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ON CERTAIN CLASSES OF HARMONIC P -VALENT FUNCTIONS BY APPLYING THE RUSCHEWEYH DERIVATIVES

A. Ebadian and A. Tehranchi

Abstract

In this paper we have introduced two new classes $\mathcal{HR}_p(\beta, \lambda, k, v)$, $\overline{\mathcal{HR}_p}(\beta, \lambda, k, v)$ of complex valued harmonic multivalent functions of the form $f = h + \bar{g}$, where h and g are analytic in the unit disk $\Delta = \{z : |z| < 1\}$ and $f(z)$ satisfying the condition

$$Re \left((1 - \lambda) \frac{D^v f}{z^p} + \lambda(1 - k) \frac{(D^v f)'}{(z^p)'} + \lambda k \frac{(D^v f)''}{(z^p)''} \right) > \frac{\beta}{p}.$$

A sufficient coefficient condition for this function in the class $\mathcal{HR}_p(\beta, \lambda, k, v)$ and a necessary and sufficient coefficient condition for the function f in the class $\overline{\mathcal{HR}_p}(\beta, \lambda, k, v)$ are determined. We investigate inclusion relations, distortion theorem, extreme points, convex combination and other interesting properties for these families.

1 Introduction

A continuous complex-valued function $f = u + iv$ in a simply connected complex domain Ω is said to be harmonic if both u and v are real harmonic functions in Ω . For u and v in $f = u + iv$, there exists analytic functions L and k such that $u = Re(L)$ and $v = Im(k)$. So we can write

$$f = u + iv = \frac{L + k}{2} + \frac{\bar{L} - \bar{k}}{2} = h + \bar{g}.$$

In this case the Jacobin of f is given by

$$\mathcal{J}_f(z) = |h'(z)|^2 - |g'(z)|^2. \quad (1.1)$$

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Then the mapping $z \rightarrow f(z)$ is orientation preserving in Ω if and only if $\mathcal{J}_f(z) > 0$ in Ω . See Clunie and Sheil [4] and Jahangiri [5], [6]. Without loss of generality, we let Ω be the open unit disk Δ because it is a simply connected domain. Let \mathcal{H}_p denote the family of functions $f = h + \bar{g}$ which are multivalent harmonic in Δ and of the form

$$h(z) = z^p + \sum_{n=m+p}^{\infty} a_n z^n, \quad g(z) = \sum_{n=m+p-1}^{\infty} b_n z^n, \quad |b_{m+p-1}| < 1. \quad (1.2)$$

Let $\overline{\mathcal{H}}_p$ be the subclass of \mathcal{H}_p consisting of function $f = h + \bar{g}$ such that

$$h(z) = z^p - \sum_{n=m+p}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=m+p-1}^{\infty} |b_n| z^n \quad |b_{m+p-1}| < 1. \quad (1.3)$$

For $f = h + \bar{g}$, given by (1.2) and $v > -1$ in Jahangiri [5], Jahangiri and et. al. [7] have defined the Ruschewyh Derivatives of harmonic functions $f = h + \bar{g}$ in \mathcal{H}_p by

$$D^v f(z) = D^v h(z) + D^v g(z) \quad (1.4)$$

where the Ruschewyh Derivative of p -valent function (see [8]), $Q(z) = z^p + \sum_{n=1}^{\infty} t_n z^n$ is given by

$$D^v Q(z) = Q(z) * \frac{z^p}{(1-z)^{v+1}} = z^p + \sum_{n=1}^{\infty} C^p(n, v) t_n z^n \quad (1.5)$$

where $v = \xi + p - 1$, $\xi > -p$ and

$$C^p(n, v) = \frac{\Gamma(v - p + 1 + n)}{\Gamma(v + 1)(n - p)!}, \quad v > -1 \quad (1.6)$$

and $*$ stands for the Hadamard product or convolution of two power series $Q(z) = z^p + \sum_{n=1}^{\infty} t_n z^n$ and $\psi(z) = z^p + \sum_{n=1}^{\infty} s_n z^n$ defined by

$$(Q * \psi)(z) = z^p + \sum_{n=1}^{\infty} (t_n s_n) z^n.$$

For fixed values of $v (v > -1)$, we let $\mathcal{HR}_p(\beta, \lambda, k, v)$ consist of harmonic functions $f = h + \bar{g}$ in \mathcal{H}_p so that

$$Re \left\{ (1 - \lambda) \frac{D^v f}{z^p} + \lambda(1 - k) \frac{(D^v f)'}{(z^p)'} + \lambda k \frac{(D^v f)''}{(z^p)''} \right\} > \frac{\beta}{p} \quad (1.7)$$

($\lambda \geq 0, 0 \leq k \leq 1, v > -1, 0 < \beta \leq p, p \in \mathcal{N}$)

where $z = re^{i\theta} \in \Delta$. We also let $\overline{\mathcal{HR}}_p(\beta, \lambda, k, v) = \mathcal{HR}_p(\beta, \lambda, k, v) \cap \overline{\mathcal{H}}_p$.

As λ changes from 0 to 1, the family $\mathcal{HR}_p(\beta, \lambda, k, v)$ produces a passage from the class of harmonic functions $\mathcal{HA}_p(\beta, k, v) \equiv \mathcal{HR}_p(\beta, 0, k, v)$ consisting of function f where

$$\operatorname{Re} \left\{ \frac{D^v f}{pz^p} \right\} \geq \beta \tag{1.8}$$

to the class of harmonic functions $\mathcal{HB}_p(\beta, k, v) \equiv \mathcal{HR}_p(\beta, 1, k, v)$ consisting of functions f where

$$\operatorname{Re} \left\{ (1-k) \frac{(D^v f)'}{p(z^p)'} + k \frac{(D^v f)''}{p(z^p)''} \right\} \geq \beta. \tag{1.9}$$

Note that if $v = 0$ and the co-analytic part $g \equiv 0$ and $k = p - 1 = 0$ then the class $\mathcal{HR}_1(\beta, \lambda, 0, 0) = F_\lambda(\alpha)$ as studied by Booshnurmah and Swamay [3]. Further if $v = k = p - 1 = 0$, $\mathcal{HB}_1(\beta, 0, 0) = N_H(\alpha)$ as studied by Ahuja and Jahangiri [2] and if $v = k = p - 1 = 0$, $\overline{\mathcal{HB}}_1(\beta, 0, 0) = N_{\overline{H}}(\alpha)$ as studied by Ahuja and Jahangiri [2]. If $v = k = 0$ the class $\mathcal{HR}_p(\beta, \lambda, 0, 0) = H_p k(n; \lambda, \alpha)$ studied by Ahuja and Jahangiri in [1], and also if $k = p - 1 = 0$ the class $\mathcal{HR}_1(\beta, \lambda, 0, v) = F_H(n, \lambda, \alpha)$ studied by Murugusundaramoorthy and Vijaya in [7].

In this paper we obtain the coefficient conditions for the classes $\mathcal{HR}_p(\beta, \lambda, k, v)$ and $\overline{\mathcal{HR}}_p(\beta, \lambda, k, v)$. A representation theorem and inclusion properties for the class $\overline{\mathcal{HR}}_p(\beta, \lambda, k, v)$ are established.

2 Coefficient Bounds

In the first theorem we give the sufficient condition for $f(z)$ to be in the class $\mathcal{HR}_p(\beta, \lambda, k, v)$.

Theorem 1 : Let $f = h + \bar{g}$ be such that h and g are given by the form (1.2), if

$$\begin{aligned} & \sum_{n=m+p}^{\infty} C^p(n, v) |p^2 - \lambda(n-p)(kn-p)| |a_n| \\ & + \sum_{n=m+p-1}^{\infty} C^p(n, v) |p^2 + \lambda(n+p)(kn-p)| |b_n| < p(p-\beta) \end{aligned} \tag{2.1}$$

where $\lambda \geq 0, v > -1, 0 \leq \beta < 1, 0 \leq k \leq 1, p \in \mathcal{N}, |b_{m+p-1}| < 1$. Then $f \in \mathcal{HR}_p(\beta, \lambda, k, v)$.

Proof : Suppose

$$A(z) = (1-\lambda) \frac{D^v f}{z^p} + \lambda(1-k) \frac{(D^v f)'}{(z^p)'} + \lambda k \frac{(D^v f)''}{(z^p)''}.$$

It suffices to show that $|p - \beta + pA(z)| \geq |p + \beta - pA(z)|$. Substituting for h and g

in $A(z)$, we obtain

$$\begin{aligned} A(z) &= 1 + \sum_{n=p+m}^{\infty} C^p(n, v) a_n \left(1 - \left(\frac{1}{p}n - 1\right)\left(\frac{1}{p}kn - 1\right)\lambda\right) \frac{z^n}{z^p} \\ &\quad + \sum_{n=p+m-1}^{\infty} C^p(n, v) b_n \left(1 + \left(\frac{1}{p}n + 1\right)\left(\frac{1}{p}nk - 1\right)\lambda\right) \frac{\overline{z^n}}{z^p} \end{aligned}$$

and then we have

$$\begin{aligned} &|p - \beta + pA(z)| - |p + \beta - pA(z)| \\ &= |2p - \beta + \sum_{n=p+m}^{\infty} C^p(n, v) a_n (p - (n - p)) \left(\frac{1}{p}kn - 1\right)\lambda| \frac{z^n}{z^p} \\ &\quad + \sum_{n=p+m-1}^{\infty} C^p(n, v) b_n \frac{\overline{z^n}}{z^p} (p + (n + p)) \left(\frac{1}{p}nk - 1\right)\lambda| \\ &\quad - |\beta - \sum_{n=p+m}^{\infty} C^p(n, v) a_n (p - (n - p)) \left(\frac{1}{p}kn - 1\right)\lambda| \frac{z^n}{z^p} \\ &\quad - \sum_{n=p+m-1}^{\infty} C^p(n, v) b_n \frac{\overline{z^n}}{z^p} (p + (n + p)) \left(\frac{1}{p}nk - 1\right)\lambda| \\ &\geq 2p - \sum_{n=p+m}^{\infty} C^p(n, v) |a_n| |p - (n - p)| \left(\frac{1}{p}kn - 1\right)\lambda \left|\frac{z^n}{z^p}\right| \\ &\quad - \sum_{n=p+m-1}^{\infty} C^p(n, v) |b_n| |p + (n + p)| \left(\frac{1}{p}nk - 1\right)\lambda \left|\frac{z^n}{z^p}\right| \\ &\quad - \sum_{n=p+1}^{\infty} C^p(n, v) |a_n| |p - (n - p)| \left(\frac{1}{p}kn - 1\right)\lambda \left|\frac{z^n}{z^p}\right| \\ &\quad - \sum_{n=p+m-1}^{\infty} C^p(n, v) |b_n| |p + (n + p)| \left(\frac{1}{p}nk - 1\right)\lambda \left|\frac{z^n}{z^p}\right| \\ &\geq 2(p - \beta) - \sum_{n=p+m}^{\infty} C^p(n, v) |a_n| |p - (n - p)| \left(\frac{1}{p}kn - 1\right)\lambda \\ &\quad - \sum_{n=p+m-1}^{\infty} C^p(n, v) |b_n| |p + (n + p)| \left(\frac{1}{p}nk - 1\right)\lambda \geq 0 \end{aligned}$$

when $z = r \rightarrow 1$ and by (2.1)

The coefficient bound (2.1) given in Theorem 1 is sharp for the function

$$f(z) = z^p + \sum_{n=m+p}^{\infty} \frac{x_n \cdot z^n}{C^p(n, v) |p - \lambda(n - p)(\frac{1}{p}kn - 1)|} + \sum_{n=m+p-1}^{\infty} \frac{\bar{y}_n \cdot \bar{z}^n}{C^p(n, v) |p + \lambda(n + p)(\frac{1}{p}kn - p)|}$$

where $\sum_{n=m+p}^{\infty} |x_n| + \sum_{n=m+p-1}^{\infty} |y_n| = p - \beta$.

Our next theorem establishes that such coefficient bounds cannot be improved.

Theorem 2 : Let $f = h + \bar{g}$ be so that h and g are given by (1.2). Then $f \in \overline{\mathcal{HR}}_p(\beta, \lambda, k, v)$ if and only if

$$\sum_{n=m+p}^{\infty} C^p(n, v) |p^2 - \lambda(n - p)(kn - p)| |a_n| + \sum_{n=m+p-1}^{\infty} C^p(n, v) |p^2 + \lambda(n + p)(kn - p)| |b_n| \leq (p - \beta)p. \quad (2.2)$$

Proof : The “if” part follows from Theorem 1, upon noting that $\overline{\mathcal{HR}}_p(\beta, \lambda, k, v) \subset \mathcal{HR}_p(\beta, \lambda, k, v)$. For the “only if” part assume that $f \in \overline{\mathcal{HR}}_p(\beta, \lambda, k, v)$. Then for $z = re^{i\theta}$ in Δ we obtain

$$\begin{aligned} & Re \left\{ (1 - \lambda) \frac{D^v f}{z^p} + \lambda(1 - k) \frac{(D^v f)'}{(z^p)'} + \lambda k \frac{(D^v f)''}{(z^p)''} \right\} \\ &= Re \left\{ (1 - \lambda) \frac{D^v f + \overline{D^v g}}{z^p} + \lambda(1 - k) \frac{z(D^v h)' - z(\overline{D^v g})'}{p z^p} \right. \\ &\quad \left. + \lambda k \frac{z^2(D^v h)'' + z(D^v h)' + \overline{z^2(D^v g)''} + \overline{z(D^v g)'}}{p^2 z^p} \right\} \\ &\geq 1 - \sum_{n=m+p}^{\infty} C^p(n, v) |a_n| \left| 1 - \lambda \left(\frac{1}{p}n - 1 \right) \left(\frac{1}{p}kn - 1 \right) \right| |z|^{n-p} \\ &\quad - \sum_{n=m+p-1}^{\infty} C^p(n, v) \left| 1 + \lambda \left(\frac{1}{p}n + 1 \right) \left(\frac{1}{p}kn - 1 \right) \right| |z|^{n-p} \geq \frac{\beta}{p}. \end{aligned}$$

The above inequality must hold for all $z \in \Delta$. In particular, letting $z = r \rightarrow 1$ yields the required condition (2.2). \square

As a special case of Theorem 2, we obtain the following two corollaries.

Corollary 1 : For class (1.8) we can write $f = h + \bar{g} \in \overline{\mathcal{HA}}_p(\beta, k, v) = \mathcal{HA}_p(\beta, k, v) \cap \overline{\mathcal{H}}_p$ if and only if

$$\sum_{n=m+p}^{\infty} C^p(n, v) \frac{p}{p - \beta} |a_n| + \sum_{n=m+p-1}^{\infty} C^p(n, v) \frac{p}{p - \beta} |b_n| \leq 1$$

Corollary 2 : For class (1.9) we can write, $f = h + \bar{g} \in \overline{\mathcal{HB}}_p(\beta, k, v) \equiv \mathcal{HB}_p(\beta, k, v) \cap \overline{\mathcal{H}}_p$ if and only if

$$\begin{aligned} & \sum_{n=m+p}^{\infty} C^p(n, v) \left(\frac{|p^2 - (n-p)(kn-p)|}{p(p-\beta)} \right) |a_n| \\ & + \sum_{n=m+p-1}^{\infty} C^p(n, v) \left(\frac{|p^2 + (n+p)(kn-p)|}{p(p-\beta)} \right) |b_n| \leq 1. \end{aligned}$$

3 Extreme Points

Theorem 3 : The function $f(z) = h(z) + \bar{g}(z) \in \overline{\mathcal{HR}}_p(\beta, \lambda, k, v)$ if and only if

$$f(z) = X_p h_p(z) + \sum_{n=m+p}^{\infty} X_n h_n(z) + \sum_{n=m+p-1}^{\infty} Y_n g_n(z), \quad z \in \Delta \quad (3.1)$$

where

$$h_p(z) = z^p, h_n(z) = z^p - \frac{p(p-\beta)}{C^p(n, v)(p^2 - \lambda(n-p)(nk-p))} z^n, \quad (n = m+p+1, \dots)$$

$$g_n(z) = z^p + \frac{p(p-\beta)}{C^p(n, v)(p^2 + \lambda(n+p)(nk-p))} \bar{z}^n, \quad (n = m+p-1, m+p, \dots)$$

$X_p \geq 0$ and $Y_{m+p-1} \geq 0, X_p + \sum_{n=m+p}^{\infty} X_n + \sum_{n=m+p-1}^{\infty} Y_n = 1$ and $X_n \geq 0, Y_n \geq 0$, for $n = m+p, m+p+1, \dots$.

Proof : For functions f of the form (3.1) we have

$$\begin{aligned} f(z) &= X_p h_p(z) + \sum_{n=m+p}^{\infty} X_n h_n(z) + \sum_{n=m+p-1}^{\infty} Y_n g_n(z), \quad z \in \Delta \\ &= X_p z^p + \sum_{n=m+p}^{\infty} \left(z^p - \frac{p(p-\beta)}{C^p(n, v)|p - \lambda(n-p)(nk-p)|} \right) X_n z^n \\ &\quad + \sum_{n=m+p-1}^{\infty} \left(z^p + \frac{p(p-\beta)}{C^p(n, v)|p^2 + \lambda(n+p)(nk-p)|} \right) Y_n \bar{z}^n \\ &= z^p + \sum_{n=m+p}^{\infty} \frac{p(p-\beta)}{C^p(n, v)|p^2 - \lambda(n-p)(nk-p)|} X_n z^n \\ &\quad + \sum_{n=m+p}^{\infty} \frac{p(p-\beta)}{C^p(n, v)|p^2 + \lambda(n+p)(nk-p)|} Y_n \bar{z}^n \end{aligned}$$

So we have

$$\begin{aligned}
 & \sum_{n=m+p}^{\infty} C^p(n, v) |p^2 - \lambda(n-p)(kn-p)| |a_n| \\
 & + \sum_{n=m+p-1}^{\infty} C^p(n, v) |p^2 + \lambda(n+p)(kn-p)| |b_n| \\
 & = \sum_{n=m+p}^{\infty} C^p(n, v) |p^2 - \lambda(n-p)(kn-p)| \frac{p(p-\beta) |X_n|}{C^p(n, v) |p^2 - \lambda(n-p)(kn-p)|} \\
 & + \sum_{n=m+p-1}^{\infty} C^p(n, v) (p^2 + \lambda(n+p)(kn-p)) \frac{p(p-\beta) |Y_n|}{C^p(n, v) |p^2 + \lambda(n+p)(kn-p)|} \\
 & = p(p-\beta) \left(\sum_{n=m+p}^{\infty} |X_n| + \sum_{n=m+p-1}^{\infty} |Y_n| \right) = p(p-\beta)(1 - X_p) \leq p(p-\beta).
 \end{aligned}$$

Consequently, $f \in \overline{\mathcal{HR}_p}(\beta, \lambda, k, v)$ by (2.1).

Conversely, suppose $f \in \overline{\mathcal{HR}_p}(\beta, \lambda, k, v)$. Letting

$$X_p = 1 - \sum_{n=m+p}^{\infty} X_n - \sum_{n=m+p-1}^{\infty} Y_n$$

where

$$X_n = \frac{C^p(n, v) |p^2 - \lambda(n-p)(kn-p)|}{p(p-\beta)} |a_n|, \quad Y_n = \frac{C^p(n, v) |p^2 + \lambda(n+p)(kn-p)|}{p(p-\beta)} |b_n|.$$

We obtain the required representation, since

$$\begin{aligned}
 f(z) & = z^p - \sum_{n=m+p}^{\infty} |a_n| z^n + \sum_{n=m+p-1}^{\infty} |b_n| \bar{z}^n \\
 & = z^p - \sum_{n=m+p}^{\infty} \frac{p(p-\beta) X_n z^n}{C^p(n, v) |p^2 - \lambda(n-p)(kn-p)|} \\
 & \quad + \sum_{n=m+p-1}^{\infty} \frac{p(p-\beta) Y_n \bar{z}^n}{C^p(n, v) |p^2 + \lambda(n+p)(kn-p)|} \\
 & = z^p - \sum_{n=m+p}^{\infty} (z^p - h_n(z)) X_n - \sum_{n=m+p-1}^{\infty} (z^p - g_n(z)) Y_n \\
 & = \left(1 - \sum_{n=m+p}^{\infty} X_n - \sum_{n=m+p-1}^{\infty} Y_n \right) z^p + \sum_{n=m+p}^{\infty} h_n(z) X_n + \sum_{n=m+p-1}^{\infty} g_n(z) Y_n \\
 & = X_p h_p(z) + \sum_{n=m+p}^{\infty} X_n h_n(z) + \sum_{n=m+p-1}^{\infty} Y_n g_n(z).
 \end{aligned}$$

□

4 Inclusion Relations

The inclusion relation between the classes $\overline{\mathcal{HB}}_p(\beta, k, v)$ and $\overline{\mathcal{HA}}_p(\beta, k, v)$ for different values of λ are not so obvious. In this section we discuss the inclusion relations between the above mentioned classes.

Theorem 7 : For $n \in \{1, 2, 3, \dots\}$ and $0 \leq \beta < p$, we have

- (i) $\overline{\mathcal{HB}}_p(\beta, k, v) \subset \overline{\mathcal{HA}}_p(\beta, k, v)$
- (ii) $\overline{\mathcal{HB}}_p(\beta, k, v) \subset \overline{\mathcal{HR}}_p(\beta, \lambda, k, v)$, $0 < \lambda \leq 1$.
- (iii) $\overline{\mathcal{HR}}_p(\beta, \lambda, k, v) \subset \overline{\mathcal{HB}}_p(\beta, k, v)$, $\lambda \geq 1$.

Proof : (i) Let $f(z) \in \overline{\mathcal{HB}}_p(\beta, k, v)$. In view of corollaries 1 and 2, we have

$$\begin{aligned} & \sum_{n=m+p}^{\infty} C^p(n, v) p^2 |a_n| + \sum_{n=m+p-1}^{\infty} C^p(n, v) p^2 |b_n| \\ & \leq \sum_{n=m+p}^{\infty} C^p(n, v) |p^2 - (n-p)(kn-p)| |a_n| + \sum_{n=m+p-1}^{\infty} C^p(n, v) |p^2 + (n+p)(kn-p)| |b_n| \\ & \leq p(p-\beta). \end{aligned}$$

(ii) Let $f(z) \in \overline{\mathcal{HB}}_p(\beta, k, v)$. For $0 \leq \lambda < 1$, we can write

$$\begin{aligned} & \sum_{n=m+p}^{\infty} C^p(n, v) |\lambda(n-p)(kn-p) - p^2| |a_n| \\ & + \sum_{n=m+p-1}^{\infty} C^p(n, v) |p^2 + \lambda(n+p)(kn-p)| |b_n| \\ & \leq \sum_{n=m+p}^{\infty} C^p(n, v) |(n-p)(kn-p) - p^2| |a_n| \\ & + \sum_{n=m+p-1}^{\infty} C^p(n, v) |p^2 + (n+p)(kn-p)| |b_n| \\ & \leq p(p-\beta) \end{aligned}$$

by Corollary 2 and so (ii) follows from Theorem 2.

(iii) By Theorem 2, if $\lambda \geq 1$, then we have

$$\begin{aligned} & \sum_{n=m+p}^{\infty} C^p(n, v) |p^2 - (n-p)(kn-p)| |a_n| \\ & + \sum_{n=m+p-1}^{\infty} C^p(n, v) |p^2 + (n+p)(kn-p)| |b_n| \\ & \leq \sum_{n=m+p}^{\infty} C^p(n, v) |\lambda(n-p)(kn-p) - p^2| |a_n| \\ & + \sum_{n=m+p-1}^{\infty} C^p(n, v) |p^2 + \lambda(n-p)(kn-p)| |b_n| \leq p(p-\beta). \end{aligned}$$

Therefore, the result follows from Corollary 2.

5 Convolution and Convex Combinations

In the next theorem we examine the convolution properties of the class $\overline{\mathcal{HR}}_p(\beta, \lambda, k, v)$.

Define the convolution of two harmonic functions

$$\begin{aligned} f(z) &= z^p - \sum_{n=m+p}^{\infty} |a_n|z^n + \sum_{n=m+p-1}^{\infty} |b_n|\overline{z^n} \\ F(z) &= z^p - \sum_{n=m+p}^{\infty} |c_n|z^n + \sum_{n=m+p-1}^{\infty} |d_n|\overline{z^n} \end{aligned}$$

by

$$(f * F)z = z^p - \sum_{m+p}^{\infty} |a_n c_n|z^n + \sum_{m+p-1}^{\infty} |b_n d_n|\overline{z^n}. \quad (5.1)$$

Theorem 8 : For $0 \leq \beta < \alpha < 1$, let $f, F \in \overline{\mathcal{HR}}_p(\alpha, \lambda, k, v)$. Then

$$f * F \in \overline{\mathcal{HR}}_p(\alpha, \lambda, k, v) \subset \mathcal{HR}_p(\beta, \lambda, k, v).$$

Proof : For $f \in \overline{\mathcal{HR}}_p(\alpha, \lambda, k, v)$ and $F \in \overline{\mathcal{HR}}_p(\beta, \lambda, k, v)$, let $f * F$ be given in (5.1). Since $|c_n| < 1$ and $|d_n| < 1$ we can write

$$\begin{aligned} & \sum_{n=m+p}^{\infty} C^p(n, v)|p^2 - \lambda(n-p)(kn-p)| |a_n c_n| \\ & + \sum_{n=m+p-1}^{\infty} C^p(n, v)|p^2 + \lambda(n+p)(kn-p)| |b_n d_n| \\ & \leq \sum_{n=m+p}^{\infty} C^p(n, v)|p^2 - \lambda(n-p)(kn-p)| |a_n| \\ & + \sum_{n=m+p-1}^{\infty} C^p(n, v)|p^2 + \lambda(n+p)(kn-p)| |b_n| \end{aligned}$$

The right hand side of the above inequality is bounded by $p(p-\alpha)$ because $f \in \overline{\mathcal{HR}}_p(\alpha, \lambda, k, v)$. Therefore $f * F \in \overline{\mathcal{HR}}_p(\alpha, \lambda, k, v) \subset \mathcal{HR}_p(\beta, \lambda, k, v)$.

Finally, we determine the convex combination properties of the members of $\overline{\mathcal{HR}}_p(\beta, \lambda, k, v)$.

Theorem 9 : The class $\overline{\mathcal{HR}}_p(\beta, \lambda, k, v)$ is closed under convex combination.

Proof : For $i = 1, 2, 3, \dots$ suppose that $f_i(z) \in \overline{\mathcal{HR}}_p(\beta, \lambda, k, v)$ where $f_i(z)$ is given by

$$f_i(z) = z^p - \sum_{n=m+p}^{\infty} |a_{n,i}|z^n + \sum_{n=m+p-1}^{\infty} |b_{n,i}|\overline{z^n}.$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combinations of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z^p - \sum_{n=m+p}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n,i}| z^n \right) + \sum_{n=m+p-1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{n,i}| z^n \right).$$

Since

$$\left\{ \sum_{n=p+m}^{\infty} \frac{|p^2 - (n-p)(kn-p)| |a_n|}{p(p-\beta)} + \sum_{m=n+p-1}^{\infty} \frac{|p^2 + (n+p)(kn-p)| |b_n|}{p(p-\beta)} \right\} \times \\ \times C^p(n, v) \leq 1.$$

So we obtain

$$\sum_{n=m+p}^{\infty} C^p(n, v) \frac{|1 - p^2 - (n-p)(kn-p)|}{p(p-\beta)} \left(\sum_{i=1}^{\infty} t_i |a_{n,i}| \right) \\ + \sum_{n=m+p-1}^{\infty} C^p(n, v) \frac{|p^2 + (n+p)(kn-p)|}{p(p-\beta)} \left(\sum_{i=1}^{\infty} t_i |b_{n,i}| \right) \\ = \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=m+p}^{\infty} C^p(n, v) \frac{|p^2 + (n+p)(kn-p)|}{p(p-\beta)} |a_{n,i}| \right. \\ \left. + \sum_{n=m+p-1}^{\infty} C^p(n, v) \frac{|p^2 + (n+p)(kn-p)|}{p(p-\beta)} |b_{n,i}| \right\} \\ \leq \sum_{i=1}^{\infty} t_i = 1$$

and so $\sum_{i=1}^{\infty} t_i f_i \in \overline{\mathcal{HR}_p}(\beta, \lambda, k, v)$. □

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