ON CERTAIN DUAL INTEGRAL EQUATIONS

by E. T. COPSON

(Received 20 May, 1960)

1. In his book on Fourier Integrals, Titchmarsh [1] gave the solution of the dual integral equations

$$\int_0^\infty \xi^{2\alpha}\psi(\xi)J_\nu(\xi\rho)\,d\xi = f(\rho) \quad (0<\rho<1), \tag{1}$$

$$\int_0^\infty \psi(\xi)J_\nu(\xi)\,d\xi = 0 \quad (\rho>1), \tag{2}$$

for the case $\alpha>0$, by some difficult analysis involving the theory of Mellin transforms. Sneddon [2] has recently shown that, in the cases $\nu = 0$, $\alpha = \pm\frac{1}{2}$, the problem can be reduced to an Abel integral equation by making the substitution

$$\psi(\xi) = \xi\int_0^1 \phi(t)\cos(\xi t)\,dt \quad (\alpha = -\tfrac{1}{2})$$

or

$$\psi(\xi) = \int_0^1 \chi(t)\sin(\xi t)\,dt, \quad \chi(0) = 0 \quad (\alpha = \tfrac{1}{2}).$$

It is the purpose of this note to show that the general case can be dealt with just as simply by putting

$$\psi(\xi) = \xi^{1-\alpha}\int_0^1 \phi(t)\,J_{\nu+\alpha}(\xi t)\,dt. \tag{3}$$

The analysis is formal: no attempt is made to supply details of rigour.

2. We need the following two lemmas.

LEMMA A. *If* $\lambda > \mu > -1$,

$$\int_0^\infty J_\lambda(at)J_\mu(bt)t^{1+\mu-\lambda}\,dt = \begin{cases} 0 & (0<a<b), \\ \dfrac{b^\mu(a^2-b^2)^{\lambda-\mu-1}}{2^{\lambda-\mu-1}a^\lambda\Gamma(\lambda-\mu)} & (0<b<a). \end{cases}$$

This result is well known [3].

LEMMA B. *Let* $f(x), f'(x)$ *be continuous in* $0 \leqq x \leqq a$. *Let* $0 < \kappa < 1$. *Then the solution of*

$$\int_0^x g(t)(x^2-t^2)^{-\kappa}\,dt = f(x) \quad (0<x<a)$$

is

$$g(x) = \frac{2\sin\pi\kappa}{\pi}\frac{d}{dx}\int_0^x tf(t)(x-t)^{\kappa-1}dt.$$

This is a simple transformation of the solution of Abel's integral equation [4].

https://doi.org/10.1017/S2040618500034249 Published online by Cambridge University Press

3. In order to determine the function $\phi(t)$ in equation (3), we have to consider separately the cases when α is positive and when α is negative. The case when α is zero obviously does not arise, since then equations (1) and (2) reduce to a single integral equation in which the right-hand side is zero when $\rho > 1$.

If $\alpha > 0$, inversion of the order of integration gives

$$\int_0^\infty \psi(\xi)J_\nu(\xi\rho)\,d\xi = \int_0^1 \phi(t)\int_0^\infty \xi^{1-\alpha}J_{\nu+\alpha}(\xi t)J_\nu(\xi\rho)\,d\xi\,dt = 0,$$

when $\rho > 1$, by Lemma A, provided that $\nu > -1$. Thus (2) is satisfied.

To deal with equation (1), we modify (3) by integration by parts. For we have

$$\begin{aligned}\psi(\xi) &= \xi^{1-\alpha}\int_0^1 \phi(t)J_{\nu+\alpha}(\xi t)\,dt \\ &= -\xi^{-\alpha}\int_0^1 \phi(t)t^{\nu+\alpha-1}\frac{d}{dt}\{t^{1-\nu-\alpha}J_{\nu+\alpha-1}(\xi t)\}\,dt \\ &= -\xi^{-\alpha}\Big[\phi(t)J_{\nu+\alpha-1}(\xi t)\Big]_0^1 + \xi^{-\alpha}\int_0^1 \Phi(t)J_{\nu+\alpha-1}(\xi t)\,dt,\end{aligned}$$

where

$$\Phi(t) = t^{1-\nu-\alpha}\frac{d}{dt}\{\phi(t)t^{\nu+\alpha-1}\}.$$

Hence if

$$\lim_{t\to+0} t^{\nu+\alpha-1}\phi(t) = 0, \tag{5}$$

we obtain

$$\psi(\xi) = \xi^{-\alpha}\int_0^1 \Phi(t)J_{\nu+\alpha-1}(\xi t)\,dt - \xi^{-\alpha}\phi(1)J_{\nu+\alpha-1}(\xi).$$

Substitution in (1) then gives

$$f(\rho) = \int_0^1 \Phi(t)\int_0^\infty \xi^\alpha J_\nu(\xi\rho)J_{\nu+\alpha-1}(\xi t)\,d\xi\,dt - \phi(1)\int_0^\infty \xi^\alpha J_\nu(\xi\rho)J_{\nu+\alpha-1}(t)\,dt,$$

where $0 < \rho < 1$. By Lemma A, this reduces to

$$f(\rho) = \frac{2^\alpha}{\Gamma(1-\alpha)}\rho^{-\nu}\int_0^\rho t^{\nu+\alpha-1}\,\Phi(t)(\rho^2-t^2)^{-\alpha}\,dt,$$

provided that $\nu+1 > \nu+\alpha > 0$, i.e. provided that $\nu > -\alpha$, $0 < \alpha < 1$. But this is an integral equation of the type given in Lemma B; its solution is

$$\Phi(\rho)\rho^{\nu+\alpha-1} = \frac{2^{1-\alpha}}{\Gamma(\alpha)}\frac{d}{d\rho}\int_0^\rho t^{\nu+1}f(t)(\rho^2-t^2)^{\alpha-1}\,dt,$$

it being assumed that $t^\nu f(t)$ and its first derivative are continuous in $0 \leqq t \leqq 1$. If we substitute for $\Phi(\rho)$ from (4), integrate and use condition (5), we find that

$$\rho^{\nu+\alpha-1}\phi(\rho) = \frac{2^{1-\alpha}}{\Gamma(\alpha)}\int_0^\rho t^{\nu+1}f(t)(\rho^2-t^2)^{\alpha-1}\,dt, \tag{6}$$

https://doi.org/10.1017/S2040618500034249 Published online by Cambridge University Press

a result which may also be written as

$$\rho^{\nu+\alpha}\phi(\rho) = \frac{2^{-\alpha}}{\Gamma(1+\alpha)}\frac{d}{d\rho}\int_0^\rho t^{\nu+1}f(t)(\rho^2-t^2)^\alpha dt. \tag{7}$$

The conditions on the parameters are $0 < \alpha < 1$, $\nu > -\alpha$.

When $\alpha = -\beta$, where $\beta > 0$, we start by integrating the integral in (3) by parts. This gives

$$\begin{aligned}\psi(\xi) &= \xi^{1+\beta}\int_0^1 \phi(t)J_{\nu-\beta}(\xi t)\,dt \\ &= \xi^\beta\int_0^1 \phi(t)t^{\beta-\nu-1}\frac{d}{dt}\{t^{\nu-\beta+1}J_{\nu-\beta+1}(\xi t)\}\,dt \\ &= \xi^\beta\Big[\phi(t)J_{\nu-\beta+1}(\xi t)\Big]_0^1 - \xi^\beta\int_0^1 \Psi(t)J_{\nu-\beta+1}(\xi t)\,dt,\end{aligned}$$

where
$$\Psi(t) = t^{\nu-\beta+1}\frac{d}{dt}\{\phi(t)t^{\beta-\nu-1}\}.$$

Hence

$$\psi(\xi) = \phi(1)\xi^\beta J_{\nu-\beta+1}(\xi) - \xi^\beta\int_0^1 \Psi(t)J_{\nu-\beta+1}(\xi t)\,dt$$

provided that
$$\lim_{t\to+0} t^{\nu-\beta+1}\phi(t) = 0.$$

It then follows from Lemma A that condition (2) is satisfied when $0 < \beta < 1$, $\nu > -1$.

If we now substitute from (3) in (1) and invert the order of integration, we obtain

$$\begin{aligned}f(\rho) &= \int_0^1 \phi(t)\int_0^\infty \xi^{1-\beta}J_\nu(\xi\rho)J_{\nu-\beta}(\xi t)\,dt\,d\xi \\ &= \frac{2^{1-\beta}}{\Gamma(\beta)}\rho^{-\nu}\int_0^\rho t^{\nu-\beta}\phi(t)(\rho^2-t^2)^{\beta-1}\,dt,\end{aligned}$$

by Lemma A. Using Lemma B, we get

$$\rho^{\nu-\beta}\phi(\rho) = \frac{2^\beta}{\Gamma(1-\beta)}\frac{d}{d\rho}\int_0^\rho t^{\nu+1}f(t)(\rho^2-t^2)^{-\beta}\,dt, \tag{8}$$

which is, in fact, equation (7) with α replaced by $-\beta$: but the conditions now are $-1 < \alpha < 0$, $\nu > -1$. The limiting condition on ϕ is evidently satisfied.

4. Having obtained formulae for $\phi(\rho)$, we can deduce the desired solution of (1) and (2). The simplest form of the solution is given by using (7), viz.

$$\psi(\xi) = \frac{(2\xi)^{1-\alpha}}{2\Gamma(1+\alpha)}\int_0^1 \rho^{-\nu-\alpha}J_{\nu+\alpha}(\xi\rho)\frac{d}{d\rho}\int_0^\rho t^{\nu+1}f(t)(\rho^2-t^2)^\alpha\,dt\,d\rho, \tag{9}$$

valid when $0 < \alpha < 1$, $\nu > -\alpha$ or when $-1 < \alpha < 0$, $\nu > -1$. From this, the results given by Sneddon for $\nu = 0$, $\alpha = \pm\frac{1}{2}$ readily follow.

To get the solution valid for $0 < \alpha < 1$, $\nu > -\alpha$, in the form given by Titchmarsh, we use (6), which gives

$$\psi(\xi) = \frac{(2\xi)^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 J_{\nu+\alpha}(\xi\rho)\rho^{1-\nu-\alpha} \int_0^\rho t^{\nu+1} f(t)(\rho^2 - t^2)^{\alpha-1}\, dt\, d\rho. \tag{10}$$

Lastly, if in addition $\nu + 2\alpha + 2 > 0$, we may integrate (9) by parts to get

$$\psi(\xi) = \frac{(2\xi)^{1-\alpha}}{2\Gamma(1+\alpha)} \left\{ J_{\nu+\alpha}(\xi) \int_0^1 t^{\nu+1} f(t)(1-t^2)^\alpha\, dt \right.$$
$$\left. + \xi \int_0^1 \rho^{-\nu-\alpha} J_{\nu+\alpha+1}(\xi\rho) \int_0^\rho t^{\nu+1} f(t)(\rho^2 - t^2)^\alpha\, dt\, d\rho \right\}. \tag{11}$$

REFERENCES

1. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals* (Clarendon Press, Oxford, 1937), pp. 334–339.

2. I. N. Sneddon, The elementary solution of dual integral equations, *Proc. Glasgow Math. Assoc.*, 4 (1960), 108–110.

3. G. N. Watson, *A treatise on the theory of Bessel functions* (University Press, Cambridge, 1944), p. 401, equations (1) and (3).

4. E. T. Whittaker and G. N. Watson, *A course of modern analysis* (University Press, Cambridge, 1920), p. 229.

St. Salvator's College
University of St. Andrews

https://doi.org/10.1017/S2040618500034249 Published online by Cambridge University Press