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On certain generalized paranormed spaces

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Abstract
In the present paper we introduce and study some generalized paranormed sequence spaces defined by Musielak-Orlicz functions as well as by a sequence of modulus functions. We also study some topological properties and prove some inclusion relations between these spaces.
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1 Introduction and preliminaries

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is convex and continuous such that $M(0) = 0$, $M(x) > 0$ for $x > 0$. Let w be the space of all real or complex sequences $x = (x_k)$. Lindenstrauss and Tzafriri [1] used the idea of the Orlicz function to define the following sequence space:

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\},$$

which is called an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

It is shown in [1] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(p \ge 1)$. An Orlicz function M satisfies the Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \le K(L)M(u)$ for all values of $u \ge 0$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [2–4]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\left\{ |v|u - M_k(u) : u \ge 0 \right\}, \quad k = 1, 2, \ldots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \left\{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0 \right\},$$

$$h_{\mathcal{M}} = \{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\left\{k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1\right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\left\{\frac{1}{k}\bigl(1 + I_{\mathcal{M}}(kx)\bigr) : k > 0\right\}.$$

A Musielak-Orlicz function (M_k) is said to satisfy the Δ_2-condition if there exist constants $a, K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in \ell_+^1$ (the positive cone of ℓ^1) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

holds for all $k \in N$ and $u \in R_+$ whenever $M_k(u) \leq a$.

A modulus function is a function $f : [0,\infty) \to [0,\infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,

(2) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,

(3) f is increasing,

(4) f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subsequently, modulus function has been discussed in [2, 5–8] and references therein.

Let l_∞, c, and c_0 denote the spaces of all bounded, convergent, and null sequences $x = (x_k)$ with complex terms, respectively. The zero sequence $(0,0,\ldots)$ is denoted by θ.

The notion of difference sequence spaces was introduced by Kızmaz [9], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [10] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$, and $c_0(\Delta^n)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [8] who studied the spaces $l_\infty(\Delta_n^m)$, $c(\Delta_n^m)$, and $c_0(\Delta_n^m)$.

Let m, n be non-negative integers, then for Z a given sequence space, we have

$$Z\bigl(\Delta_m^n\bigr) = \bigl\{x = (x_k) \in w : \bigl(\Delta_m^n x_k\bigr) \in Z\bigr\}$$

for $Z = c, c_0$ and l_∞ where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}.$$

Taking $m = 1$, we get the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$, and $c_0(\Delta^n)$ studied by Et and Çolak [10]. Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ introduced and studied by Kızmaz [9]. For more details as regards sequence spaces, see [6, 11–23] and references therein.

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Let (X, q) be a space seminormed by q. In the present paper we define the following sequence spaces:

$$w_0(\mathcal{M}, \Delta^n_m, p, q, u) = \biggl\{ x = (x_k) : \frac{1}{n}\sum_{k=1}^{n}\biggl[M_k\biggl(\frac{q(u_k\Delta^n_m x_k)}{\rho}\biggr)\biggr]^{p_k} \to 0 \text{ as } n \to \infty, \text{ for some } \rho > 0 \biggr\},$$

$$w(\mathcal{M}, \Delta^n_m, p, q, u) = \biggl\{ x = (x_k) : \frac{1}{n}\sum_{k=1}^{n}\biggl[M_k\biggl(\frac{q(u_k\Delta^n_m x_k - L)}{\rho}\biggr)\biggr]^{p_k} \to 0 \text{ as } n \to \infty, \text{ for some } L \in X, \rho > 0 \biggr\},$$

and

$$w_\infty(\mathcal{M}, \Delta^n_m, p, q, u) = \biggl\{ x = (x_k) : \sup_n \frac{1}{n}\sum_{k=1}^{n}\biggl[M_k\biggl(\frac{q(u_k\Delta^n_m x_k)}{\rho}\biggr)\biggr]^{p_k} < \infty, \text{for some } \rho > 0 \biggr\}.$$

If we take $\mathcal{M}(x) = x$, we get

$$w_0(\Delta^n_m, p, q, u) = \biggl\{ x = (x_k) : \frac{1}{n}\sum_{k=1}^{n}\biggl[\biggl(\frac{q(u_k\Delta^n_m x_k)}{\rho}\biggr)\biggr]^{p_k} \to 0 \text{ as } n \to \infty, \text{for some } \rho > 0 \biggr\},$$

$$w(\Delta^n_m, p, q, u) = \biggl\{ x = (x_k) : \frac{1}{n}\sum_{k=1}^{n}\biggl[\biggl(\frac{q(u_k\Delta^n_m x_k - L)}{\rho}\biggr)\biggr]^{p_k} \to 0 \text{ as } n \to \infty, \text{for some } L \in X, \rho > 0 \biggr\},$$

and

$$w_\infty(\Delta^n_m, p, q, u) = \biggl\{ x = (x_k) : \sup_n \frac{1}{n}\sum_{k=1}^{n}\biggl[\biggl(\frac{q(u_k\Delta^n_m x_k)}{\rho}\biggr)\biggr]^{p_k} < \infty, \text{for some } \rho > 0 \biggr\}.$$

If we take $p = (p_k) = 1$, $\forall k$, we get

$$w_0(\mathcal{M}, \Delta^n_m, q, u) = \biggl\{ x = (x_k) : \frac{1}{n}\sum_{k=1}^{n}\biggl[M_k\biggl(\frac{q(u_k\Delta^n_m x_k)}{\rho}\biggr)\biggr] \to 0 \text{ as } n \to \infty, \text{for some } \rho > 0 \biggr\},$$

$$w(\mathcal{M},\Delta_m^n,q,u)=\left\{x=(x_k):\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q(u_k\Delta_m^n x_k-L)}{\rho}\right)\right]\to 0 \text{ as } n\to\infty,\right.$$
$$\left.\text{for some } L\in X,\rho>0\right\},$$

and

$$w_\infty(\mathcal{M},\Delta_m^n,q,u)=\left\{x=(x_k):\sup_n\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q(u_k\Delta_m^n x_k)}{\rho}\right)\right]<\infty,\text{for some } \rho>0\right\}.$$

If we take $u=(u_k)=1$, $\forall k$, we get

$$w_0(\mathcal{M},\Delta_m^n,p,q)=\left\{x=(x_k):\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q(\Delta_m^n x_k)}{\rho}\right)\right]^{p_k}\to 0 \text{ as } n\to\infty,\right.$$
$$\left.\text{for some } \rho>0\right\},$$
$$w(\mathcal{M},\Delta_m^n,p,q)=\left\{x=(x_k):\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q(\Delta_m^n x_k-L)}{\rho}\right)\right]^{p_k}\to 0 \text{ as } n\to\infty,\right.$$
$$\left.\text{for some } L\in X,\rho>0\right\},$$

and

$$w_\infty(\mathcal{M},\Delta_m^n,p,q)=\left\{x=(x_k):\sup_n\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q(\Delta_m^n x_k)}{\rho}\right)\right]^{p_k}<\infty,\text{for some } \rho>0\right\}.$$

The following inequality will be used throughout the paper. If $0\le p_k\le\sup p_k=K$, $D=\max(1,2^{K-1})$ then

$$|a_k+b_k|^{p_k}\le D\left\{|a_k|^{p_k}+|b_k|^{p_k}\right\}\tag{1.1}$$

for all k and $a_k,b_k\in\mathbb{C}$. Also $|a|^{p_k}\le\max(1,|a|^K)$ for all $a\in\mathbb{C}$.

The aim of this paper is to study some topological and algebraic properties of the above sequence spaces.

2 Main results

Theorem 2.1 *Suppose $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be any bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then the spaces $w_0(\mathcal{M},\Delta_m^n,p,q,u)$, $w(\mathcal{M},\Delta_m^n,p,q,u)$ and $w_\infty(\mathcal{M},\Delta_m^n,p,q,u)$ are linear spaces over the complex field $\mathbb{C}$.*

Proof Let $x=(x_k),y=(y_k)\in w_\infty(\mathcal{M},\Delta_m^n,p,q,u)$ and $\alpha,\beta\in\mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\sup_n\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q(u_k\Delta_m^n x_k)}{\rho_1}\right)\right]^{p_k}<\infty$$

and

$$\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q(u_k\Delta_m^n y_k)}{\rho_2}\right)\right]^{p_k} < \infty.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since (M_k) is non-decreasing, convex and so by using inequality (1.1), we have

$$\begin{aligned}
&\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q(\alpha u_k\Delta_m^n x_k + \beta u_k\Delta_m^n y_k)}{\rho_3}\right)\right]^{p_k} \\
&\leq \sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q(\alpha u_k\Delta_m^n x_k)}{\rho_3} + \frac{q(\beta u_k\Delta_m^n y_k)}{\rho_3}\right)\right]^{p_k} \\
&\leq \sup_n \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^{p_k}}\left[M_k\left(\frac{q(u_k\Delta_m^n x_k)}{\rho_1}\right)\right]^{p_k} + \sup_n \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^{p_k}}\left[M_k\left(\frac{q(u_k\Delta_m^n y_k)}{\rho_2}\right)\right]^{p_k} \\
&\leq D\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q(u_k\Delta_m^n x_k)}{\rho_1}\right)\right]^{p_k} + D\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q(u_k\Delta_m^n y_k)}{\rho_2}\right)\right]^{p_k} \\
&< \infty.
\end{aligned}$$

Thus, $\alpha x + \beta y \in w_\infty(\mathcal{M}, \Delta_m^n, p, q, u)$. Hence $w_\infty(\mathcal{M}, \Delta_m^n, p, q, u)$ is a linear space. Similarly, we can prove $w(\mathcal{M}, \Delta_m^n, p, q, u)$ and $w_0(\mathcal{M}, \Delta_m^n, p, q, u)$ are linear spaces over the field of complex numbers. □

Theorem 2.2 *Suppose $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then the space $w_\infty(\mathcal{M}, \Delta_m^n, p, q, u)$ is a paranormed space with the paranorm defined by*

$$g(x) = \inf\left\{\rho^{\frac{p_k}{H}} : \sup_n\left(\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho}\right)\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, \rho > 0\right\},$$

where $H = \max(1, \sup_k p_k)$.

Proof (i) Clearly, $g(x) \geq 0$ for $x = (x_k) \in w_\infty(\mathcal{M}, \Delta_m^n, p, q, u)$. Since $M_k(0) = 0$, we get $g(\theta) = 0$.

(ii) $g(-x) = g(x)$.

(iii) Let $x = (x_k), y = (y_k) \in w_\infty(\mathcal{M}, \Delta_m^n, p, q, u)$ then there exist $\rho_1, \rho_2 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho_1}\right)\right)\right]^{p_k} \leq 1$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{u_k\Delta_m^n y_k}{\rho_2}\right)\right)\right]^{p_k} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$, then by Minkowski's inequality, we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho}\right)\right)\right]^{p_k} \leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{\rho_1}\right)\right)\right]^{p_k} + \left(\frac{\rho_2}{\rho_1+\rho_2}\right)\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{u_k\Delta_m^n y_k}{\rho_2}\right)\right)\right]^{p_k}$$

and thus

$$\begin{aligned} &g(x+y)\\ &\quad = \inf\left\{(\rho_1+\rho_2)^{\frac{p_k}{H}} : \sup_n\left(\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k + u_k\Delta_m^n y_k}{\rho}\right)\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, \rho > 0\right\}\\ &\quad \leq g(x) + g(y). \end{aligned}$$

(iv) Finally we prove that scalar multiplication is continuous. Let λ be any complex number by definition

$$\begin{aligned} g(\lambda x) &= \inf\left\{(\rho)^{\frac{p_k}{H}} : \sup_n\left(\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{u_k\Delta_m^n \lambda x_k}{\rho}\right)\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, \rho > 0\right\}\\ &= \inf\left\{(|\lambda| r)^{\frac{p_k}{H}} : \sup_n\left(\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k}{r}\right)\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, \rho > 0\right\}, \end{aligned}$$

where $r = \frac{\rho}{|\lambda|}$. Hence, $w_\infty(\mathcal{M}, \Delta_m^n, p, q, u)$ is a paranormed space. □

Theorem 2.3 *If $0 < p_k \leq r_k < \infty$ for each k, then $Z(\mathcal{M}, \Delta_m^n, p, q, u) \subseteq Z(\mathcal{M}, \Delta_m^n, r, q, u)$ for $Z = w_0, w, w_\infty$.*

Proof Let $x = (x_k) \in w(\mathcal{M}, \Delta_m^n, p, q, u)$. Then there exist some $\rho > 0$ and $L \in X$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k - L}{\rho}\right)\right)\right]^{p_k} \to 0 \quad \text{as } n \to \infty.$$

This implies that

$$\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k - L}{\rho}\right)\right)\right]^{p_k} < \epsilon \quad (0 < \epsilon < 1)$$

for sufficiently large k. Hence we get

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k - L}{\rho}\right)\right)\right]^{r_k} &\leq \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{u_k\Delta_m^n x_k - L}{\rho}\right)\right)\right]^{p_k}\\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This implies that $x = (x_k) \in w(\mathcal{M}, \Delta_m^n, r, q, u)$. This completes the proof. Similarly, we can prove for the cases $Z = w_0, w_\infty$. □

Theorem 2.4 *Suppose $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ are Musielak-Orlicz functions satisfying the Δ_2-condition, then we have the following results*:

(i) *If $p = (p_k)$ is a bounded sequence of positive real numbers then $Z(\mathcal{M}', \Delta^n_m, p, q, u) \subseteq Z(\mathcal{M}'' \circ \mathcal{M}', \Delta^n_m, p, q, u)$ for $Z = w_0, w,$ and w_∞.*

(ii) *$Z(\mathcal{M}', \Delta^n_m, p, q, u) \cap Z(\mathcal{M}'', \Delta^n_m, p, q, u) \subseteq Z(\mathcal{M}' + \mathcal{M}'', \Delta^n_m, p, q, u)$ for $Z = w_0, w,$ and w_∞.*

Proof (i) If $x = (x_k) \in w_0(\mathcal{M}', \Delta^n_m, p, q, u)$, then there exists some $\rho > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\biggl[M'_k\biggl(q\biggl(\frac{u_k\Delta^n_m x_k}{\rho}\biggr)\biggr)\biggr]^{p_k} \to 0 \quad \text{as } n \to \infty.$$

Suppose

$$y_k = M'_k\biggl(q\biggl(\frac{u_k\Delta^n_m x_k}{\rho}\biggr)\biggr)$$

for all $k \in \mathbb{N}$. Choose $0 < \delta < 1$, then for $y_k \geq \delta$ we have $y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$. Now (M''_k) satisfies the Δ_2-condition so that there exists $J \geq 1$ such that

$$M''_k(y_k) < \frac{Jy_k}{2\delta}M''_k(2) + \frac{Jy_k}{2\delta}M''_k(2) = \frac{Jy_k}{\delta}M''_k(2).$$

We obtain

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\biggl[M''_k \circ M'_k\biggl(q\biggl(\frac{u_k\Delta^n_m x_k}{\rho}\biggr)\biggr)\biggr]^{p_k} &= \frac{1}{n}\sum_{k=1}^{n}\biggl[M''_k\biggl\{M'_k\biggl(q\biggl(\frac{u_k\Delta^n_m x_k}{\rho}\biggr)\biggr)\biggr\}\biggr]^{p_k} \\ &= \frac{1}{n}\sum_{k=1}^{n}\bigl[M''_k(y_k)\bigr]^{p_k} \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Similarly we can prove the other cases.

(ii) Suppose $x = (x_k) \in w_0(M'_k, \Delta^n_m, p, q, u) \cap w_0(M''_k, \Delta^n_m, p, q, u)$, then there exist $\rho_1, \rho_2 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\biggl[M'_k\biggl(q\biggl(\frac{u_k\Delta^n_m x_k}{\rho_1}\biggr)\biggr)\biggr]^{p_k} \to 0, \quad \text{as } n \to \infty.$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\biggl[M''_k\biggl(q\biggl(\frac{u_k\Delta^n_m x_k}{\rho_2}\biggr)\biggr)\biggr]^{p_k} \to 0, \quad \text{as } n \to \infty.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The remaining proof follows from the inequality

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\biggl[\bigl(M'_k + M''_k\bigr)\biggl(q\biggl(\frac{u_k\Delta^n_m x_k}{\rho}\biggr)\biggr)\biggr]^{p_k} \leq D\biggl\{&\frac{1}{n}\sum_{k=1}^{n}\biggl[M'_k\biggl(q\biggl(\frac{u_k\Delta^n_m x_k}{\rho_1}\biggr)\biggr)\biggr]^{p_k} \\ &+ \frac{1}{n}\sum_{k=1}^{n}\biggl[M''_k\biggl(q\biggl(\frac{u_k\Delta^n_m x_k}{\rho_2}\biggr)\biggr)\biggr]^{p_k}\biggr\}.\end{aligned}$$

Hence, $w_0(M'_k, \Delta^n_m, p, q, u) \cap w_0(M''_k, \Delta^n_m, p, q, u) \subseteq w_0(M'_k + M''_k, \Delta^n_m, p, q, u)$. Similarly we can prove the other cases. □

Theorem 2.5 (i) *If* $0 < \inf p_k \le p_k < 1$, *then*

$$w_\infty(\mathcal{M}, \Delta^n_m, p, q, u) \subset w_\infty(\mathcal{M}, \Delta^n_m, q, u).$$

(ii) *If* $1 \le p_k \le \sup p_k < \infty$, *then*

$$w_\infty(\mathcal{M}, \Delta^n_m, q, u) \subset w_\infty(\mathcal{M}, \Delta^n_m, p, q, u).$$

Proof (i) Let $x = (x_k) \in w_\infty(\mathcal{M}, \Delta^n_m, p, q, u)$. Since $0 < \inf p_k \le 1$, we have

$$\sup_n \left\{ \frac{1}{n} \sum_{k=1}^{n} \left[M_k\left(q\left(\frac{u_k \Delta^n_m x_k}{\rho} \right) \right) \right] \right\} \le \sup_n \left\{ \frac{1}{n} \sum_{k=1}^{n} \left[M_k\left(q\left(\frac{u_k \Delta^n_m x_k}{\rho} \right) \right) \right]^{p_k} \right\}$$

and hence $x = (x_k) \in w_\infty(\mathcal{M}, \Delta^n_m, q, u)$.

(ii) Let $p_k \ge 1$ for each k and $\sup_k p_k < \infty$. Let $x = (x_k) \in w_\infty(\mathcal{M}, \Delta^n_m, q, u)$, then for each $\epsilon > 0$ such that $0 < \epsilon < 1$, there exists a positive integer $n \in \mathbb{N}$ such that

$$\sup_n \left\{ \frac{1}{n} \sum_{k=1}^{n} \left[M_k\left(q\left(\frac{u_k \Delta^n_m x_k}{\rho} \right) \right) \right] \right\} \le \epsilon < 1.$$

This implies that

$$\sup_n \left\{ \frac{1}{n} \sum_{k=1}^{n} \left[M_k\left(q\left(\frac{u_k \Delta^n_m x_k}{\rho} \right) \right) \right]^{p_k} \right\} \le \sup_n \left\{ \frac{1}{n} \sum_{k=1}^{n} \left[M_k\left(q\left(\frac{u_k \Delta^n_m x_k}{\rho} \right) \right) \right] \right\}.$$

Thus, $x = (x_k) \in w_\infty(\mathcal{M}, \Delta^n_m, p, q, u)$ and this completes the proof. □

Theorem 2.6 *The sequence space* $w_\infty(\mathcal{M}, \Delta^n_m, p, q, u)$ *is solid.*

Proof Let $x = (x_k) \in w_\infty(\mathcal{M}, \Delta^n_m, p, q, u)$, *i.e.*

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M_k\left(q\left(\frac{u_k \Delta^n_m x_k}{\rho} \right) \right) \right]^{p_k} < \infty.$$

Let (α_k) be a sequence of scalars such that $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$. Thus we have

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M_k\left(q\left(\frac{\alpha_k u_k \Delta^n_m x_k}{\rho} \right) \right) \right]^{p_k} \le \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M_k\left(q\left(\frac{u_k \Delta^n_m x_k}{\rho} \right) \right) \right]^{p_k} < \infty.$$

This shows that $(\alpha_k x_k) \in w_\infty(\mathcal{M}, \Delta^n_m, p, q, u)$ for all sequences of scalars (α_k) with $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$, whenever $(x_k) \in w_\infty(\mathcal{M}, \Delta^n_m, p, q, u)$. Hence the space $w_\infty(\mathcal{M}, \Delta^n_m, p, q, u)$ is a solid sequence space. □

Theorem 2.7 *The sequence space* $w_\infty(\mathcal{M}, \Delta^n_m, p, q, u)$ *is monotone.*

Proof The proof of the theorem is obvious and so we omit it. □

Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Let (X, q) be a space seminormed by q. Now, we define the following sequence spaces:

$$w_0\big(F, \Delta_m^n, p, q, u\big) = \Bigg\{x = (x_k) : \frac{1}{n}\sum_{k=1}^{n}\bigg[f_k\bigg(\frac{q(u_k \Delta_m^n x_k)}{\rho}\bigg)\bigg]^{p_k} \to 0 \text{ as } n \to \infty,$$

$$\text{for some } \rho > 0\Bigg\},$$

$$w\big(F, \Delta_m^n, p, q, u\big) = \Bigg\{x = (x_k) : \frac{1}{n}\sum_{k=1}^{n}\bigg[f_k\bigg(\frac{q(u_k \Delta_m^n x_k - L)}{\rho}\bigg)\bigg]^{p_k} \to 0 \text{ as } n \to \infty,$$

$$\text{for some } \rho > 0 \text{ and } L \in X\Bigg\},$$

and

$$w_\infty\big(F, \Delta_m^n, p, q, u\big) = \Bigg\{x = (x_k) : \sup_n \frac{1}{n}\sum_{k=1}^{n}\bigg[f_k\bigg(\frac{q(u_k \Delta_m^n x_k)}{\rho}\bigg)\bigg]^{p_k} < \infty, \text{for some } \rho > 0\Bigg\}.$$

Theorem 2.8 *Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then the spaces $w_0(F, \Delta_m^n, p, q, u)$, $w(F, \Delta_m^n, p, q, u)$, and $w_\infty(F, \Delta_m^n, p, q, u)$ are linear spaces over the complex field $\mathbb{C}$.*

Proof The proof of Theorem 2.1 holds along the same lines for this theorem and so we omit it. □

Theorem 2.9 *Let $F = (f_k)$ be a sequence of modulus function, $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then $w_\infty(F, \Delta_m^n, p, q, u)$ is a paranormed space with the paranorm defined by*

$$g(x) = \inf\Bigg\{\rho^{\frac{p_k}{H}} : \sup_n\bigg(\frac{1}{n}\sum_{k=1}^{n}\bigg[f_k\bigg(q\bigg(\frac{u_k \Delta_m^n x_k}{\rho}\bigg)\bigg)\bigg]^{p_k}\bigg)^{\frac{1}{H}} \le 1, \rho > 0\Bigg\}, \tag{2.1}$$

where $H = \max(1, \sup_k p_k)$.

Proof The proof follows from Theorem 2.2 and so we omit it. □

Theorem 2.10 *Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then*

$$w_0\big(F, \Delta_m^n, p, q, u\big) \subset w\big(F, \Delta_m^n, p, q, u\big) \subset w_\infty\big(F, \Delta_m^n, p, q, u\big),$$

and the inclusions are strict.

Proof The proof is obvious. □

Theorem 2.11 *Let $F = (f_k)$ and $G = (g_k)$ be any two sequences of modulus functions. For any bounded sequences $p = (p_k)$ of positive real numbers and for any two seminorms q and r. Then*

(i) $w_Z(F, \Delta^n_m, q, u) \subset w_Z(F \circ G, \Delta^n_m, q, u)$,

(ii) $w_Z(F, \Delta^n_m, p, q, u) \cap w_Z(F, \Delta^n_m, p, r, u) \subset w_Z(F, \Delta^n_m, p, q + r, u)$,

(iii) $w_Z(F, \Delta^n_m, p, q, u) \cap w_Z(G, \Delta^n_m, p, q, u) \subset w_Z(F + G, \Delta^n_m, p, q, u)$, *where* $Z = 0, 1, \infty$.

Proof (i) We shall prove it for the relation $w_0(F, \Delta^n_m, q, u) \subset w_0(F \circ G, \Delta^n_m, q, u)$. For $\epsilon > 0$, we choose δ, $0 < \delta < 1$, such that $f_k(t) < \epsilon$ for $0 \leq t \leq \delta$ and all $k \in \mathbb{N}$. We write $y_k = g_k(\frac{q(\Delta^m_n u_k x_k)}{\rho})$ and consider

$$\sum_{k=1}^{n} \bigl[f_k(y_k)\bigr] = \sum_{1} \bigl[f_k(y_k)\bigr] + \sum_{2} \bigl[f_k(y_k)\bigr],$$

where the first summation is over $y_k \leq \delta$ and the second summation is over $y_k > \delta$. Since F is continuous, we have

$$\sum_{1} \bigl[f_k(y_k)\bigr] < n\epsilon. \tag{2.2}$$

By the definition of F, we have the following relation for $y_k > \delta$:

$$f_k(y_k) < 2 f_k(1) \frac{y_k}{\delta}.$$

Hence,

$$\frac{1}{n} \sum_{2} \bigl[f_k(y_k)\bigr] \leq 2\delta^{-1} f_k(1) \frac{1}{n} \sum_{k=1}^{n} y_k. \tag{2.3}$$

It follows from (2.2) and (2.3) that $w_0(F, \Delta^n_m, q, u) \subset w_0(F \circ G, \Delta^n_m, q, u)$. Similarly, we can prove $w(F, \Delta^n_m, q, u) \subset w(F \circ G, \Delta^n_m, q, u)$ and $w_\infty(F, \Delta^n_m, q, u) \subset w_\infty(F \circ G, \Delta^n_m, q, u)$.

The proof of (ii) and (iii) follows from (i). □

Corollary 2.12 *Let f be a modulus function. Then*

$$w_Z\bigl(\Delta^n_m, q, u\bigr) \subset w_Z\bigl(f, \Delta^n_m, q, u\bigr), \quad \textit{for } Z = 0, 1, \infty.$$

Theorem 2.13 *Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then $w_\infty(F, \Delta^n_m, p, q, u)$ is complete and seminormed by* (2.1).

Proof Suppose (x^n) is a Cauchy sequence in $w_\infty(F, \Delta^n_m, p, q, u)$, where $x^n = (x^n_k)^\infty_{k=1}$ for all $n \in \mathbb{N}$. So that $g(x^i - x^j) \to 0$ as $i, j \to \infty$. Suppose $\epsilon > 0$ is given and let s and x_0 be such that $\frac{\epsilon}{s x_0} > 0$ and $f_k(\frac{s x_0}{2}) \geq \sup_{k \geq 1}(p_k)$. Since $g(x^i - x^j) \to 0$, as $i, j \to \infty$, which means that there exists $n_0 \in \mathbb{N}$ such that

$$g\bigl(x^i - x^j\bigr) < \frac{\epsilon}{s x_0}, \quad \text{for all } i, j \geq n_0.$$

This gives $g(x_1^i - x_1^j) < \frac{\epsilon}{sx_0}$ and

$$\inf\left\{\rho^{\frac{p_k}{H}} : \sup_{k\geq 1}\left(f_k\left(\frac{q(u_k\Delta_m^n x_k^i - u_k\Delta_m^n x_k^j)}{\rho}\right)\right) \leq 1, \rho > 0\right\} < \frac{\epsilon}{sx_0}. \tag{2.4}$$

It shows that (x_1^i) is a Cauchy sequence in X. Thus, (x_1^i) is convergent in X because X is complete. Suppose $\lim_{i\to\infty} x_1^i = x_1$ then $\lim_{j\to\infty} g(x_1^i - x_1^j) < \frac{\epsilon}{sx_0}$, we get

$$g\left(x_1^i - x_1\right) < \frac{\epsilon}{sx_0}.$$

Thus, we have

$$f_k\left(\frac{q(u_k\Delta_m^n x_k^i - u_k\Delta_m^n x_k^j)}{g(x^i - x^j)}\right) \leq 1.$$

This implies that

$$f_k\left(\frac{q(u_k\Delta_m^n x_k^i - u_k\Delta_m^n x_k^j)}{g(x^i - x^j)}\right) \leq f_k\left(\frac{sx_0}{2}\right)$$

and thus

$$q\left(u_k\Delta_m^n x_k^i - u_k\Delta_m^n x_k^j\right) < \frac{sx_0}{2}\cdot\frac{\epsilon}{sx_0} < \frac{\epsilon}{2},$$

which shows that $(u_k\Delta_m^n x_k^i)$ is a Cauchy sequence in X for all $k \in \mathbb{N}$. Therefore, $(u_k\Delta_m^n x_k^i)$ converges in X. Suppose $\lim_{i\to\infty} \Delta_m^n x_k^i = y_k$ for all $k \in \mathbb{N}$. Also, we have $\lim_{i\to\infty} u_k\Delta_m^n x_2^i = y_1 - x_1$. On repeating the same procedure, we obtain $\lim_{i\to\infty} u_k\Delta_m^n x_{k+1}^i = y_k - x_k$ for all $k \in \mathbb{N}$. Therefore by continuity of f_k, we get

$$\lim_{j\to\infty}\sup_{k\geq 1} f_k\left(\frac{q(u_k\Delta_m^n x_k^i - u_k\Delta_m^n x_k^j)}{\rho}\right) \leq 1,$$

so that

$$\sup_{k\geq 1} f_k\left(\frac{q(u_k\Delta_m^n x_k^i - u_k\Delta_m^n x_k)}{\rho}\right) \leq 1.$$

Let $i \geq n_0$ and taking the infimum of each ρ, we have

$$g\left(x^i - x\right) < \epsilon.$$

So $(x^i - x) \in w_\infty(F, \Delta_m^n, p, q, u)$. Hence $x = x^i - (x^i - x) \in w_\infty(F, \Delta_m^n, p, q, u)$, since $w_\infty(F, \Delta_m^n, p, q, u)$ is a linear space. Hence, $w_\infty(F, \Delta_m^n, p, q, u)$ is a complete paranormed space. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally during the development of manuscript and the authors read and approved the final manuscript.

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