

# ON CERTAIN CLASSES OF ALMOST PRODUCT STRUCTURES

A. Montesinos

**1. Introduction.** A. M. Naveira [2] gave a classification of Riemannian almost product structures  $(M, g, P)$  attending to the invariances of  $\nabla P$  under the action of  $O(p) \times O(q)$ . The essential conditions defining the classes are  $F$  (foliation),  $C_1$  (Vidal's),  $C_2$  (minimal),  $C_3$  (umbilical). O. Gil-Medrano [1] gave an interpretation of  $C_i$  under the general assumption of integrability.

We first show the transversal nature of the conditions  $C_i$  when integrability is assumed. Then, we give a geometric interpretation of these conditions without integrability by expressing them in terms of Lie derivatives.

Condition  $C_2$  turns out to depend only on the volume form induced by  $g$  on the distribution  $\mathcal{H}$ . It can be rephrased in terms of the *expansion* of  $\mathcal{H}$ , which in certain sense is dual to the divergence of the complementary distribution  $\mathcal{V}$ , and becomes the *complementary form* of Vaisman [5] when  $\mathcal{V}$  is integrable.

We see that  $C_3$  can be written as  $C_1$  at each point by a conformal transformation, and give an example. If in addition  $\mathcal{V}$  is integrable, we have a conformal foliation.

If  $\mathcal{V}$  is a conformal foliation of codimension  $q \geq 3$ , S. Nishikawa and H. Sato [3] have proved that  $\text{Pont}^k(\mathcal{H}; \mathbf{R}) = 0$  in cohomology for  $k > q$ , by using Cartan connections and classifying spaces. In a forthcoming paper on the conformal curvature of a conformal foliation we shall give a differential geometric proof of that result for arbitrary  $q$ . Another proof with standard techniques, less conceptual but more direct, could be given from Proposition 5.1.

**2. General set-up.** Let  $(M, g, P)$  be a Riemannian almost-product structure, i.e.  $g$  is a Riemannian metric on  $M$  and  $P$  is an  $(1, 1)$  tensor field such that  $P^2 = 1$ ,  $g(P, P) = g$ . Let  $\mathcal{V}$  and  $\mathcal{H}$  be the *vertical* and *horizontal* distributions, corresponding to the projectors  $v = \frac{1}{2}(I + P)$ ,  $h = \frac{1}{2}(I - P)$ , and assume  $\dim \mathcal{V} = p$ ,  $\dim \mathcal{H} = q \neq 0$ . The capitals  $A, B, C, \dots; X, Y, Z, \dots; Q, S, T, \dots$  will denote vector fields that are, respectively, vertical, horizontal and unrestricted. All objects are supposed  $C^\infty$ .

Let  $\nabla$  be the Levi-Civita connection and put  $\alpha(Q, S, T) = g((\nabla_Q P)S, T)$ . Then

$$(1) \quad \alpha(Q, S, T) = \alpha(Q, T, S) = -\alpha(Q, PS, PT).$$

Let  $\{e_u\}$  ( $u: p+1, \dots, p+q$ ) denote in the sequel an orthonormal local base of horizontal vector fields. Then the 1-form  $\lambda$  is globally well defined through the local expression  $\lambda(Q) = (1/q) \sum_u \alpha(e_u, e_u, Q)$ , and it is clear from (1) that  $\lambda = \lambda v$ .

---

Received April 22, 1981. Revision received June 31, 1982.  
Michigan Math. J. 30 (1983).

We say that  $\mathfrak{F}$  is:

- i)  $C_1$ , if  $\alpha(X, X, A) = 0$  (*Vidal* [6]);
- ii)  $C_2$ , if  $\lambda = 0$  (*minimal*);
- iii)  $C_3$ , if  $\alpha(X, X, A) = g(X, X)\lambda(A)$  (*umbilical*).

Apart from *foliation*, whose interpretation is obvious, these are the essential conditions leading to the Naveira classification [2]. Now, we write them in terms of Lie derivatives.

PROPOSITION 2.1.  $\alpha(X, X, A) = (L_A g)(X, X)$ , and

$$\lambda(A) = -\frac{2}{q}(L_{e_u} \theta^u)(A),$$

where  $\theta^u = g(e_u, \cdot)$ .

*Proof.* We have:

$$\alpha(X, X, A) = g((\nabla_X P)X, A) = g(\nabla_X PX, A) - g(P\nabla_X X, A) = -2g(\nabla_X X, A).$$

Since  $g(X, A) = 0$ , we get:

$$-2g(\nabla_X X, A) = 2g(X, \nabla_X A) = 2g(X, \nabla_A X) - 2g(X, L_A X) = (L_A g)(X, X).$$

Now

$$\begin{aligned} q\lambda(A) &= \sum_u \alpha(e_u, e_u, A) = \sum_u (L_A g)(e_u, e_u) = -2 \sum_u g(L_A e_u, e_u) \\ &= 2\theta^u(L_{e_u} A) = -2(L_{e_u} \theta^u)(A). \end{aligned}$$

COROLLARY 2.2. *Conditions  $C_i$  are equivalently written*

$$C_1: (L_A g)(X, X) = 0$$

$$C_2: (L_{e_u} \theta^u)v = 0$$

$$C_3: (L_A g)(X, X) = g(X, X)\lambda(A).$$

These conditions refer more to the normal bundle  $\nu$  of  $\mathfrak{V}$  than to  $\mathfrak{F}$ . This will be clear when  $\mathfrak{V}$  is a foliation after the following result.

PROPOSITION 2.3. *Let  $(M, g, P)$  be a Riemannian almost-product structure with integrable vertical distribution  $\mathfrak{V}$ , and let  $\mathfrak{H}$  be a complementary distribution, i.e.  $\mathfrak{V} \oplus \mathfrak{H} = TM$ . If  $\mathfrak{F}$  is respectively  $C_1, C_2, C_3$ , then it is possible to choose a metric  $g'$  such that  $(M, g', \mathfrak{V} \oplus \mathfrak{H})$  is a Riemannian almost-product structure, and that  $\mathfrak{H}$  is  $C_1, C_2, C_3$ .*

*Proof.* Let  $v'$  and  $h'$  be the vertical and horizontal projectors corresponding to  $\mathfrak{V} \oplus \mathfrak{H}$ . We put  $g'(Q, S) = g(v'Q, v'S) + g(h'Q, h'S)$ . Then, if  $g'(Q, Q) = 0$ , we have  $h'Q = 0$ , and  $Q$  is vertical; hence  $v'Q = Q$ , and so  $Q = 0$ . Therefore,  $g'$  is Riemannian. Also  $g'(h'Q, v'S) = g(hh'Q, hv'S) = 0$ ; thus  $g'$  is adapted to  $\mathfrak{V} \oplus \mathfrak{H}$ . Now, if  $Z$  is  $h'$ -horizontal and basic (here we need the integrability):

$$(L_A g')(Z, Z) = L_A g'(Z, Z) = L_A g(hZ, hZ) = (L_A g)(hZ, hZ).$$

Then, in the cases  $C_1$  or  $C_3$ :

$$(L_A g')(Z, Z) = g(hZ, hZ)\lambda(A) = g'(Z, Z)\lambda(A),$$

and our claim follows. As for the case  $C_2$ , if  $\{e'_u\}$  is an orthonormal base of  $h'$ -horizontal vectors, we have  $\delta_{uv} = g'(e'_u, e'_v) = g(he'_u, he'_v)$ . Hence:

$$\sum_u (L_A g')(e'_u, e'_u) = \sum_u (L_A g)(he'_u, he'_u) = 0.$$

**3. The Vidal condition.** The condition  $C_1$  was stated by E. Vidal and E. Vidal-Costa [6] under the form  $(D_A g)(X, X) = 0$ , where  $D$  is the Vaisman connection (see also [5]). Its form as  $\alpha(X, X, A) = 0$  is due to A. M. Naveira [2].

Let us give a geometric interpretation. If  $m \in M$ , for computing  $(L_A X)_m$  it is enough to know the values of  $X$  upon the integral curve of  $A$  by  $m$ . Let  $\phi_t$  be the flow of  $A$  and  $X_m \in \mathcal{F}C_m$ . Then,  $X_t = \phi_t^* X_m$  represents the dragging of  $X_m$  along the integral curve  $\phi_t(m)$ . Thus  $(L_A X_t)_m = 0$  and  $(L_A g(X_t, X_t))_m = (L_A g)(X_m, X_m)$ . If  $C_1$  holds, this is zero. Hence,  $C_1$  says that *the transport of  $X_m$  by means of the flow of  $A$  makes the length of  $X_t$  stationary at  $m$ .*

In pictorial terms, the ribbon  $X_t$ , whose sides are the integral curves of  $A$  passing by the cue and the tip of  $X_m$ , twists but not widens at  $m$ . The Vidal condition is a generalization of the Reinhart's [4] in the sense that the former drops the integrability of  $\mathfrak{V}$ . A Reinhart space can be viewed as a foliation whose leaves maintain constant distance. Now we have no leaves, but certainly have curves in  $\mathfrak{V}$  (1-leaves in  $\mathfrak{V}$ ). In this sense, our interpretation generalizes that of the Reinhart structure.

As far as I am aware, there are no examples in the literature of a Riemannian almost-product structure with the condition  $C_1$  only. The following is one. Let  $S^3 \subset \mathbf{R}^4$  be parametrized by  $(x, y, z, w)$ , with  $x^2 + y^2 + z^2 + w^2 = 1$ , and  $S^1$  parametrized by  $\theta$ . Let  $U_1, U_2, U_3$  be the parallelization of  $S^3$  given by

$$U_1 = (-y, x, -w, z), \quad U_2 = (w, z, -y, -x), \quad U_3 = (-z, w, x, -y).$$

We take for  $S^3 \times S^1$  the Riemannian structure

$$g = s^1 \otimes s^1 + s^2 \otimes s^2 + s^3 \otimes s^3 + fd\theta \otimes d\theta - (s^1 \otimes d\theta + d\theta \otimes s^1),$$

where  $f = 4 + wx - yz$ , and  $\{s^i, d\theta\}$  is the dual of  $\{U_i, \partial/\partial\theta\}$ . We put  $\mathfrak{V} = \{U_3, U_1 + \partial/\partial\theta\}$ ,  $\mathcal{F}C = \{U_1, U_2\}$ .

We have  $[U_i, U_j] = -2U_k$ ,  $L_{U_i} s^j = -L_{U_j} s^i = -2s^k$  if  $i, j, k$  is a cyclic permutation of 1, 2, 3; the remaining Lie derivatives are zero. Hence, neither  $\mathfrak{V}$  nor  $\mathcal{F}C$  are foliations. Now, since  $U_1(f) = 0$ , we have:

$$\begin{aligned} L_{U_1} g &= 0 \\ L_{U_2} g &= U_2(f) d\theta \otimes d\theta - (s^3 \otimes d\theta + d\theta \otimes s^3) \\ L_{U_3} g &= U_3(f) d\theta \otimes d\theta + (s^2 \otimes d\theta + d\theta \otimes s^2) \\ L_{\partial/\partial\theta} g &= 0. \end{aligned}$$

Hence  $(L_{U_3} g)(X, X) = (L_{U_1 + \partial/\partial\theta} g)(X, X) = 0$ , for  $X \in \mathcal{F}C$ ; thus  $\mathcal{F}C$  is  $C_1$ . Also,  $(L_{U_2} g)(U_3, U_3) = 0$ ,  $(L_{U_2} g)(U_1 + \partial/\partial\theta, U_1 + \partial/\partial\theta) = U_2(f) \neq 0$ . Therefore  $\mathfrak{V}$  is not  $C_1$ , nor  $C_2$ , nor  $C_3$ .

**4. Minimal distributions.** Let  $\mathcal{F}$  be a  $q$ -dimensional distribution on  $M$  and  $\omega$  a volume form on  $\mathcal{F}$ , that is a  $q$ -form such that  $\omega(X_1, \dots, X_q) \neq 0$  if  $\{X_1, \dots, X_q\} = \mathcal{F}_m$  for arbitrary  $m$ . Let  $\mathcal{V}_m = \{Q \in M_m \mid \omega(Q, \cdot) = 0\}$ . Then,  $m \rightarrow \mathcal{V}_m$  defines a  $p$ -dimensional distribution on  $M$  such that  $\mathcal{V} \oplus \mathcal{F} = TM$ . In other words, the pair  $(\mathcal{F}, \omega)$  defines an almost-product structure  $P$  on  $M$ .

Let  $\{X_u\}$  ( $u: 1, \dots, q$ ) be a set of vector fields on  $U \subset M$  generating  $\mathcal{F}$  on  $U$ , and such that  $\omega(X_1, \dots, X_q) = 1$  on  $U$ . Then, we define the *expansion* of  $\mathcal{F}$  with respect to  $\omega$ ,  $\text{Ex}_\omega$ , as the 1-form given on  $U$  by

$$\text{Ex}_\omega(Q) = (L_{vQ}\omega)(X_1, \dots, X_q).$$

It is clear that  $\text{Ex}_\omega$  is globally well defined. Let  $\{\theta^u\}$  ( $u: 1, \dots, q$ ) be the dual of  $\{X_u\}$ , i.e.  $\theta^u = -\theta^u P$ ,  $\theta^u(X_v) = \delta_v^u$ . Then  $\text{Ex}_\omega = \frac{1}{2}\theta^u L_{X_u} P$ . In fact, we have

$$\begin{aligned} (L_{vQ}\omega)(X_1, \dots, X_q) &= -\sum_u \omega(X_1, \dots, L_{vQ}X_u, \dots, X_q) \\ &= -\theta^u (L_{vQ}X_u)\omega(X_1, \dots, X_q) = \theta^u (L_{X_u}vQ) \\ &= (\theta^u L_{X_u}v)(Q) = -(L_{X_u}\theta^u)(vQ) = \frac{1}{2}(\theta^u L_{X_u}P)(Q). \end{aligned}$$

**COROLLARY 4.1.** *Let  $(M, g, P)$  be a Riemannian almost-product structure. Then  $\mathcal{F}$  is  $C_2$  if and only if  $\text{Ex}_\omega = 0$ , where  $\omega$  is the volume form induced by  $g$  on  $\mathcal{F}$ .*

Now, let  $\{A_a, X_u\}$  be an adapted frame of  $P$  on  $U \subset M$  such that

$$\omega(X_1, \dots, X_q) = 1$$

on  $U$ . Let  $\{\alpha^a, \theta^u\}$  be its dual and  $\tau = \alpha^1 \wedge \dots \wedge \alpha^p \wedge \theta^1 \wedge \dots \wedge \theta^q$ . Then:

**PROPOSITION 4.2.**  $(A_1 \wedge \dots \wedge A_p)(\text{Ex}_\omega, \cdot) = v(\text{div}_\tau(A_1 \wedge \dots \wedge A_p))$ .

*Proof.* We have  $L_{A_a}\tau = (\alpha^b([A_b, A_a]) + \theta^u([X_u, A_a]))\tau$ , whence

$$\begin{aligned} \text{div}_\tau(A_1 \wedge \dots \wedge A_p) &= \sum_a (-1)^{a+1} \theta^u([X_u, A_a]) A_1 \wedge \dots \wedge \hat{A}_a \wedge \dots \wedge A_p \\ &\quad - \sum_{a < b} (-1)^{a+b} \theta^u([A_a, A_b]) X_u \wedge A_1 \wedge \dots \wedge \hat{A}_a \wedge \dots \wedge \hat{A}_b \wedge \dots \wedge A_p \end{aligned}$$

Therefore

$$v(\text{div}_\tau(A_1 \wedge \dots \wedge A_p)) = \sum_a (-1)^{a+1} \theta^u([X_u, A_a]) A_1 \wedge \dots \wedge \hat{A}_a \wedge \dots \wedge A_p.$$

Now

$$(A_1 \wedge \dots \wedge A_p)(\text{Ex}_\omega, \cdot) = \sum_a (-1)^{a+1} \text{Ex}_\omega(A_a) A_1 \wedge \dots \wedge \hat{A}_a \wedge \dots \wedge A_p,$$

and  $\text{Ex}_\omega(A_a) = \theta^u([X_u, A_a])$ , whence our claim follows.  $\square$

In this sense,  $\text{Ex}_\omega$  is dual to the divergence of  $\mathfrak{V}$ .

The geometric meaning of  $\text{Ex}_\omega$  is clear. Let  $m \in M$  and  $X_u \in \mathfrak{C}_m$  be such that  $\omega_m(X_1, \dots, X_q) = 1$ ; thus we have at  $m$  a horizontal parallelepiped of unit volume. Take a vertical field  $A$ , that is a field transversal to  $\mathfrak{C}$  with respect to  $\omega$ . Drag the parallelepiped along the flow of  $A$ , and compute at  $m$  the rate of growth of its volume; the result is  $\text{Ex}_\omega(A)_m$ . Thus  $\mathfrak{C}$  is minimal, in the sense of stationary volume along vertical directions, if  $\text{Ex}_\omega = 0$ .

REMARK.  $\omega$  is a volume form on the normal bundle  $\nu$  of  $\mathfrak{V}$ ; if  $\mathfrak{V}$  happens to be a foliation, one can do all this after replacing  $\mathfrak{C}$  by  $\nu$ , cf. 2.3. Then,  $\text{Ex}_\omega$  becomes the *complementary form* of Vaisman [5].

**5. Conformal foliations.** Let  $\mathfrak{C}$  be  $C_3$ . Then  $(L_A g)(X, X) = g(X, X)\lambda(A)$ . If  $m \in M$ , there is some function  $f$  on  $M$  such that  $2(df)_m = -\lambda_m$ . Therefore  $(L_A e^{2f}g)(X, X)_m = 0$ . In other words,  $C_1$  can be realized at  $m$  by a conformal change of  $g$ . Hence, the condition  $C_3$  is a conformal invariant (cf. [1]). Thus, the geometric interpretation of  $C_3$  reduces to that of  $C_1$ .

An interesting case arises when  $\mathfrak{V}$  is a foliation.

PROPOSITION 5.1. *Let  $\mathfrak{C}$  be  $C_3$  and  $\mathfrak{V}$  a foliation. Then, for each  $m \in M$  there is some open neighborhood  $U$  of  $m$  on which the given Riemannian almost-product structure is conformally Reinhart.*

*Proof.* Let  $\{dx^u\}$  be a coordinate base of horizontal 1-forms on  $U$  and  $\{X_u\}$  its dual base of horizontal vector fields; let  $\omega$  be the volume form on  $\mathfrak{C}$  and  $2qf = \ln \omega(X_1, \dots, X_q)^2$ . We have

$$\lambda(A) = -\frac{2}{q}(L_{e_u}\theta^u)(A),$$

when  $\{e_u\}$  is orthonormal; if  $e_u = B_u^v X_v$  and  $\theta^u = \underline{B}_v^u dx^v$ , where the matrix  $\underline{B}_v^u$  is the inverse of  $B_u^v$ , then we obtain by substitution:

$$q\lambda(A) = -2(\underline{B}_w^u dB_u^w)(A) = -A(\ln(\det B_u^w)^2) = -A(2qf).$$

Hence  $(L_A e^{2f}g)(X, X) = 0$  on  $U$ . □

Then, we have proved that  $\mathfrak{V}$  is a conformal foliation (cf. [3], [5]) if and only if  $\mathfrak{C}$  is  $C_3$ . Not every conformal foliation admits a global conformal transformation making it a Reinhart structure, as it is known [3]. The following is another example; it allows to visualize clearly the global obstruction. Let  $M = S^1 \times \mathbf{R}$  be parametrized by  $(\theta, x)$ . Take  $g = d\theta \otimes d\theta + dx \otimes dx$ ,  $\mathfrak{V} = \{\partial/\partial\theta + x\partial/\partial x\}$ ,  $\mathfrak{C} = \{-x\partial/\partial\theta + \partial/\partial x\}$ . Then  $\mathfrak{C}$  is trivially  $C_3$ . However, since the leaf  $l_0$  of  $\mathfrak{V}$  at  $x = 0$  is a circle and the nearby leaves approach more and more that circle after whole turns, it is impossible to take a global metric making constant the distance from  $l_0$  to a nearby leaf. In other words, that structure is not Reinhart whatever may be  $g$ .

## REFERENCES

1. O. Gil-Medrano, *On the geometric properties of some classes of almost product structures*, preprint.
2. A. M. Naveira, *A classification of Riemannian almost product manifolds*, Rend. Mat., to appear.
3. S. Nishikawa and H. Sato, *On characteristic classes of Riemannian, conformal, and projective foliations*, J. Math. Soc. Japan 28 (1976), 223–241.
4. B. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. (2) 69 (1959), 119–132.
5. I. Vaisman, *Conformal foliations*, Kodai Math. J. 2 (1979), 26–37.
6. E. Vidal and E. Vidal-Costa, *Special connections and almost foliated metrics*, J. Differential Geometry 8 (1973), 297–304.

Departamento de Geometría y Topología  
Facultad de Matemáticas  
Burjasot (Valencia), Spain