

ON CERTAIN PAIRS OF AUTOMORPHISMS OF RINGS

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Abstract

In this paper we prove algebraic generalizations of some results of C. J. K. Batty and A. B. Thaheem, concerned with the identity $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ where α and β are automorphisms of a C^* -algebra. The main result asserts that if automorphisms α , β of a semiprime ring R satisfy $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ then there exist invariant ideals U_1 , U_2 and U_3 of R such that $U_i \cap U_j = 0$, $i \neq j$, $U_1 \oplus U_2 \oplus U_3$ is an essential ideal, $\alpha = \beta$ on U_1 , $\alpha = \beta^{-1}$ on U_2 , and $\alpha^2 = \beta^2 = \alpha^{-2}$ on U_3 . Furthermore, if the annihilator of any ideal in R is a direct summand (in particular, if R is a von Neumann algebra), then $U_1 \oplus U_2 \oplus U_3 = R$.

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Introduction

Over the last ten years a lot of work has been done on the operator equation

$$(*) \quad \alpha + \alpha^{-1} = \beta + \beta^{-1}$$

where α and β are $*$ -automorphisms of a von Neumann algebra. We refer to some recent papers [1, 5] for a detailed discussion on this equation and a more comprehensive bibliography.

It seems that the culminating results in the series of papers concerning the equation (*) can be found in the paper [1] of Batty, where the treatment of this problem was extended from von Neumann algebras to C^* -algebras. The

main result in [1] gives a condition which is both necessary and sufficient for solution of the equation in C^* -algebras, and which produces as corollaries the necessary conditions which we will establish purely algebraically.

Our work had been motivated by the following two results of Batty [1; Corollary 3.2, Corollary 3.5]:

THEOREM A. *Suppose that $*$ -automorphisms α and β of a C^* -algebra R satisfy $(*)$. Then there exist ideals I_1 , I_2 and I_3 of R , each invariant under α , β , α^{-1} and β^{-1} , such that $I_1 \cap I_2 \cap I_3 = 0$, $I_3 \subseteq I_1 + I_2$ and, for every x in R , $\beta(x) - \alpha(x) \in I_1$, $\beta(x) - \alpha^{-1}(x) \in I_2$, $\beta^2(x) - \alpha^2(x) \in I_3$, $\beta^2(x) - \alpha^{-2}(x) \in I_3$.*

The next theorem was also proved by Thaheem in [4, 5].

THEOREM B. *Suppose that $*$ -automorphisms α and β of a von Neumann algebra R satisfy $(*)$. Then $R = U_1 \oplus U_2 \oplus U_3$, where U_1 , U_2 and U_3 are von Neumann subalgebras of R , invariant under α and β , such that $\alpha = \beta$ on U_1 , $\alpha = \beta^{-1}$ on U_2 , and $\alpha^2 = \beta^2 = \alpha^{-2}$ on U_3 .*

In this paper we will generalize both Theorems A and B. Our methods are much more elementary than those employed by the other authors. Roughly speaking, we will show that the presence of analysis in the study of equation $(*)$ is sometimes superfluous. We will see that Theorem A remains true if R is an arbitrary semiprime ring, and that Theorem B holds if R is a semiprime ring in which the annihilator of any ideal is a direct summand. Moreover, if we no longer insist that $U_1 \oplus U_2 \oplus U_3$ is R but rather just a “large piece” of R (more precisely, an essential ideal), then Theorem B holds in any semiprime ring R .

In particular, our results imply that the assumption that α and β preserve adjoints, which is required in Theorems A and B, can be removed.

We remark that the study of equation $(*)$ is much simpler if one assumes that α and β commute. It turns out that in this case the presence of the ideal U_3 in Theorem B is not necessary (see, for example, [3, 6]). An algebraic generalization of this result is presented in our forthcoming paper [2].

Preliminaries

We recall a few definitions and easy results. Let R be a ring. Then R is said to be prime if $aRb = 0$ implies $a = 0$ or $b = 0$. A von Neumann algebra is prime if and only if it is a factor (that is, its center consists of

scalar multiples of the identity). If $aRa = 0$ implies $a = 0$, then R is called semiprime. Every C^* -algebra R is semiprime (for $0 \neq aa^*a \in aRa$ if $a \neq 0$).

REMARK A. Let R be semiprime. Suppose that $a, b \in R$ satisfy $aRb = 0$. Then we also have $(bRa)R(bRa) = 0, abRab = 0, baRba = 0$, and therefore $bRa = 0, ab = 0, ba = 0$ by the semiprimeness of R . Observe that the left and the right annihilators of an ideal U of R coincide. It will be denoted by $\text{Ann}(U)$. Note that $U \cap \text{Ann}(U) = 0$, and that $U \oplus \text{Ann}(U)$ is an essential ideal.

We will be especially concerned with semiprime rings R in which the annihilator of any ideal is a direct summand; that is, $\text{Ann}(U) \oplus \text{Ann}(\text{Ann}(U)) = R$ for any ideal U of R . Note that every von Neumann algebra has this property; namely, the annihilator of any ideal is of the form pR for some central projection p in R . More generally, the same is true for AW*-algebras.

REMARK B. Let α be an automorphism of a ring R . Suppose that the ideal I of R is invariant under α and α^{-1} , that is, α maps I onto itself. One can easily verify that in this case the two-sided annihilator $\text{Ann}(I)$ of I is also invariant under α and α^{-1} .

The results

We begin our investigation of the equation (*) by considering a somewhat more general situation where automorphisms α, β and γ satisfy $\alpha + \gamma = \beta + 1$.

LEMMA 1. Let α, β, γ be automorphisms of a ring R . If $\alpha + \gamma = \beta + 1$ then

- (1) $(\alpha - 1)(x)R((\beta + 1)(\alpha - \beta))(w)R(\alpha - \beta)(z) = 0,$
- (2) $(\alpha - 1)(x)R((\beta + 1)(\alpha - 1))(w)R(\alpha - \beta)(z) = 0$ for all $x, w, z \in R.$

PROOF. From $\alpha - \beta = 1 - \gamma$ it follows that

$$\begin{aligned}
 &(\alpha - \beta)(x)\alpha(y) + \beta(x)(\alpha - \beta)(y) \\
 &= (\alpha - \beta)(xy) = (1 - \gamma)(xy) \\
 &= (1 - \gamma)(x)y + \gamma(x)(1 - \gamma)(y) \\
 &= (\alpha - \beta)(x)y + \gamma(x)(\alpha - \beta)(y).
 \end{aligned}$$

Thus $(\alpha - \beta)(x)(\alpha - 1)(y) + (\beta - \gamma)(x)(\alpha - \beta)(y) = 0$. That is,

$$(3) \quad (\alpha - \beta)(x)(\alpha - 1)(y) + (\alpha - 1)(x)(\alpha - \beta)(y) = 0 \quad \text{for all } x, y \in R$$

since $\beta - \gamma = \alpha - 1$ by assumption. Replacing y by yz in (3) we obtain

$$(\alpha - \beta)(x)(\alpha - 1)(y)\alpha(z) + (\alpha - \beta)(x)y(\alpha - 1)(z) + (\alpha - 1)(x)(\alpha - \beta)(y)\alpha(z) + (\alpha - 1)(x)\beta(y)(\alpha - \beta)(z) = 0.$$

By (3) this relation reduces to

$$(4) \quad (\alpha - \beta)(x)y(\alpha - 1)(z) + (\alpha - 1)(x)\beta(y)(\alpha - \beta)(z) = 0 \quad \text{for all } x, y, z \in R.$$

Replacing y by $y(\alpha - \beta)(w)u$ in (4) we get

$$(\alpha - \beta)(x)y(\alpha - \beta)(w)u(\alpha - 1)(z) = -(\alpha - 1)(x)\beta(y)(\beta(\alpha - \beta))(w)\beta(u)(\alpha - \beta)(z).$$

But on the other hand, using (4) twice we obtain

$$\begin{aligned} &(\alpha - \beta)(x)y\{(\alpha - \beta)(w)u(\alpha - 1)(z)\} \\ &= -\{(\alpha - \beta)(x)y(\alpha - 1)(w)\}\beta(u)(\alpha - \beta)(z) \\ &= (\alpha - 1)(x)\beta(y)(\alpha - \beta)(w)\beta(u)(\alpha - \beta)(z). \end{aligned}$$

Comparing the last two relations we obtain (1). In a similar fashion, by substituting $y(\alpha - 1)(w)u$ for y in (4), one shows that (2) holds.

COROLLARY 1. *Suppose that automorphisms α, β, γ of a prime ring R satisfy $\alpha + \gamma = \beta + 1$. If $\alpha \neq \beta$ and $\alpha \neq 1$ then $\alpha = \beta\gamma, \gamma = \beta\alpha$, and $\beta^2 = 1$.*

PROOF. From (1) it follows immediately that $\beta(\alpha - \beta) = -(\alpha - \beta)$. By assumption, $\alpha - \beta = 1 - \gamma$, therefore this relation yields $\beta - \alpha = \beta(1 - \gamma) = \beta - \beta\gamma$ which means that $\alpha = \beta\gamma$. Similarly, by (2) we have $\beta(\alpha - 1) = -(\alpha - 1)$; since $1 - \alpha = \gamma - \beta$ it follows that $\gamma = \beta\alpha$. According to both identities, $\alpha = \beta\gamma$ and $\gamma = \beta\alpha$, we are forced to conclude that $\beta^2 = 1$.

REMARK 1. The next simple example illustrates Corollary 1 (compare with [1; Proposition 2.1]). Let R be an algebra with unit element e , and let b in R be such that $b^2 = e$. Define the inner automorphism β by $\beta(x) = bxb$. Let λ be any scalar different from 1 and -1 and define the inner automorphism γ by $\gamma(x) = (1 - \lambda^2)^{-1}(e + \lambda b)x(e - \lambda b)$. Note that $\beta\gamma + \gamma = \beta + 1$.

As a special case of Corollary 1 we obtain an extension of [1; Corollary 3.3].

COROLLARY 2. *Suppose that automorphisms α and β of a prime ring R satisfy $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. If $\alpha \neq \beta$ and $\alpha \neq \beta^{-1}$ then $\alpha^2 = \beta^2 = \alpha^{-2}$.*

PROOF. We have $\alpha\beta + \alpha^{-1}\beta = \beta^2 + 1$; now apply Corollary 1.

LEMMA 2. *If automorphisms α and β of a semiprime ring R satisfy $\alpha + \alpha^{-1} = \beta + \beta^{-1}$, then*

$$(5) \quad (\alpha - \beta^{-1})(x)R(\alpha^2 - \beta^2)(y) = 0 \quad \text{for all } x, y \in R,$$

$$(6) \quad (\alpha - \beta)(x)R(\alpha^2 - \beta^{-2})(y) = 0 \quad \text{for all } x, y \in R.$$

PROOF. We have $\alpha\beta + \alpha^{-1}\beta = \beta^2 + 1$. Therefore Lemma 1 implies that

$$(\alpha\beta - 1)(x)R((\beta^2 + 1)(\alpha\beta - \beta^2))(w)R(\alpha\beta - \beta^2)(z) = 0$$

for all $x, w, z \in R$. Since β is onto we then also have

$$(7) \quad (\alpha - \beta^{-1})(x)R((\beta^2 + 1)(\alpha - \beta))(w)R(\alpha - \beta)(z) = 0 \quad \text{for all } x, w, z \in R.$$

We have

$$\beta^2(\alpha - \beta) + (\alpha - \beta) = \beta^2(\beta^{-1} - \alpha^{-1}) + (\alpha - \beta) = \alpha - \beta^2\alpha^{-1} = (\alpha^2 - \beta^2)\alpha^{-1},$$

therefore it follows from (7) that

$$(8) \quad (\alpha - \beta^{-1})(x)R(\alpha^2 - \beta^2)(y)R(\alpha - \beta)(z) = 0 \quad \text{for all } x, y, z \in R.$$

The range of $\alpha^2 - \beta^2$ is contained in the range of $\alpha - \beta$; indeed, we have $\alpha^2 - \beta^2 = \alpha(\alpha + \alpha^{-1}) - \beta(\beta + \beta^{-1}) = (\alpha - \beta)(\alpha + \alpha^{-1})$. Hence (8) yields

$$(\alpha - \beta^{-1})(x)R(\alpha^2 - \beta^2)(y)R(\alpha^2 - \beta^2)(z) = 0 \quad \text{for all } x, y, z \in R.$$

But then (5) holds by the semiprimeness of R . Noting that $\alpha^2 + \alpha^{-2} = \beta^2 + \beta^{-2}$, and then using the analogous approach as in the proof of (5), one proves (6).

LEMMA 3. *If automorphisms α and β of a semiprime ring R satisfy $\alpha + \alpha^{-1} = \beta + \beta^{-1}$, then α commutes with β^2 and β commutes with α^2 .*

PROOF. Let us show that α commutes with β^2 . The initial hypothesis yields

$$(9) \quad \alpha(\alpha - \beta) = (\alpha - \beta)\beta^{-1}.$$

By (6) it follows that

$$(\alpha(\alpha - \beta))(x)\alpha(R)(\alpha(\alpha^2 - \beta^{-2}))(y) = 0 \quad \text{for all } x, y \in R.$$

In view of (9) this relation implies that

$$(\alpha - \beta)(x)R(\alpha^3 - \alpha\beta^{-2})(y) = 0 \quad \text{for all } x, y \in R.$$

Substituting $\alpha(y)$ for y in (6) we obtain

$$(\alpha - \beta)(x)R(\alpha^3 - \beta^{-2}\alpha)(y) = 0 \quad \text{for all } x, y \in R.$$

Comparing the last two relations we get

$$(10) \quad (\alpha - \beta)(x)R(\alpha\beta^{-2} - \beta^{-2}\alpha)(y) = 0 \quad \text{for all } x, y \in R.$$

Multiply the identity $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ from the left by β and from the right by α ; then we get $\beta\alpha^2 - \beta^2\alpha = \alpha - \beta$. Since $\alpha - \beta = \beta^{-1} - \alpha^{-1}$, and since $\beta^2 = \alpha^2 + \alpha^{-2} - \beta^{-2}$, it follows that $\beta^{-1} + \alpha^3 - \beta\alpha^2 = \beta^{-2}\alpha$. Consequently

$$\alpha\beta^{-2} - \beta^{-2}\alpha = (\alpha - \beta)(\beta^{-2} - \alpha^2).$$

Therefore (10) implies that

$$(\alpha\beta^{-2} - \beta^{-2}\alpha)(y)R(\alpha\beta^{-2} - \beta^{-2}\alpha)(y) = 0$$

for every $y \in R$. But then $\alpha\beta^{-2} = \beta^{-2}\alpha$ since R is semiprime. Thus α and β^2 commute. For the sake of symmetry we omit the proof of the commutativity of α^2 and β .

COROLLARY 3. *Let R be a semiprime ring with unit element and containing the element $1/2$. If inner automorphisms α, β of R satisfy $\alpha + \alpha^{-1} = \beta + \beta^{-1}$, then they commute.*

PROOF. Let $a, b \in R$ be such that $\alpha(x) = axa^{-1}$ and $\beta(x) = bxb^{-1}$. By assumption, $axa^{-1} + a^{-1}xa = bxb^{-1} + b^{-1}xb$ for all $x \in R$. In particular, $2a = bab^{-1} + b^{-1}ab$. Multiplying from the right by b we obtain

$$(11) \quad 2ab = ba + b^{-1}ab^2.$$

By Lemma 3, α and β^2 commute. Hence $ab^2 = cb^2a$ for some c in the center of R . By (11) we then have $2ab = (1 + c)ba$. Since R contains the element $1/2$ it follows that $ab = c_1ba$, where c_1 is an invertible element in the center of R . But then α and β commute.

REMARK 2. The case where commuting automorphisms α, β of a semi-prime ring R satisfy $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ is considered in our paper [2]. In particular, it was shown that if R is prime of characteristic not 2 then either $\alpha = \beta$ or $\alpha = \beta^{-1}$. Combining this with Corollary 3 we obtain the following

result which generalizes [3; Corollary 2.1]: Let R be a prime ring with unit element and containing the element $1/2$. If inner automorphisms α, β of R satisfy $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ then either $\alpha = \beta$ or $\alpha = \beta^{-1}$.

We now come to the main result of this paper.

THEOREM 1. *Let α and β be automorphisms of a semiprime ring R such that $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. Then there exists ideals U_1, U_2 and U_3 of R such that*

- (i) $U_i \cap U_j = 0, i \neq j$, and $U_1 \oplus U_2 \oplus U_3$ is an essential ideal of R . Moreover, if the annihilator of any ideal in R is a direct summand (in particular, if R is a von Neumann algebra), then $U_1 \oplus U_2 \oplus U_3 = R$,
- (ii) U_i are invariant under $\alpha, \beta, \alpha^{-1}$ and β^{-1} ,
- (iii) $\alpha = \beta$ on U_1 ,
- (iv) $\alpha = \beta^{-1}$ on U_2 ,
- (v) $\alpha^2 = \beta^2 = \alpha^{-2}$ on U_3 .

REMARK 3. In [5] Thaheem constructed an example of automorphisms α and β satisfying $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ on a von Neumann algebra R but there is no decomposition of R for which $\alpha = \beta$ on the one part and $\alpha = \beta^{-1}$ on the other part. Thus the presence of an ideal U_3 in Theorem 1 is really necessary. In Thaheem's example the algebra R was not prime. We do not know whether the equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ has any nontrivial solutions in prime rings (in the sense that $\alpha \neq \beta$ and $\alpha \neq \beta^{-1}$). In order to find such a solution one can assume that $\alpha^2 = \beta^2 = \alpha^{-2}$ (Corollary 2) and that α, β are not both inner (Remark 2).

PROOF OF THEOREM 1. Let U_0 be an ideal of R generated by all $(\alpha^2 - \beta^{-2})(x), x \in R$. We set $V = \text{Ann}(U_0)$ and $U_1 = \text{Ann}(V)$. By Remark A we have $U_1 \cap V = 0$ and $U_1 \oplus V$ is an essential ideal. From Lemma 3 we see that the mapping $\alpha^2 - \beta^{-2}$ commutes with $\alpha, \beta, \alpha^{-1}$ and β^{-1} . Simple calculations show that this implies that U_0 is invariant under $\alpha, \beta, \alpha^{-1}$ and β^{-1} . But then, by Remark B, the same is true for ideals V and U_1 .

Take $u_1 \in U_1$. Since U_1 is invariant under α and $\beta, (\alpha - \beta)(u_1)$ lies in U_1 . However, from Lemma 2 (and Remark A) it follows that the range of $\alpha - \beta$ lies in $\text{Ann}(U_0) = V$. Since $U_1 \cap V = 0$ we then have $\alpha(u_1) = \beta(u_1)$. Thus we have proved (iii).

Let V_1 be an ideal of R generated by all $(\beta^2 - \alpha^{-2})(v), v \in V$. Of course, $V_1 \subseteq V$. We define $U_3 = \text{Ann}(V_1) \cap V$ and $U_2 = \text{Ann}(U_3) \cap V$. Since $U_2 \subseteq \text{Ann}(U_3)$, we have $U_2 \cap U_3 = 0$. Next, since U_2 and U_3 are

contained in V , we also have $U_2 \cap U_1 = 0$ and $U_3 \cap U_1 = 0$. Let us show that the ideal $W = U_1 \oplus U_2 \oplus U_3$ is essential. We have to show that

$$\text{Ann}(W) = \text{Ann}(U_1) \cap \text{Ann}(U_2) \cap \text{Ann}(U_3)$$

is equal to zero. Since $U_2 = \text{Ann}(U_3) \cap V$, we have

$$\text{Ann}(W) \cap V = \text{Ann}(U_1) \cap \text{Ann}(U_2) \cap U_2 = 0.$$

Hence $\text{Ann}(W) \subseteq \text{Ann}(V)$. But on the other hand we have $\text{Ann}(W) \subseteq \text{Ann}(U_1)$. Since $\text{Ann}(V) \cap \text{Ann}(U_1) = \text{Ann}(V \oplus U_1) = 0$ (namely, the ideal $V \oplus U_1$ is essential) it follows that $\text{Ann}(W) = 0$. That is, W is essential.

Assume that the annihilator of any ideal in R is a direct summand. Then $U_1 \oplus V = R$. We want to show that $U_2 \oplus U_3 = V$. By assumption, $\text{Ann}(V_1)$ is a direct summand. Thus $R = \text{Ann}(V_1) \oplus Z$ for some ideal Z of R . Since $V_1 \subseteq V$, we have

$$Z = \text{Ann}(\text{Ann}(V_1)) \subseteq \text{Ann}(\text{Ann}(V)) = \text{Ann}(U_1) = V.$$

Thus Z is contained in V . Pick $v \in V$. There exist elements $w \in \text{Ann}(V_1)$ and $z \in Z$ such that $v = w + z$. We claim that $w \in U_3$ and $z \in U_2$. Since $z \in Z \subseteq V$, we also have $w \in V$. Thus $w \in \text{Ann}(V_1) \cap V = U_3$. The ideal U_3 is contained in $\text{Ann}(V_1)$, therefore $Z = \text{Ann}(\text{Ann}(V_1)) \subseteq \text{Ann}(U_3)$. Hence $z \in \text{Ann}(U_3) \cap V = U_2$. With this we have proved that $U_1 \oplus U_2 = V$, and, therefore, $U_1 \oplus U_2 \oplus U_3 = R$. The proof of (i) is thus complete.

Since $\alpha, \beta, \alpha^{-1}$ and β^{-1} commute with $\beta^2 - \beta^{-2}$ (Lemma 3), and since all these automorphisms leave V invariant, it follows easily that V_1 is also invariant under $\alpha, \beta, \alpha^{-1}$ and β^{-1} . Using Remark B we see the same is true for the ideal $U_3 = \text{Ann}(V_1) \cap V$. Similarly we argue about the ideal U_2 . Thus (ii) is proved.

Let us prove (iv). Given $v \in V$, we have $\alpha^2(v) - \beta^{-2}(v) \in V$ since V is invariant under α^2 and β^{-2} . But on the other hand, $(\alpha^2 - \beta^{-2})(v)$ is contained in U_0 . Since $U_0 \cap V = 0$ it follows that $\alpha^2(v) = \beta^{-2}(v)$. Lemma 2 then yields

$$(\alpha - \beta^{-1})(x)R(\beta^2 - \beta^{-2})(v) = 0 \quad \text{for all } x \in R, v \in V.$$

This means that the range of $\alpha - \beta^{-1}$ is contained in $\text{Ann}(V_1)$. Since $(\alpha - \beta^{-1})(u_2) \in U_2$ if $u_2 \in U_2$, and since $U_2 \cap \text{Ann}(V_1) = U_2 \cap \text{Ann}(V_1) \cap V = U_2 \cap U_3 = 0$, it follows that $\alpha(u_2) = \beta^{-1}(u_2)$.

It remains to prove (v). Pick $u_3 \in U_3$. On the one hand we have $(\beta^2 - \beta^{-2})(u_3) \in U_3$, and on the other hand, by the definition of V_1 , $(\beta^2 - \beta^{-2})(u_3) \in V_1$. However, $U_3 \cap V_1 = V \cap \text{Ann}(V_1) \cap V_1 = 0$, and hence

$\beta^2(u_3) = \beta^{-2}(u_3)$. We have proved that $\alpha^2 = \beta^{-2}$ on V , and therefore also on $U_3 \subseteq V$. With this (v) is proved.

In the case that the annihilator of any ideal of R is a direct summand, we see from Theorem 1 that the range of $\alpha - \beta$ is contained in the ideal $I_1 = U_2 \oplus U_3$, the range of $\alpha - \beta^{-1}$ is contained in $I_2 = U_1 \oplus U_3$, and the union of the ranges of $\alpha^2 - \beta^2$ and $\alpha^2 - \beta^{-2}$ is contained in $I_3 = U_1 \oplus U_2$. This result is in accordance with Theorem A. With the aid of Lemmas 2 and 3, even if we do not assume that the annihilator of any ideal of R is a direct summand, it is not difficult to prove the following generalization of Theorem A.

THEOREM 2. *Let α and β be automorphisms of a semiprime ring R such that $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. Then there exists ideals I_1, I_2 and I_3 of R , each invariant under $\alpha, \alpha^{-1}, \beta$ and β^{-1} , such that $I_1 \cap I_2 \cap I_3 = 0, I_3 \subseteq I_1 \cap I_2$ and, for each x in R ,*

$$\begin{aligned} \beta(x) - \alpha(x) &\in I_1, & \beta(x) - \alpha^{-1}(x) &\in I_2, \\ \beta^2(x) - \alpha^2(x) &\in I_3, & \beta^2(x) - \alpha^{-2}(x) &\in I_3. \end{aligned}$$

PROOF. Let J be an ideal of R generated by all $(\alpha^2 - \beta^{-2})(x), x \in R$, and set $I_1 = \text{Ann}(J)$. From Lemma 2 we see that $(\alpha - \beta)(x) \in I_1$ for every $x \in R$. Since the mapping $\alpha^2 - \beta^{-2}$ commutes with $\alpha, \beta, \alpha^{-1}$ and β^{-1} (Lemma 3) it follows that J is invariant under $\alpha, \beta, \alpha^{-1}$ and β^{-1} . But then Remark B tells us that the same is true for the ideal I_1 .

We introduce L to be an ideal of R generated by all $(\alpha^2 - \beta^2)(x), x \in R$, and let $I_2 = \text{Ann}(L)$. Similarly as above one deduces that the range of $\alpha - \beta^{-1}$ is contained in I_2 , and that L and I_2 are invariant under $\alpha, \beta, \alpha^{-1}$ and β^{-1} .

The union of the ranges of $\alpha^2 - \beta^{-2}$ and $\alpha^2 - \beta^2$ is certainly contained in the ideal $I_3 = J + L$. Of course, I_3 is also invariant under $\alpha, \beta, \alpha^{-1}$ and β^{-1} . Next,

$$I_1 \cap I_2 \cap I_3 = \text{Ann}(J) \cap \text{Ann}(L) \cap (J + L) = \text{Ann}(J + L) \cap (J + L) = 0$$

by Remark A. From $\alpha^2 - \beta^2 = \alpha(\alpha + \alpha^{-1}) - \beta(\beta + \beta^{-1}) = (\alpha - \beta)(\beta + \beta^{-1})$ we see that the range of $\alpha^2 - \beta^2$ is contained in the range of $\alpha - \beta$. Therefore, L is contained in I_1 . Similarly we see that J is contained in I_2 . Consequently I_3 is contained in $I_1 + I_2$. The proof of the theorem is complete.

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