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ON CERTAIN HOMOTOPY PROPERTIES OF SOME SPACES OF LINEAR AND PIECEWISE LINEAR HOMEOMORPHISMS. II

BY

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ABSTRACT. In his study of the smoothings of p. l. manifolds, R. Thom considered the homotopy groups of a certain space L_n of p.l. homeomorphisms on an n-simplex. N. H. Kuiper showed in 1965 that the higher homotopy groups of L_n were in general nontrivial. The main result in this paper is that $\pi_0(L_2) = \pi_1(L_2) = 0$.

The proof of this result is based on a theorem of S. S. Cairns in 1944 on the deformation of rectilinear complexes in R^2 and a theorem established in Part I of this paper.

I. **Introduction.** In this part of the paper, we shall consider certain spaces L_n of piecewise linear homeomorphisms on an n-simplex. These spaces, besides being natural objects to study in geometric topology, arise naturally in various places in differential topology. For instance, in terms of the homotopy groups of these spaces, R. Thom formulated a sufficient condition for the existence of a differentiable structure on a triangulated topological manifold [9].

The spaces L_n and their homotopy groups are defined as follows: Let s_n be a fixed n-simplex in the Euclidean space R^n. For a simplicial subdivision K of s_n, we shall let $L(K)$ be the space of all homeomorphisms from s_n onto s_n which are linear on each simplex of K and are pointwise fixed on $\mathrm{Bd}(s_n)$. $L(K)$ is equipped with the compact open topology. The space L_n is then defined to be the union of the $L(K)$'s obtained from all possible subdivisions K of s_n. Their homotopy groups $\pi_k(L_n)$ are defined to be the inductive limit of $\pi_k(L(K))$'s with respect to the directed system of all subdivisions K of s_n.

There are, of course, various other ways to topologize the spaces L_n. The resulting homotopy groups will be quite different from ours. For instance, it has been established that with the compact open topology, all the spaces L_n are contractible [3]. However, with the homotopy groups defined above, to deform a loop in L_n into another loop, our definition forces the deformation to be carried out with respect to a fixed subdivision K of s_n.

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There are many open questions concerning the homotopy properties of these spaces but very few results are known. An outstanding exception is N. H. Kuiper's result in 1965. Using Milnor's 7-sphere, he showed that the homotopy groups of L_n are in general not trivial, in fact, the direct sum $\pi_0(L_6) + \pi_1(L_5) + \pi_2(L_4) + \pi_3(L_3) \neq 0$ [7]. However, nothing further about these groups has been shown. Our main result in this paper is that $\pi_0(L_2) = \pi_1(L_2) = 0$.

In the following, we shall let S be a fixed 2-simplex in the plane. A simplicial subdivision K of S is called a *proper* subdivision if K has only three vertices on Bd(S). The proof of our main result is based primarily on the following two theorems.

Theorem 1.1 (Cairns). *For any proper subdivision K of S, $\pi_0(L(K)) = 0$.*

Theorem 1.2. *For any proper subdivision K of S, $\pi_1(L(K)) = 0$.*

Theorem 1.1 is a classical result of S. S. Cairns ([1], [2] see also Proposition 2.19 of [5]). Theorem 1.2 is the main theorem of Part I of this paper [5]. To establish our main result, we need only prove the following theorem.

Theorem 1.3. *Let K be an arbitrary simplicial subdivision of S (not necessarily proper). For each compact subset A of the space $L(K)$, there exist subdivisions K_1, K_2 of S such that*

1. *K_1 is a proper subdivision of S. K_2 is a common subdivision of K and K_1. (Note that we may therefore consider $L(K_1)$ and $L(K)$ as subspaces of $L(K_2)$.)*

2. *The set A may be deformed in $L(K_2)$ into a set A' contained in $L(K_1)$. Furthermore, if the identity element e of $L(K)$ is contained in A, the deformation of A into A' described above will always keep the element e fixed.*

For if we let the compact set A be the set consisting of a single element of an arbitrary $L(K)$, Theorem 1.3 combined with Theorem 1.1 proves at once that $\pi_0(L_2) = 0$. Similarly, if we let A be the image of a loop in an arbitrary $L(K)$ based at e, Theorem 1.3 combines with Theorem 1.2 to prove at once that $\pi_1(L_2) = 0$. The remaining part of this paper will be devoted to the proof of Theorem 1.3.

II. **Outline of the proof.** We shall first make some trivial observations. Let K be a simplicial subdivision of S. For any simplicial subdivision K' of K, note that the space $L(K)$ may be considered in a natural way as a subspace of $L(K')$. The following lemma may also be proved easily.

Lemma 2.1. *Let K be a simplicial subdivision of S. Let $f\colon S \longrightarrow S$ be any map whose restriction to each simplex of K is linear and whose restriction to*

$\mathrm{Bd}(S)$ *is the identity map. Then the map* $f \in L(K)$ *if and only if* f *is injective on* $\mathrm{St}(v, K)$ *for each vertex* v *of* K.

Proof. Since the map f is injective on the star of each vertex, it must be a covering map. But the triangle S is simply connected, hence f must be a homeomorphism. See, for example, Theorem 6.5.12 of [4].

Henceforth, let K be an arbitrarily given simplicial subdivision of S. Theorem 1.3 will be proven in the following three steps.

Step I. For each $f \in L(K)$, certain numbers $\delta > 0$ are to be defined (called the *allowable distances* for f) such that

a. If δ is an allowable distance for f, each positive number $\delta' < \delta$ will also be an allowable distance for f.

b. For each $f \in L(K)$, there is a neighborhood N of f in $L(K)$ and a number $\delta > 0$ such that δ is an allowable distance for each g in N.

Notation 2.2. For a sufficiently small positive number δ, we shall let T_δ be the (solid) triangle lying inside S and concentric to S such that the distance between the corresponding sides of T_δ and S equals to δ.

We shall also let B_δ be the circular strip $\mathrm{Cl}(S - T_\delta)$ where $\mathrm{Cl}(X)$ means the closure of X with respect to S.

Step II. Let δ be a given positive number and let

$$W = \{f \in L(K) \mid \delta \text{ is an allowable distance for } f\}.$$

We shall construct a subdivision K_δ of K such that W may be deformed in $L(K_\delta)$ by a homotopy $F\colon W \times I \to L(K_\delta)$ where $F(\ ,0)\colon W \to L(K_\delta)$ is the inclusion map, and, for each $f \in F(W, 1)$, $f \mid B_\delta =$ identity map.

Moreover, if the identity element e belongs to W, the deformation F may be constructed in such a way that $F(e, t) = e$ for each t in I.

Remark 2.3. Suppose A is a compact subset of $L(K)$. By Step I and the compactness of A, there exists a positive number δ which is an allowable distance for every f in A. Then by Step II, a subdivision K_δ as described above may be constructed and the corresponding homotopy F will give rise to a deformation carrying A in $L(K_\delta)$ into a compact set A' such that, for each $f' \in A'$, $f' \mid B_\delta =$ identity map.

Step III. We shall then construct a proper subdivision K_1 out of K_δ such that if K_2 is a common subdivision of K_1 and K_δ, the set A' (considered as a subset of $L(K_2)$) is actually contained in the subspace $L(K_1)$.

Note that this is sufficient for finishing the proof of Theorem 1.3, for the deformation of A to A' was indeed carried out in the space $L(K_2)$ (since $L(K_\delta) \subset L(K_2)$), and hence, the subdivisions K_1, K_2 and the set A' do have all the properties given in the conclusion of Theorem 1.3.

III. Establishing Step I.

Definition 3.1. If J_1 and J_2 are two rectilinear cell complexes (in the sense of [8, p. 74] or [6, p. 5]) with $|J_1| = |J_2| = S$, we shall let $J_1 \cap J_2$ be the cell complex which is the common subdivision $\{\sigma \cap \tau | \ \sigma \in J_1, \ \tau \in J_2\}$ of J_1 and J_2.

If J is a (rectilinear) cell subdivision of S (i.e., J is a cell complex with $|J| = S$), a cell σ of J is called an *inner cell* (or an *inner simplex* if J happens to be a simplicial subdivision) provided that σ is not contained in $\mathrm{Bd}(S)$.

Finally, for each small positive number δ, we shall let $R(\delta)$ be the cell subdivision of S obtained by letting T_δ be a 2-cell and by cutting up the region B_δ into three more 2-cells by connecting each vertex of T_δ to the closest vertex of S by a 1-cell.

Now let a simplicial subdivision K of S be given.

Notation 3.2. We shall henceforth let α be a fixed positive number such that

1. All the inner vertices of K are contained in $\mathrm{Int}(T_\alpha)$.

2. If q is an inner 1-simplex of K with both vertices on $\mathrm{Bd}(S)$, then $q \cap \mathrm{Int}(T_\alpha) \neq \emptyset$.

Notation 3.3. In the following, we shall always let J be the cell complex $K \cap R(\alpha)$.

Now, let $f \in L(K)$ be given. We shall first define an allowable neighborhood with respect to f of each vertex v lying on $\mathrm{Bd}(S)$. The allowable distances for f will then be obtained from the allowable neighborhoods.

Consider a vertex v of K lying on $\mathrm{Bd}(S)$. Let $v_1, v_2, \cdots, v_k$ be all the vertices of J such that each simplex $\langle v, v_i \rangle$ is an inner 1-cell of J. Note that $v_1, v_2, \cdots, v_k$ must then lie on $\mathrm{Bd}(T_\alpha)$ (the set theoretic boundary, hence, the triangle enclosing T_α). We shall assume that these k vertices are enumerated consecutively in the clockwise order. We shall also let $h_1, h_2, \cdots, h_{k+1}$ be the 1-cells of J lying on $\mathrm{Bd}(T_\alpha)$ such that each h_i has some v_j as one (or both) of its vertices. The 1-cells h_i are also enumerated in the clockwise order.

Note that in the image of f, each $f(h_i)$ determines uniquely an open half plane H_i of R^2 containing v such that $f(h_i)$ lies on $\mathrm{Bd}(H_i)$.

Definition 3.4. We now define the allowable neighborhood, $A(v, f)$, of v with respect to f to be the set $A(v, f) = \bigcap_{i=1}^{k+1} H_i$.

Observe that given a map $f \in L(K)$, an *allowable neighborhood* $A(v, f)$ is defined for each vertex $v \in \mathrm{Bd}(S)$ of K. Each $A(v, f)$ is clearly a nonempty open set containing v.

To define allowable distances for f, we first observe the following fact. Let δ be any positive number less than α and less than $\mathrm{dist}\,(f(v), \mathrm{Bd}(S))$ for each inner vertex v of K. Consider a vertex v of K which lies on $\mathrm{Bd}(S)$. Let $v_1, v_2, \cdots, v_k$ be all the vertices of J lying on the triangle $\mathrm{Bd}(T_\alpha)$ such that

$\langle v, v_i \rangle$ is an inner 1-cell of J for each i. Then for each such vertex v_i, the triangle $\mathrm{Bd}(T_\delta)$ intersects the 1-cell $\langle v, v_i \rangle$ at a point v_i' and also intersects the 1-cell $\langle v, f(v_i) \rangle$ in the image $f(K)$ at a point w_i. Clearly the smaller is δ, the closer are each v_i' and w_i to v.

Definition 3.5. A number δ $(0 < \delta < \alpha)$ is an *allowable distance* for f if the following conditions are satisfied:

1. $\delta < \mathrm{dist}(f(v), \mathrm{Bd}(S))$ for each inner vertex v of J.

2. For each vertex $v \in \mathrm{Bd}(S)$ of K with corresponding vertices $v_1, v_2, \cdots, v_k$ lying on the triangle $\mathrm{Bd}(T_\alpha)$ such that each $\langle v, v_i \rangle$ is a 1-cell of J, the points $v_i' = \langle v, v_i \rangle \cap \mathrm{Bd}(T_\delta)$ and $w_i = \langle v, f(v_i) \rangle \cap \mathrm{Bd}(T_\delta)$ are all contained in the allowable neighborhood $A(v, f)$.

Remark 3.6. Observe that the allowable distances do possess the properties listed in Step I: the property a listed there is clearly satisfied. As for the property b, the argument goes as follows:

Let $f \in L(K)$ be given. We set

$$a = \inf\{\mathrm{dist}(f(v), \mathrm{Bd}(S)) \mid v \text{ an inner vertex of } J\}.$$

We then find a small open disk $D(v)$ around each vertex $v \in \mathrm{Bd}(S)$ of K such that $D(v) \subset A(v, f)$.

Now, let N be a neighborhood of f in $L(K)$ such that each $g \in N$ is so close to f that

1. $\mathrm{dist}(g(v), f(v)) < \tfrac{1}{2}a$ for each inner vertex v of J.
2. $D(v) \subset A(v, g)$ for each vertex $v \in \mathrm{Bd}(S)$ of K.

Therefore, we need only choose a $\delta > 0$ small enough such that

1. $\delta < \tfrac{1}{2}a$.
2. For each vertex $v \in \mathrm{Bd}(S)$ of K with corresponding vertices $v_1, v_2, \cdots, v_k$ lying on $\mathrm{Bd}(T_\alpha)$, the points v_i' and w_i $(i = 1, 2, \cdots, k)$ all belong to $D(v)$.

Such a δ is clearly an allowable distance for each g in N.

IV. **Establishing Step II.** Let a positive number δ be given and let

$$W = \{f \in L(K) \mid \delta \text{ is an allowable distance for } f\}.$$

We shall now construct a subdivision K_δ of K and a homotopy $F\colon W \times I \to L(K_\delta)$ which deforms W into a set containing only maps f such that $f \mid B_\delta$ = identity.

We first construct K_δ. Note that if $\delta \geq \alpha$ the set W is empty. Any subdivision and any homotopy vacuously satisfy the required conditions. Henceforth, we assume that $\delta < \alpha$. Let $J(\delta) = J \cap R(\delta)$. $J(\delta)$ is a subdivision of K into a rectilinear cell complex. Let K_δ be an arbitrary simplicial subdivision of $J(\delta)$ with no extra vertices added. This can always be done (see [6, Lemma 1.4]). Clearly, K_δ is a subdivision of K.

Remark 4.1. K_δ and $J(\delta)$ have the same vertices. To define the homotopy $F\colon W \times I \to L(K_\delta)$, one needs only indicate the position of each $F(f, t)(v)$ where $(f, t) \in W \times I$ and v is a vertex of $J(\delta)$ since for each f and t, a linear extension of the images $\{F(f, t)(v)\}$ determines uniquely the element $F(f, t)$ in $L(K_\delta)$.

Roughly speaking, the homotopy F will be defined as follows: for each $f \in W$, we shall hold each $f(v)$ fixed when $v \notin \mathrm{Bd}(T_\delta)$, but when $v \in \mathrm{Bd}(T_\delta)$, we shall pull $f(v)$ carefully back to v.

The desired homotopy F will be constructed in two stages. We first construct an $F_1\colon W \times I \to L(K_\delta)$ such that for each $f \in W$, F_1 carries $f(v)$ to a point on the triangle $\mathrm{Bd}(T_\delta)$ if v is itself a vertex lying on $\mathrm{Bd}(T_\delta)$ and F_1 leaves $f(v)$ fixed for all other vertices $v \in K_\delta$. Then, we construct an F_2 which carries each $F_1(f, 1)(v)$ back to the vertex v when v is in $\mathrm{Bd}(T_\delta)$. We shall finally show that the homotopy F obtained by successively applying F_1 and F_2 is the desired homotopy.

We now construct F_1. Consider a vertex $v \in \mathrm{Bd}(S)$ of $J(\delta)$. Let $v_1, v_2, \cdots, v_k$ be the vertices of J lying on the triangle $\mathrm{Bd}(T_\alpha)$ such that each $\langle v, v_i\rangle$ is an inner 1-cell of J. We assume that these vertices are labelled in the clockwise order.

Now, we let $v_1', v_2', \cdots, v_k'$ be the vertices of $J(\delta)$ lying on the triangle $\mathrm{Bd}(T_\delta)$ such that v_i' is the unique point $\langle v, v_i\rangle \cap \mathrm{Bd}(T_\delta)$.

Consider an arbitrary $f \in W$. Since δ is an allowable distance for f, $f(v_i) \in \mathrm{Int}(T_\delta)$ for each such vertex v_i. Therefore, the triangle $\mathrm{Bd}(T_\delta)$ cuts each segment $\langle f(v), f(v_i)\rangle$ exactly once, say at the point w_i. We now define $F_1(f, t)$ for each $(f, t) \in W \times I$ as follows:

$$F_1(f, t)(v_i') = (1 - t)f(v_i') + tw_i \text{ for each vertex } v_i' \text{ of } K_\delta \text{ lying on } \mathrm{Bd}(T_\delta),$$

$$F_1(f,t)(v) = f(v) \text{ for all other vertices } v \text{ of } K_\delta.$$

We note that, for each $(f, t) \in W \times I$, $F_1(f, t)$ is indeed an element of $L(K_\delta)$. This is because that for each $v \in \mathrm{Bd}(S)$, the corresponding $f(v_i')$'s all move along the segment $\langle v, f(v_i)\rangle$ to the point w_i. Regardless how K_δ was obtained from $J(\delta)$, as long as no extra vertices are added, $F_1(f, t)$ must be injective on the star of each vertex with respect to K_δ. Therefore, each $F_1(f, t)$ is indeed in $L(K_\delta)$ by 2.1. The homotopy F_1 is clearly continuous since we defined F_1 by moving vertices of K_δ along continuous paths.

To construct the homotopy F_2, we may now assume that each $f(v_i')$ is already deformed into the point w_i. We shall now construct a homotopy which carries each $f(v_i')$ back to the vertex v_i' for each f in W.

Let us consider *all* the vertices v_i' of $J(\delta)$ which lie on the triangle $\mathrm{Bd}(T_\delta)$

(not just those corresponding to a single vertex v in $\mathrm{Bd}(S)$). We may consider them as being ordered by some cyclic ordering. Note that the vertices $f(v_i')$'s must have the same ordering on $\mathrm{Bd}(T_\delta)$ in the sense that if we travel along the triangle $\mathrm{Bd}(T_\delta)$, say clockwise, we shall encounter the vertices $f(v_i')$'s in exactly the same order as we encounter v_i's.

We shall assume for the moment that we have the following lemma, upon which the construction of F_2 will be based. The lemma itself will be proven at the end of this section.

Lemma 4.2. *Let V be the set of the vertices v_i' of $J(\delta)$ lying on $\mathrm{Bd}(T_\delta)$. There exists a map $G: I \times V \times W \to \mathrm{Bd}(T_\delta)$ such that*

1. *For each $v_i' \in V$ and each $f \in W$, $G_t(v_i', f)$ moves continuously on $\mathrm{Bd}(T_\delta)$ from $f(v_i')$ to v_i', and as t increases, the point $G_t(v_i', f)$ moves always toward v_i' (i.e., the arc length along $\mathrm{Bd}(T_\delta)$ between $G_t(v_i', f)$ and v_i' decreases monotonically as t increases).*

2. *For each $t \in I$ and $f \in W$, the vertices $G_t(v_i', f)$'s have the same ordering on $\mathrm{Bd}(T_\delta)$ as the vertices v_i''s.*

With this map G, we simply define $F_2: W \times I \to L(K_\delta)$ as follows: For each $(f, t) \in W \times I$,

$$F_2(f, t)(v_i') = G_t(v_i', f) \quad \text{for each vertex } v_i' \text{ of } K_\delta \text{ lying on } \mathrm{Bd}(T_\delta),$$

$$F_2(f, t)(v) = f(v) \quad \text{for all other vertices } v \text{ of } K_\delta.$$

To check if F_2 is well-defined we shall again use Lemma 2.1 to see if for each $(f, t) \in W \times I$, $F_2(f, t)$ is injective on the star of v for each vertex v of K_δ. This is indeed the case as is guaranteed by the following two facts:

A. For each $v \in \mathrm{Bd}(S)$, the corresponding $F_2(f, t)(v_i')$'s always belong to the allowable neighborhood $A(v, f)$. This is because of condition 1 of Lemma 4.2 and the fact that the starting point $F_2(f, 0)(v_i') = w_i$ and the end point $F_2(f, 1)(v_i') = v_i'$ are both points in $A(v, f)$ and $A(v, f)$ is a convex set.

B. For each $f \in W$ and each fixed $t \in I$, the vertices $F_2(f, t)(v_i')$ are in the same ordering on $\mathrm{Bd}(T_\delta)$ as the vertices v_i's. This follows from condition 2 of Lemma 4.2.

Regardless how K_δ was formed from $J(\delta)$ by inserting 1-simplices, fact A ensures us that in the process of moving the vertices $F_2(f, t)(v_i')$ from w_i to v_i', the 1-cells $f(h_i)$'s do not cause any trouble. Fact B ensures us that the vertices $F_2(f, t)(v_i')$ do not run into each other in the process of moving.

To sum up, we have constructed two homotopies $F_1, F_2: W \times I \to L(K_\delta)$. We now let F be the homotopy obtained by applying first F_1 then F_2. For each $f \in W$, F then moves $f(v_i')$ back to v_i' for each vertex $v_i' \in \mathrm{Bd}(T_\delta)$ of K_δ, while all the other $f(v)$ $(v \notin \mathrm{Bd}(T_\delta))$ are kept fixed. Note that the only inner vertices of K_δ which are on the circular strip $B_\delta = \mathrm{Cl}(S - T_\delta)$ are exactly those v_i' on $\mathrm{Bd}(T_\delta)$ and for such vertices, the corresponding $f(v_i')$ are pushed back to v_i' by F. Hence, for $f \in F(W, 1)$, $f \mid B_\delta =$ identity map.

We further observe that if the identity element e of $L(K_\delta)$ happens to be in W, both F_1 and F_2 will keep e fixed. This follows immediately from the fact that for each vertex $v_i' \in \mathrm{Bd}(T_\delta)$, $e(v_i')$ already coincides with v_i', hence, both F_1 and F_2 keep e unchanged.

Therefore, K_δ and F do possess all the properties promised in Step II. We now prove Lemma 4.2 which was quoted above.

Proof of Lemma 4.2. We shall prove the lemma by carrying over the problem from $\mathrm{Bd}(T_\delta)$ to a circle. Let C be the unit circle in the plane. Assuming the origin of the plane is contained in $\mathrm{Int}(T_\delta)$, we let h: $\mathrm{Bd}(T_\delta) \longrightarrow C$ be the homeomorphism $h(x) = x/\|x\|$. Let $V' = h(V)$ and $W' = \{hfk \mid f \in W\}$ where $k = (h|\mathrm{Bd}(T_\delta))^{-1}$. Note that V' is a finite set of points on C and W' is a collection of maps: $V' \longrightarrow C$ such that for each $v \in V'$, $f(v)$ and v are not antipodal points on C. We further note that each $f \in W'$ is order preserving in the sense that the points $\{f(v) \mid v \in V'\}$ lie on the circle C in the same cyclic order as the points in V'. We shall consider W' as a topological space under the metric $d(f, g) = \mathrm{Max}\{\|f(v) - g(v)\| \mid v \in V'\}$.

We now construct a map G': $I \times V' \times W' \longrightarrow C$ as follows. For each $t \in I$,

$$G_t'(v, f) = (tv + (1-t)f(v))/\|tv + (1-t)f(v)\|.$$

For each $v \in V'$ and each $f \in W'$, $G_t(v, f)$ clearly moves continuously on C from $f(v)$ to v. Furthermore, it can be shown that for each $t \in I$ and $f \in W'$, the map $G_t'(\ , f)$: $V' \longrightarrow C$ is order preserving.

Now consider the map G: $I \times V \times W \longrightarrow \mathrm{Bd}(T_\delta)$ given by $G(t, v, f) = h^{-1}G_t'(h(v), hfk)$. G clearly satisfies the conditions given in Lemma 4.2. This settles Step II.

V. **Establishing Step III.** Given the subdivision K, we shall now construct a proper subdivision K_1 of S with the following property: if K_2 is a common subdivision of K_1 and K_δ (hence, $L(K_1)$ and $L(K_\delta)$ are subsets of $L(K_2)$) and if A_1' is a subset of $L(K_\delta)$ such that for each $f' \in A_1'$, $f' \mid B_\delta =$ identity, then the set A_1' is actually contained in $L(K_1)$.

Let $v_1v_2v_3$ be the three vertices of S. We shall label the vertices of the concentric triangle $\mathrm{Bd}(T_\delta)$ inside S by $w_1\,w_2\,w_3$ in such a way that w_i is the vertex of $\mathrm{Bd}(T_\delta)$ which is closest to v_i. We now define K_1 to be the triangulation of S which is the same as K_δ on the closed region T_δ. As for the circular strip B_δ, we chop it up into smaller triangles by running a 1-simplex from v_1 to each vertex of K_1 on the side $\langle w_1, w_2\rangle$ and a 1-simplex from v_2 to each vertex on the side $\langle w_2, w_3\rangle$, and finally, a 1-simplex from v_3 to each vertex on the side $\langle w_3, w_1\rangle$. This defines a proper subdivision of S.

For each $f \in A'$, we claim that $f \in L(K_1)$. Clearly, f is linear on each simplex σ of K_1 lying in T_δ, for σ is also a simplex of K_δ. f is also linear on each simplex σ of K_1 lying on B_δ, for f is the identity map there. Hence, $f \in L(K_1)$. Therefore, the subdivision K_1 does satisfy all the properties required in Step III.

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