of Theorem 1, we have

$$\{1+\beta|b|\}\,\frac{c(\lambda+p)}{a}\sum_{k=1}^{\infty} a_{p+k} \le \sum_{k=1}^{\infty}\{k+\beta|b|\}\,\frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k}\;a_{p+k} \le \beta|b|, \tag{3.3}$$

which readily yields

$$\sum_{k=1}^{\infty} a_{p+k} \le \frac{a\beta|b|}{c(\lambda+p)(1+\beta|b|)}. \tag{3.4}$$

Making use of (2.1) again, in conjunction with (3.4), we get

$$\sum_{k=1}^{\infty}(k+p)\frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k}\;a_{p+k} + \sum_{k=1}^{\infty}(\beta|b|-p)\frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k}\;a_{p+k} \le \beta|b|,$$

or

$$\begin{aligned}\frac{c(\lambda+p)}{a}\sum_{k=1}^{\infty}(k+p)a_{p+k} &\le \beta|b| + (p-\beta|b|)\frac{c(\lambda+p)}{a}\sum_{k=1}^{\infty}a_{p+k}\\ &\le \beta|b| + \frac{\beta|b|(p-\beta|b|)}{(1+\beta|b|)} = \frac{(1+p)\beta|b|}{(1+\beta|b|)}.\end{aligned}$$

Hence

$$\sum_{k=1}^{\infty}(k+p)a_{p+k} \le \frac{a(p+1)\beta|b|}{c(\lambda+p)(1+\beta|b|)} \qquad (p > |b|), \tag{3.5}$$

which, by means of (1.13), establishes the inclusion (3.1) asserted by Theorem 2. □

Putting (i) $\lambda = c = 1,\ a = p+1$ and (ii) $\lambda = c = 1,\ a = p$ in Theorem 2, we obtain the following results.

Corollary 1. *Let*

$$\delta = \frac{(p+1)\beta|b|}{(1+\beta|b|)} \qquad (p > |b|), \tag{3.6}$$

then

$$S_p(b,\beta) \subset N_\delta(h). \tag{3.7}$$

Corollary 2. *Let*

$$\delta = \frac{p\beta|b|}{(1+\beta|b|)} \qquad (p > |b|), \tag{3.8}$$

then

$$C_p(b,\beta) \subset N_\delta(h). \tag{3.9}$$

4. δ-neighborhoods for the class $H_p^{(\alpha)}(a,b,c,\lambda,\beta)$.

In this section, we determine the neighborhood for the class $H_p^{(\alpha)}(a,b,c,\lambda,\beta)$, which define as follows. A function $f(z) \in T_p$ is said to be in the class $H_p^{\alpha}(a,b,c,\lambda,\beta)$ if there exists a functional $g(z) \in H_p(a,b,c,\lambda,\beta)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < p - \alpha \qquad (z \in U; 0 \le \alpha < p).$$

Theorem 3. *Let $g(z) \in H_p(a,b,c,\lambda,\beta)$ and*

$$\alpha = p - \frac{\delta c(\lambda+p)(1+\beta|b|)}{(p+1)[c(\lambda+p)(1+\beta|b|) - a\beta|b|]}, \tag{4.1}$$

then

$$N_\delta(g) \subset H_p^{(\alpha)}(a,b,c,\lambda,\beta). \tag{4.2}$$

Proof. Let $f(z) \in N_\delta(g)$. We find from (1.12)

$$\sum_{k=1}^{\infty}(p+k)|a_{p+k} - b_{p+k}| \le \delta, \tag{4.3}$$

which readily implies that

$$\sum_{k=1}^{\infty} |a_{p+k} - b_{p+k}| \leq \frac{\delta}{(p+1)} \qquad (p \in N). \tag{4.4}$$

Next, since $g(z) \in H_p(a, b, c, \lambda, \beta)$, we have from Theorem 1

$$\sum_{k=1}^{\infty} b_{p+k} \leq \frac{a\beta|b|}{c(\lambda+p)(1+\beta|b|)}, \tag{4.5}$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| \leq \frac{\sum_{k=1}^{\infty} |a_{p+k} - b_{p+k}|}{1 - \sum_{k=1}^{\infty} b_{p+k}} \leq \frac{\delta c(\lambda+p)(1+\beta|b|)}{(p+1)[c(\lambda+p)(1+\beta|b|) - a\beta|b|]} = (p - \alpha), \tag{4.6}$$

provided that α is given by (4.1). Thus $f(z) \in H_p(a, b, c, \lambda, \beta)$. This evidently proves Theorem 3. □

Putting (i) $\lambda = c = 1$, $a = p + 1$ and (ii) $\lambda = c = 1$, $a = p$ in Theorem 3, we obtain the following results.

Corollary 3. *Let* $g(z) \in S_p(b, \beta)$ *and*

$$\alpha = p - \frac{\delta(1+\beta|b|)}{(p+1)}, \tag{4.7}$$

then

$$N_\delta(g) \subset S_p^{(\alpha)}(b, \beta). \tag{4.8}$$

Corollary 4. *Let* $g(z) \in C_p(b, \beta)$ *and*

$$\alpha = p - \frac{\delta(1+\beta|b|)}{1+p+\beta|b|}, \tag{4.9}$$

then

$$N_\delta(g) \subset C_p^{(\alpha)}(b, \beta). \tag{4.10}$$

5. Subordination results

The function $f(z)$ is said to be subordinate to $g(z)$ in U written $f(z) \prec g(z)$, if there exist a function $w(z)$ analytic in U such that $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z))$.

Definition 2. A sequence $\{b_{p+k}\}_{k=0}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if for any regular and convex function

$$g(z) = \sum_{k=0}^{\infty} c_{p+k} z^{p+k},$$

with $c_p = 1, z \in U$,

$$\sum_{k=0}^{\infty} b_{p+k} c_{p+k} z^{p+k} \prec g(z) \quad (z \in U). \tag{5.1}$$

In 1961, wilf [10] gave following necessary and sufficient conditions for a sequence to be a subordinating factor sequens:

Lemma 1. *The sequence $\{b_{p+k}\}_{k=0}^{\infty}$ is a subordinating factor sequens if and only if*

$$Re\left\{1 + 2\sum_{k=0}^{\infty} b_{p+k} z^{p+k}\right\} > 0 \quad (z \in U). \tag{5.2}$$

Theorem 4. *Let $f(z) \in H_p(a, b, c, \lambda, \beta)$ of the form (1.2) and*

$$g(z) = \sum_{k=0}^{\infty} c_{p+k} z^{p+k}, \qquad c_p = 1$$

be regular and convex function in U, then

$$\frac{c(\lambda + p)(1 + \beta|b|)}{2[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]}(f * g) \prec g(z), \tag{5.3}$$

where

$$(z \in U;\ p \in N;\ \lambda > -p;\ b \in C\backslash\{0\};\ 0 < \beta \leq 1).$$

Moreover,

$$Re\{f(z)\} > (-1)^p\ \frac{\{a\beta|b| + c(\lambda + p)(1 + \beta|b|)\}}{c(\lambda + p)(1 + \beta|b|)}, \tag{5.4}$$

and the subordinating result (5.3) is sharp for the maximum factor

$$\frac{c(\lambda + p)(1 + \beta|b|)}{2[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]}. \tag{5.5}$$

Proof. Let $f(z) \in H_p(a,b,c,\lambda,\beta)$ of the form (1.2) and

$$g(z) = \sum_{k=0}^{\infty} c_{p+k} z^{p+k}, \qquad c_p = 1$$

be regular and convex function in U. To show subordination result (5.3), we need to show that

$$\left\{ \frac{c(\lambda+p)(1+\beta|b|)a_{p+k}}{2[c(\lambda+p)(1+\beta|b|)+a\beta|b|]} \right\}_{k=0}^{\infty}$$

is a subordinating factor with $a_p = 1$ which in view of Lemma 1 is true if

$$Re\left\{ 1 + \sum_{k=0}^{\infty} \frac{c(\lambda+p)(1+\beta|b|)a_{p+k}z^{p+k}}{[c(\lambda+p)(1+\beta|b|)+a\beta|b|]} \right\} > 0 \quad (z \in U). \tag{5.6}$$

Since

$$\{k+\beta|b|\} \frac{(\lambda+p)_k (c)_k}{(1)_k (a)_k} \geq \{1+\beta|b|\} \frac{c(\lambda+p)}{a} > 0 \qquad (k \geq 1),$$

on using Theorem 1, we have for $|z| = r < 1$,

$$\begin{aligned}
&Re\left\{ 1 + \frac{c(\lambda+p)(1+\beta|b|)}{[c(\lambda+p)(1+\beta|b|)+a\beta|b|]} \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \right\} \\
=&Re\left\{ 1 + \frac{c(\lambda+p)(1+\beta|b|)}{[c(\lambda+p)(1+\beta|b|)+a\beta|b|]} z^p \right. \\
&\left. + \frac{1}{[c(\lambda+p)(1+\beta|b|)+a\beta|b|]} \sum_{k=0}^{\infty} c(\lambda+p)(1+\beta|b|)a_{p+k}z^{p+k} \right\} \\
\geq& 1 - \frac{c(\lambda+p)(1+\beta|b|)|z^p|}{[c(\lambda+p)(1+\beta|b|)+a\beta|b|]} - \frac{\sum_{k=1}^{\infty} a\{k+\beta|b|\} \frac{(\lambda+p)_k (c)_k}{(1)_k (a)_k} \, a_{p+k}|z^{p+k}|}{[c(\lambda+p)(1+\beta|b|)+a\beta|b|]} \\
\geq& 1 - \frac{c(\lambda+p)(1+\beta|b|)r^p}{[c(\lambda+p)(1+\beta|b|)+a\beta|b|]} - \frac{a\beta|b|r^{p+1}}{[c(\lambda+p)(1+\beta|b|)+a\beta|b|]} \\
\geq& 1 - \frac{c(\lambda+p)(1+\beta|b|)}{[c(\lambda+p)(1+\beta|b|)+a\beta|b|]} - \frac{a\beta|b|}{[c(\lambda+p)(1+\beta|b|)+a\beta|b|]} = 0.
\end{aligned}$$

This evidently proves the inequality (5.6) and hence the subordination result (5.3). taking $g(z) = \sum_{k=0}^{\infty} z^{p+k}$ in the subordination result (5.3), we easily get the result (5.4), and for the function

$$f(z) = z^p - \frac{a\beta|b|}{c(\lambda+p)(1+\beta|b|)} z^{p+1} \in H_p(a,b,c,\lambda,\beta),$$

it can be verified that $\frac{c(\lambda+p)(1+\beta|b|)}{[c(\lambda+p)(1+\beta|b|)+a\beta|b|]}$ is a maximum factor for the subordination result (4.3). □

6. Partial sums

In this section, we determine inequalities involving partial sums of $f(z) \in T_p$ where the partial sums of $f(z) \in T_p$ of the form (1.2) is defined as follows:

$$f_0(z) = z^p \text{ and } f_n(z) = z^p - \sum_{k=1}^{n} a_{p+k} z^{p+k} \quad (a_{p+k} \geq 0;\ n \geq 1). \tag{6.1}$$

Theorem 5. *Let the function $f(z) \in T_p$ be defined by (1.2) belongs to $H_p(a,b,c,\lambda,\beta)$, then*

$$Re\left\{\frac{f(z)}{f_n(z)}\right\} > 1 - \frac{1}{\psi_{n+1}(p,a,b,c,\lambda,\beta)}, \tag{6.2}$$

and

$$Re\left\{\frac{f_n(z)}{f(z)}\right\} > \frac{\psi_{n+1}(p,a,b,c,\lambda,\beta)}{1+\psi_{n+1}(p,a,b,c,\lambda,\beta)}, \tag{6.3}$$

where

$$\psi_{n+1}(p,a,b,c,\lambda,\beta) = \{n+1+\beta|b|\}\,\frac{(\lambda+p)_{n+1}(c)_{n+1}}{(1)_{n+1}(a)_{n+1}\beta|b|}. \tag{6.4}$$

$$(z \in U;\ p \in N;\ a,c \in R\backslash Z_0^-;\ \lambda > -p;\ b \in C\backslash\{0\};\ 0 < \beta \leq 1).$$

Proof. Let the function $f(z) \in T_p$ be defined by (1.2) belongs to $H_p(a,b,c,\lambda,\beta)$, then from Theorem 1 and using

$$\psi_{n+1}(p,a,b,c,\lambda,\beta) > \psi_n(p,a,b,c,\lambda,\beta) > 1, \tag{6.5}$$

we get

$$\sum_{k=1}^{n} a_{p+k} + \psi_{n+1}(p,a,b,c,\lambda,\beta) \sum_{k=n+1}^{\infty} a_{p+k} < \sum_{k=1}^{\infty} \psi_k(p,a,b,c,\lambda,\beta) a_{p+k} \leq 1. \tag{6.6}$$

Set

$$g_1(z) = \psi_{n+1}(p,a,b,c,\lambda,\beta)\left\{\frac{f(z)}{f_n(z)} - \left(1 - \frac{1}{\psi_{n+1}(p,a,b,c,\lambda,\beta)}\right)\right\}, \tag{6.7}$$

which is analytic in U and $g_0(z)$. If (6.5) holds we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| = \left|\frac{\psi_{n+1}(p,a,b,c,\lambda,\beta)\sum\limits_{k=n+1}^{\infty} a_{p+k}z^k}{2+2\sum\limits_{k=1}^{n} a_{p+k}z^k + \psi_{n+1}(p,a,b,c,\lambda,\beta)\sum\limits_{k=n+1}^{\infty} a_{p+k}z^k}\right|$$

$$\leq \frac{\psi_{n+1}(p,a,b,c,\lambda,\beta)\sum\limits_{k=n+1}^{\infty} a_{p+k}}{2-2\sum\limits_{k=1}^{n} a_{p+k} - \psi_{n+1}(p,a,b,c,\lambda,\beta)\sum\limits_{k=n+1}^{\infty} a_{p+k}}$$

$$\leq 1,$$

which readily yields that $Re(g_1(z)) > 0$, and hence from (6.6) assertion (6.2) of Theorem 5 is obtained.
Similarly, if we set

$$g_2(z) = (1+\psi_{n+1}(p,a,b,c,\lambda,\beta))\left\{\frac{f_n(z)}{f(z)} - \frac{\psi_{n+1}(p,a,b,c,\lambda,\beta)}{1+\psi_{n+1}(p,a,b,c,\lambda,\beta)}\right\}, \tag{6.8}$$

$$= \left\{1 - \frac{(1+\psi_{n+1}(p,a,b,c,\lambda,\beta))\sum\limits_{k=n+1}^{\infty} a_{p+k}z^k}{1+\sum\limits_{k=1}^{\infty} a_{p+k}z^k}\right\},$$

and making use of (6.5), we find that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| = \left|\frac{(1+\psi_{n+1}(p,a,b,c,\lambda,\beta))\sum\limits_{k=n+1}^{\infty} a_{p+k}z^k}{2+2\sum\limits_{k=1}^{n} a_{p+k}z^k - (1+\psi_{n+1}(p,a,b,c,\lambda,\beta))\sum\limits_{k=n+1}^{\infty} a_{p+k}z^k}\right|$$

$$\leq \frac{(1+\psi_{n+1}(p,a,b,c,\lambda,\beta))\sum\limits_{k=n+1}^{\infty} a_{p+k}}{2-2\sum\limits_{k=1}^{n} a_{p+k} - (\psi_{n+1}(p,a,b,c,\lambda,\beta)-1)\sum\limits_{k=n+1}^{\infty} a_{p+k}}$$

$$\leq 1,$$

which proves the assertion (6.3). □

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