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On Certain Classes of Rings of Formal Matrices

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Abstract—We consider a problem on isomorphism of rings of formal matrices of order three with values in a ring R. We study conditions of regularity and complete idempotence for rings of generalized matrices.

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Let $R_1, R_2, \dots, R_n$ be rings and M_{ij} be some (R_i, R_j)-bimodules, moreover $M_{ii} = R_i$ for all $1 \le i, j \le n$. Assume also that $\varphi_{ijk} : M_{ij} \otimes_{R_j} M_{jk} \to M_{ik}$ are (R_i, R_k)-bimodule homomorphisms such that φ_{iij} and φ_{ijj} are canonical isomorphisms for all $1 \le i, j \le n$. We introduce the notation $a \circ b = \varphi_{ijk}(a \otimes b)$ for $a \in M_{ij}$, $b \in M_{jk}$. The letter K stands for the set of all $n \times n$-matrices (m_{ij}) with elements $m_{ij} \in M_{ij}$ for all $1 \le i, j \le n$. Direct calculation shows that K is a ring with respect to the usual addition and multiplication operations if and only if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a \in M_{ik}$, $b \in M_{kl}$, $c \in M_{lj}$, $1 \le i, k, l, j \le n$. If K is a ring, we say that it is a *ring of formal matrices* of order n and denote it by $K(\{M_{ij}\} : \{\varphi_{ikj}\})$. A ring of formal matrices $K(\{M_{ij}\} : \{\varphi_{ikj}\})$ of order n such that $M_{ij} = R$ for all $1 \le i, j \le n$ is said to be a ring of formal matrices over R of order n and is denoted by $K_n(R)$ or $K_n(R : \{\varphi_{ikj}\})$.

Let $K_n(R : \{\varphi_{ikj}\})$ be a formal matrix ring over R of order n. Consider $\eta_{ijk} = \varphi_{ijk}(1 \otimes 1)$ for all $1 \le i, j \le n$. Then $a \circ b = \varphi_{ijk}(a \otimes b) = \eta_{ijk} ab$ for all $a, b \in R$. Now for any $a \in R$ $a\eta_{ijk} = \varphi_{ijk}(a \otimes 1) = \varphi_{ijk}(1 \otimes a) = \eta_{ijk} a$. Thus, η_{ijk} belongs to the center $C(R)$ of the ring R. The following also holds true:

1) $\eta_{iij} = \eta_{ijj} = 1$, $1 \le i, j \le n$,

2) $\eta_{ijk}\eta_{ikl} = \eta_{ijl}\eta_{jkl}$, $1 \le i, j, k, l \le n$,

3) $\eta_{iji} = \eta_{jij}$, $1 \le i, j \le n$,

4) $\eta_{iji} = \eta_{ijk}\eta_{jik} = \eta_{kij}\eta_{kji}$, $1 \le i, j, k \le n$.

The first item holds because φ_{iij} and φ_{ijj} are canonical isomorphisms. Associativity of operation $\circ$ yields $\eta_{ijk}\eta_{ikl} abc = \eta_{ijl}\eta_{jkl} abc$ for all $a, b, c \in R$. We put $a = b = c = 1$ and obtain the second item. The other items are direct consequences of the first two.

At the same time we may put $\varphi_{ijk}(a \otimes b) = \eta_{ijk} ab$ for any set $\{\eta_{ijk} \mid 1 \le i, j, k \le n\}$ of central elements from R meeting the first and the second conditions and any $a, b \in R$. Direct calculation shows that $K_n(R : \{\varphi_{ikj}\})$ is a ring of formal matrices over R of order n. Thus, the formal matrix ring $K_n(R : \{\varphi_{ikj}\})$ is uniquely defined by the central elements set $\{\eta_{ijk} \mid 1 \le i, j, k \le n\}$. In this case the formal matrix ring $K_n(R : \{\varphi_{ikj}\})$ will be denoted by $K_n(R : \{\eta_{ikj}\})$.

Here we study the rings of formal matrices with values in some ring R. In the first Section we give the preliminary results and general examples of the formal matrix rings with values in some ring R. The second Section is devoted to study of regularity and complete idempotency for the formal matrix ring. In the third Section we study the formal matrix ring of order 3 with values in some ring R isomorphism problem. Note that the isomorphism problem of ring of formal matrices of order 2 with values in some ring R was already solved in [1].

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