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On certain types of sets in ideal topological spaces

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Abstract. In the present article we introduce certain typical sets in an ideal topological space, some such corresponding versions in topological spaces being already there in the literature. We prove several properties of the introduced classes of sets, and finally as application, we initiate the study of a kind of separation axiom, termed $*$-$T_{\frac{1}{2}}$-property.

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1 Introduction

The concept of ideals in general topological spaces is found in the classic text by Kuratowski [4] and also in [11]. A collection $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an ideal on X if it satisfies the following two conditions:
(i) $A \in \mathcal{I}$ and $A \supseteq B \Rightarrow B \in \mathcal{I}$, and
(ii) $A \in \mathcal{I}$, $B \in \mathcal{I} \Rightarrow A \bigcup B \in \mathcal{I}$.

A topological space (X, τ) with an ideal $\mathcal{I}$ on X is denoted by $(X, \tau, \mathcal{I})$, called an ideal topological space. For a subset A of X, an operator $(.)^* : \mathcal{P}(X) \to \mathcal{P}(X)$ (where $\mathcal{P}(X)$ denotes the power set of X), called a local function[4] of A and denoted by $A^*(\mathcal{I}, \tau)$ or simply A^*, is defined by the set $\{x \in X : U \bigcap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. In

[3,4], it was also shown that the operator $cl^*(.)$, defined by $cl^*(A) = A \bigcup A^*$, is a Kuratowski closure operator and hence generates a topology $\tau^*(\mathfrak{I})$ or simply τ^* on X, called $*$-topology, finer than τ. The members of τ^* are called τ^*-open or simply $*$-open sets and the complement of a $*$-open set is called a $*$-closed set or equivalently, a subset A of X is called $*$-closed if $A^* \subseteq A$. For a subset A of topological space (X,τ), H. Maki [6] introduced the following notations: $A^\wedge = \bigcap\{U : A \subseteq U$ and U is open $\}$ and $A^\vee = \bigcup\{F : F \subseteq A$ and F is closed $\}$. A subset A of X is said to be a $\wedge$-set ($\vee$-set) if $A = A^\wedge$ (resp. $A = A^\vee$). A subset A of X is said to be g-closed [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X; and the complement of a g-closed subset in X is called a g-open set in X. For further details regarding g-closed sets and similar such sets one may refer to [8–10]. A subset A of an ideal space $(X,\tau,\mathfrak{I})$ is said to be $*$-g-closed [7] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $*$-open. A subset A of X is said to be $*$-g-open if $X \setminus A$ is $*$-g-closed. It was shown in [7] that the the class of $*$-g-closed sets lies strictly between the class of closed sets and the class of g-closed sets. It was also shown in the same paper that the class of g-closed sets in (X,τ) is not same as that of $*$-g-closed sets in an ideal topological space $(X,\tau,\mathfrak{I})$.

In Section 2, we introduce two types of sets viz. $\wedge_*$-sets and $\vee_*$-sets in a way analogous to $\wedge$-sets and $\vee$-sets studied in [6]. Certain descriptions of such sets along with those in connection with the $*$-g-open and $*$-g-closed sets of [7] are incorporated in this section.

Section 3 includes the introduction and study of two other typical sets, termed generalized $\wedge_*$-sets and $\vee_*$-sets. These sets are defined and investigated in terms of the operators introduced in Section 2.

Dunham [2] studied $T_{\frac{1}{2}}$-separation axiom in a topological space. We have, in our turn, initiated in Section 4 the study of $*$-$T_{\frac{1}{2}}$-spaces in an ideal topological space, where we apply in the process the different types of sets introduced in Sections 2 and 3. We show that the class of $*$-$T_{\frac{1}{2}}$-spaces contains the class of $T_{\frac{1}{2}}$-spaces and is contained in that of T_1-spaces.

2 $\wedge_*$-and $\vee_*$-sets

The intent of this section is to introduce two types of sets viz. $\wedge_*$-sets and $\vee_*$-sets, and characterize $*$-g-closed sets with the help of these sets. To that end, we define the following two operators:

Definition 2.1. *Let $(X,\tau,\mathfrak{I})$ be an ideal topological space and $A \subseteq X$. We define*

$A^{\wedge}_{*} = \bigcap\{U : A \subseteq U$ *and* U *is a* $*$*-open set* $\}$.
$A^{\vee}_{*} = \bigcup\{F : F \subseteq A$ *and* F *is a* $*$*-closed set*$\}$.

Some elementary but basic results concerning the above two types of sets are obtained in the following theorem:

Theorem 2.1. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* A, B, $A_\alpha (\alpha \in \Delta)$ *be subsets of* X. *Then the following are true:*
(a) $A \subseteq A^{\wedge}_{*}$.
(b) *If* A *is* $*$*-open then* $A = A^{\wedge}_{*}$.
(c) $A \subseteq B \Rightarrow A^{\wedge}_{*} \subseteq B^{\wedge}_{*}$.
(d) $(A^{\wedge}_{*})^{\wedge}_{*} = A^{\wedge}_{*}$.
(e) $(\bigcup\{A_\alpha : \alpha \in \Delta\})^{\wedge}_{*} = \bigcup\{(A_\alpha)^{\wedge}_{*} : \alpha \in \Delta\}$.
(f) $(\bigcap\{A_\alpha : \alpha \in \Delta\})^{\wedge}_{*} \subseteq \bigcap\{(A_\alpha)^{\wedge}_{*} : \alpha \in \Delta\}$.

Proof. (a) and (b) follow from the above definition.
(c) Let $x \notin B^{\wedge}_{*}$. Then there exists a $*$-open set U such that $B \subseteq U$ and $x \notin U$. Since $A \subseteq B$, $x \notin A^{\wedge}_{*}$.
(d) Clearly $(A^{\wedge}_{*})^{\wedge}_{*} \supseteq A^{\wedge}_{*}$.
By definition, $A^{\wedge}_{*} \subseteq U$ for every $*$-open set U with $A \subseteq U$. Then $(A^{\wedge}_{*})^{\wedge}_{*} \subseteq U^{\wedge}_{*} = U$ (by using (b) and (c)). Thus $(A^{\wedge}_{*})^{\wedge}_{*} \subseteq A^{\wedge}_{*}$.
(e) In view of (c), it is sufficient to show that $(\bigcup\{A_\alpha : \alpha \in \Delta\})^{\wedge}_{*} \subseteq \bigcup\{(A_\alpha)^{\wedge}_{*} : \alpha \in \Delta\}$.
Let $x \notin \bigcup\{(A_\alpha)^{\wedge}_{*} : \alpha \in \Delta\}$. Then for each $\alpha \in \Delta$, there exists a $*$-open set U_α such that $A_\alpha \subseteq U_\alpha$ and $x \notin U_\alpha$. Let $U = \bigcup\{U_\alpha : \alpha \in \Delta\}$. Then U is a $*$-open set containing $\bigcup\{A_\alpha : \alpha \in \Delta\}$ and $x \notin U$. Hence $x \notin (\bigcup\{A_\alpha : \alpha \in \Delta\})^{\wedge}_{*}$.
(f) Follows from (c) above. □

Remark 2.1. In (f) of the above theorem, equality does not hold in general, even if Δ is a finite index set. We show this by the following example.

Example 2.1. Consider an ideal topological space $(X, \tau, \mathcal{I})$, where $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\phi, \{c\}\}$.
Then $\tau^* = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Let us consider $A = \{a, b\}$ and $B = \{a, c\}$. Then $A^{\wedge}_{*} = \{a, b\}$, $B^{\wedge}_{*} = X$ and $(A \bigcap B)^{\wedge}_{*} = \{a\}$. Thus $(A \bigcap B)^{\wedge}_{*} \subsetneqq A^{\wedge}_{*} \bigcap B^{\wedge}_{*}$.

Lemma 2.2. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. Then* $(X \setminus A)^{\wedge}_{*} = X \setminus A^{\vee}_{*}$ *for each* $A \subseteq X$.

Proof. We have $X \setminus A^{\vee}_{*} = X \setminus (\bigcup\{F : F \subseteq A$ and F is a $*$-closed set$\}) = \bigcap\{X \setminus F : X \setminus A \subseteq X \setminus F$ and $X \setminus F$ is a $*$-open set$\} = (X \setminus A)^{\wedge}_{*}$. □

Using the above lemma and Theorem 2.1, we have the following result:

Theorem 2.3. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and A, B, $A_\alpha(\alpha \in \Delta)$ be subsets of X. Then the following are true:*
(a) $A_*^\vee \subseteq A$.
(b) If A is $*$-closed then $A = A_*^\vee$.
(c) $A \subseteq B \Rightarrow A_*^\vee \subseteq B_*^\vee$.
(d) $(A_*^\vee)_*^\vee = A_*^\vee$.
(e) $(\bigcap\{A_\alpha : \alpha \in \Delta\})_*^\vee = \bigcap\{(A_\alpha)_*^\vee : \alpha \in \Delta\}$.
(f) $\bigcup\{(A_\alpha)_*^\vee : \alpha \in \Delta\} \subseteq (\bigcup\{A_\alpha : \alpha \in \Delta\})_*^\vee$.

Definition 2.2. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is said to be a*
(i) $\wedge_*$-set if $A = A_*^\wedge$.
(ii) $\vee_*$-set if $A = A_*^\vee$.
Thus a subset A of X is a $\wedge_*$-set if and only if $X \setminus A$ is a $\vee_*$-set.

Theorem 2.4. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then the following statements hold:*
(a) ϕ and X are $\wedge_*$-sets and $\vee_*$-sets.
(b) Every union of $\wedge_*$-sets is a $\wedge_*$-set.
(c) Every intersection of $\vee_*$-sets is a $\vee_*$-set.

Proof. (a) Obvious.
(b) Let $\{A_\alpha : \alpha \in \Delta\}$ be a family of $\wedge_*$-sets. Then $A_\alpha = (A_\alpha)_*^\wedge$, for each $\alpha \in \Delta$. Let $A = \bigcup\{A_\alpha : \alpha \in \Delta\}$. Then by Theorem 2.1$(e)$, $A_*^\wedge = A \Rightarrow A$ is a $\wedge_*$-set.
(c) Follows from (b) above and Lemma 2.2. □

Theorem 2.5. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is $*$-g-closed if and only if $cl(A) \subseteq A_*^\wedge$.*

Proof. Let A be a $*$-g-closed set. Let $x \in cl(A)$. If $x \notin A_*^\wedge$, then there exists a $*$-open set U containing A such that $x \notin U$. Now, A is $*$-g-closed and $A \subseteq U$, where U is $*$-open $\Rightarrow cl(A) \subseteq U \Rightarrow x \notin cl(A)$, a contradiction. Thus $cl(A) \subseteq A_*^\wedge$.
Conversely, suppose that $cl(A) \subseteq A_*^\wedge$. Let $A \subseteq U$, where U is $*$-open. Then $A_*^\wedge \subseteq U$ and so $cl(A) \subseteq U$. Hence A is $*$-g-closed. □

Corollary 2.6. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is $*$-g-open if and only if $A_*^\vee \subseteq int(A)$.*

Corollary 2.7. *Let A be a $\wedge_*$-set in an ideal topological space $(X,\tau,\mathcal{I})$. Then A is $*$-g-closed if and only if A is closed in (X,τ).*

Proof. Suppose that A is $*$-g-closed. Then by Theorem 2.5, $cl(A) \subseteq A^{\wedge}_{*} = A$ $\Rightarrow A$ is closed.
Converse is obvious. □

Corollary 2.8. *Let A be a $\vee_*$-set in an ideal topological space $(X, \tau, \mathcal{I})$. Then A is $*$-g-open if and only if A is open in (X, τ).*

Theorem 2.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. If $A^{\wedge}_{*}$ is $*$-g-closed then A is $*$-g-closed.*

Proof. Let $A^{\wedge}_{*}$ be $*$-g-closed. Suppose that $A \subseteq U$, where U is $*$-open. Then $A^{\wedge}_{*} \subseteq U$. Now $A^{\wedge}_{*}$ is $*$-g-closed $\Rightarrow cl(A^{\wedge}_{*}) \subseteq U \Rightarrow cl(A) \subseteq cl(A^{\wedge}_{*}) \subseteq U \Rightarrow A$ is a $*$-g-closed set. □

Remark 2.2. The converse of the above result is not true, in general. The example, given below, justifies it.

Example 2.2. Consider an ideal topological space $(X, \tau, \mathcal{I})$, where $X = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, X\}$ and $\mathcal{I} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Then $\tau^* = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$. Let $A = \{d\}$. Then A is a $*$-g-closed set. But $A^{\wedge}_{*} = \{a, d\}$ is not a $*$-g-closed set.

3 Generalized $\wedge_*$- and $\vee_*$-sets

In this section, we introduce and study two other types of sets viz. $g.\wedge_*$-sets, $g.\vee_*$-sets. We discuss several properties of $g.\wedge_*$-sets and $g.\vee_*$ sets, a few of which involve sets introduced in the previous section. We start this section by recalling the following definition from [6]:

Definition 3.1. *A subset A of a space (X, τ) is said to be a generalized $\wedge$-set ($g.\wedge$-set, for short) if $A^{\wedge} \subseteq F$ whenever $A \subseteq F$ and F is closed in X.*
A subset A of X is said to be a $g.\vee$-set if $X \setminus A$ is a $g.\wedge$-set.

In an analogous way we define generalized $\wedge_*$-sets as follows:

Definition 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is said to be a generalized $\wedge_*$-set ($g.\wedge_*$-set, in short) if $A^{\wedge}_{*} \subseteq F$ whenever $A \subseteq F$ and F is closed in X.*
A subset A of X is said to be a $g.\vee_*$-set if $X \setminus A$ is a $g.\wedge_*$-set.

Remark 3.1. (i) Clearly every $g.\wedge$-set is a $g.\wedge_*$-set. That the converse is false is shown by the example below.
(ii) Every $\wedge_*$-set is a $g.\wedge_*$-set. That the converse is false is shown by Example 3.1(b) below.

Example 3.1. (a) Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}$ and $\mathcal{I} = \{\phi, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$. Then $(X, \tau, \mathcal{I})$ is an ideal topological space. Consider a set $A = \{a\}$. Then $A^{\wedge} = \{a, b\}$ and $A^{\wedge}_* = \{a\}$, as A is $*$-open. Thus it is noted that A is a $g.\wedge_*$-set but not a $g.\wedge$-set.
(b) Consider the ideal topological space $(X, \tau, \mathcal{I})$, where $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. Let $A = \{a, c\}$. Then $A^{\wedge}_* = X$. Thus A is a $g.\wedge_*$-set, but not a $\wedge_*$-set.

Theorem 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is a $g.\vee_*$-set if and only if $U \subseteq A^{\vee}_*$ whenever $U \subseteq A$ and U is open in X.*

Proof. Let A be a $g.\wedge_*$-set and $U \subseteq A$, where U is open in X. Then $X \setminus A \subseteq X \setminus U$ and $X \setminus U$ is closed in $X \Rightarrow (X \setminus A)^{\wedge}_* \subseteq X \setminus U \Rightarrow X \setminus A^{\vee}_* \subseteq X \setminus U$(by Lemma 2.2) $\Rightarrow U \subseteq A^{\vee}_*$.
Conversely, suppose that A is a subset of X such that $U \subseteq A^{\vee}_*$ whenever $U \subseteq A$ and U is open in X. Let $X \setminus A \subseteq F$, where F is closed in X. Then $X \setminus F \subseteq A$ and $X \setminus F$ is open in X and so by hypothesis, $X \setminus F \subseteq A^{\vee}_*$ and thus $X \setminus A^{\vee}_* \subseteq F$. Again, by using Lemma 2.2, $(X \setminus A)^{\wedge}_* \subseteq F \Rightarrow X \setminus A$ is a $g.\wedge_*$-set $\Rightarrow A$ is a $g.\vee_*$-set. □

Theorem 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. If A is a $g.\vee_*$-set, then $F = X$ whenever $A^{\vee}_* \bigcup (X \setminus A) \subseteq F$ and F is closed in X.*

Proof. Let A be a $g.\vee_*$-set and $A^{\vee}_* \bigcup (X \setminus A) \subseteq F$, where F is closed in X. Then $X \setminus F \subseteq X \setminus (A^{\vee}_* \bigcup (X \setminus A)) \Rightarrow X \setminus F \subseteq (X \setminus A^{\vee}_*) \bigcap A \Rightarrow X \setminus F \subseteq A \Rightarrow X \setminus F \subseteq A^{\vee}_*$ (by Theorem 3.1). Thus $X \setminus F \subseteq (X \setminus A^{\vee}_*) \bigcap A^{\vee}_* \Rightarrow X \setminus F = \phi$ $\Rightarrow F = X$. □

Theorem 3.3. *Let $A(\subseteq X)$ be a $g.\vee_*$-set of an ideal topological space $(X, \tau, \mathcal{I})$. Then $A^{\vee}_* \bigcup (X \setminus A)$ is closed if and only if A is a $\vee_*$-set.*

Proof. Let A be a $g.\vee_*$-set such that $A^{\vee}_* \bigcup (X \setminus A)$ is closed in X. Then by Theorem 3.2, $A^{\vee}_* \bigcup (X \setminus A) = X \Rightarrow A \subseteq A^{\vee}_*$. Thus by Theorem 2.3($a$), it follows that $A = A^{\vee}_*$ and hence A is a $\vee_*$-set.
Conversely, let A be a $\vee_*$-set. Then $A^{\vee}_* \bigcup (X \setminus A) = A \bigcup (X \setminus A) = X$ which is closed. □

Corollary 3.4. *Let $A(\subseteq X)$ be a $g.\wedge_*$-set in an ideal topological space $(X, \tau, \mathcal{I})$. Then $A^{\wedge}_* \bigcup (X \setminus A)$ is open if and only if A is a $\wedge_*$-set.*

Theorem 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then for each $x \in X$, either $\{x\}$ is an open set in X or a $g.\vee_*$-set.*

Proof. Let $x \in X$. If $\{x\}$ is not open in X, then X is the only closed set containing $X \setminus \{x\}$. Therefore $X \setminus \{x\}$ is a $g.\wedge_*$-set. Hence $\{x\}$ is a $g.\vee_*$-set. □

Theorem 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then every singleton of X is a $g.\wedge_*$-set if and only if $U = U_*^{\vee}$ for every open set U in X.*

Proof. Let every singleton set of X be a $g.\wedge_*$-set in $(X, \tau, \mathcal{I})$. Let U be an open set in X and $x \in X \setminus U$. Since $\{x\}$ is a $g.\wedge_*$-set, we have $\{x\}_*^{\wedge} \subseteq X \setminus U$. Thus $\bigcup\{\{x\}_*^{\wedge} : x \in X \setminus U\} \subseteq X \setminus U$ and hence using Theorem 2.1(e), we have $(\bigcup\{\{x\} : x \in X \setminus U\})_*^{\wedge} \subseteq X \setminus U$. Therefore $(X \setminus U)_*^{\wedge} \subseteq X \setminus U$ and hence $X \setminus U = (X \setminus U)_*^{\wedge} = X \setminus U_*^{\vee}$ (by Lemma 2.2). Thus $U = U_*^{\vee}$.
Conversely, suppose that $x \in X$ and $x \in F(\subseteq X)$, where F is a closed set in X. Then $X \setminus F$ is open in X and so by hypothesis, $X \setminus F = (X \setminus F)_*^{\vee} = X \setminus F_*^{\wedge}$ (by Lemma 2.2) $\Rightarrow F = F_*^{\wedge}$. Hence $\{x\}_*^{\wedge} \subseteq F_*^{\wedge} = F \Rightarrow \{x\}$ is a $g.\wedge_*$-set. □

4 $*$-$T_{\frac{1}{2}}$-Spaces

Dunham [2] defined a kind of separation axiom viz. $T_{\frac{1}{2}}$-property in a topological space. It is shown in [2] that the class of $T_{\frac{1}{2}}$-spaces lies between the classes of T_0-spaces and T_1-spaces. The intent of this section is to introduce a similar type of separation axiom, termed $*$-$T_{\frac{1}{2}}$-property which is strictly stronger than $T_{\frac{1}{2}}$-property, but is weaker than the T_1-axiom. Such a separation axiom is characterized here in terms of the types of sets introduced in earlier sections.

We begin by recalling the definition of $T_{\frac{1}{2}}$-spaces as given in [2]

Definition 4.1. *A topological space (X, τ) is said to be a $T_{\frac{1}{2}}$-space if every g-closed set is closed in X.*

Our proposed definition of $*$-$T_{\frac{1}{2}}$-spaces goes as follows.

Definition 4.2. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be a $*$-$T_{\frac{1}{2}}$-space if every $*$-g-closed set is closed in X.*

Remark 4.1. It is easy to see that $T_{\frac{1}{2}}$-property $\Rightarrow$ $*$-$T_{\frac{1}{2}}$-property. In the example below we now show that $*$-$T_{\frac{1}{2}}$-property $\not\Rightarrow$ $T_{\frac{1}{2}}$-property.

Example 4.1. Consider the space (X, τ) of Example 2.10 in [2], where $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Then (X, τ) is not a $T_{\frac{1}{2}}$-space [2]. Let $\mathcal{I} = \{\phi, \{b\}\}$. Then $\tau^* = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ and the ideal topological space $(X, \tau, \mathcal{I})$ is a $*$-$T_{\frac{1}{2}}$-space (refer to Theorem 4.1).

Theorem 4.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $(X, \tau, \mathcal{I})$ is a $*$-$T_{\frac{1}{2}}$-space if and only if for each $x \in X$, either $\{x\}$ is $*$-closed or open in X.*

Proof. Let $(X, \tau, \mathcal{I})$ be a $*$-$T_{\frac{1}{2}}$-space. If for $x \in X$, $\{x\}$ is not $*$-closed then X is the only $*$-open set containing $X \setminus \{x\}$ and hence $X \setminus \{x\}$ is $*$-g-closed. Since $(X, \tau, \mathcal{I})$ is a $*$-$T_{\frac{1}{2}}$-space, $X \setminus \{x\}$ is closed in X and hence $\{x\}$ is open in X.
Conversely, let A be a $*$-g-closed set and $x \in cl(A)$. Then by hypothesis, the singleton $\{x\}$ is either $*$-closed or open in X.
Case I: Suppose that $\{x\}$ is $*$-closed. Since A is $*$-g-closed, by Theorem 2.6 of [7] we have, $cl(A) \setminus A$ does not contain any nonempty $*$-closed set. Therefore $x \notin cl(A) \setminus A \Rightarrow x \in A$.
Case II: Suppose that $\{x\}$ is open in X. Then $x \in cl(A) \Rightarrow \{x\} \bigcap A \neq \phi \Rightarrow x \in A$.
Thus combining both the cases, we have that A is closed in X. Hence $(X, \tau, \mathcal{I})$ is a $*$-$T_{\frac{1}{2}}$-space. □

Corollary 4.2. *An ideal topological space $(X, \tau, \mathcal{I})$ is a $*$-$T_{\frac{1}{2}}$-space if and only if every subset A of X is the intersection of all those sets which are $*$-open or closed sets containing A.*

Proof. Let $(X, \tau, \mathcal{I})$ be a $*$-$T_{\frac{1}{2}}$-space and $A \subseteq X$. Since $A = \bigcap\{X \setminus \{x\} : x \notin A\}$, the result follows in view of the above theorem.
Conversely, suppose that the condition holds. Then for each $x \in X$, $X \setminus \{x\}$ is the intersection of all sets that are $*$-open sets or closed sets containing it. Therefore $X \setminus \{x\}$ is either $*$-open or closed in X and hence by the above theorem, $(X, \tau, \mathcal{I})$ is a $*$-$T_{\frac{1}{2}}$-space. □

Remark 4.2. From Theorem 4.1 it readily follows that every T_1-space is a $*$-$T_{\frac{1}{2}}$-space. That the converse fails is shown by the following example.

Example 4.2. Consider the ideal topological space $(X, \tau, \mathcal{I})$ of Example 2.2. Since for each $x \in X$, either $\{x\}$ is $*$-closed or open in X, it follows that $(X, \tau, \mathcal{I})$ is a $*$-$T_{\frac{1}{2}}$-space. But (X, τ) is not a T_1-space.

Theorem 4.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $(X, \tau, \mathcal{I})$ is a $*$-$T_{\frac{1}{2}}$-space if an only if every $g.\vee_*$-set is a $\vee_*$-set.*

Proof. Let $(X, \tau, \mathcal{I})$ be a $*$-$T_{\frac{1}{2}}$-space. If possible, let there exists a $g.\vee_*$-set A which is not a $\vee_*$-set. Then there exists an element $x \in A$ such that $x \notin A_*^{\vee}$. Therefore by definition of $A_*^{\vee}$, $\{x\}$ is not $*$-closed. Thus X is the only $*$-open set containing $X \setminus \{x\} \Rightarrow X \setminus \{x\}$ is $*$-g-closed and hence by hypothesis, $X \setminus \{x\}$ is closed in X. Since $x \in A$ and $x \notin A_*^{\vee}$, $A_*^{\vee} \bigcup (X \setminus A) \subseteq X \setminus \{x\}$. Thus by Theorem 3.2, $X \setminus \{x\} = X$, a contradiction.
Conversely, let every $g.\vee_*$-set be a $\vee_*$-set. If $(X, \tau, \mathcal{I})$ is not a $*$-$T_{\frac{1}{2}}$-space, then there exists a $*$-g-closed set A which is not closed in X. Therefore there exists an element $x \in X$ with $x \in cl(A)$ but $x \notin A$. Now by Theorem 3.5, $\{x\}$ is either an open set or a $g.\vee_*$-set.
Case I: If $\{x\}$ is open, then $x \in cl(A) \Rightarrow \{x\} \bigcap A \neq \phi \Rightarrow x \in A$, a contradiction.
Case II: If $\{x\}$ is a $g.\vee_*$-set, then by hypothesis $\{x\}$ is a $\vee_*$-set and hence a $*$-closed set. Thus $A \subseteq X \setminus \{x\} \Rightarrow cl(A) \subseteq X \setminus \{x\}$ which contradicts that $x \in cl(A)$. □

Corollary 4.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $(X, \tau, \mathcal{I})$ is a $*$-$T_{\frac{1}{2}}$-space if and only if every $g.\wedge_*$-set is a $\wedge_*$-set.*

Proof. Follows from Definition 2.2 and 3.2, and Theorem 4.3 above. □

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