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character amenable Banach algebras

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by

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CHARACTER AMENABLE BANACH ALGEBRAS

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Abstract

In this thesis, we survey the properties of character amenable Banach algebras. Character amenability is a cohomological property weaker than the classical amenability introduced by B.E. Johnson. We give characterization of character amenability in terms of bounded approximate identities and certain topologically invariant elements of the second dual. In addition, we obtain equivalent characterizations of character amenability of Banach algebras in terms of variances of the approximate diagonal and the virtual diagonal. We show that character amenability for either the group algebra $L^1(G)$ or the Herz–Figà-Talamanca algebra $A_p(G)$ is equivalent to the amenability of the underlying group G. We also discuss hereditary properties of character amenability. In the case of uniform algebras we obtain complete characterization of character amenability in term of the Choquet boundary of the underlying space. In addition, we discuss character amenable version of the reduction of dimension formula and splitting properties of modules over character amenable Banach algebras.

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Contents

CHAPTER 1

Introduction and Preliminaries

1.1. Introduction

Homological algebra is a powerful tool in mathematics, used mainly for classifying objects with various categories with algebraic structures. Amenability is a cohomological property introduced by B.E. Johnson and, independently by Y. Helemskii [**17**] in 1972. Banach algebras that are amenable have, by definition, trivial first cohomology groups, provided that the coefficients of the group are taken in dual Banach modules. The first cohomology group measures the obstruction to a continuous derivation on A to be an inner derivation. Therefore, amenable Banach algebras are those for which every continuous derivation into a dual Banach module is automatically inner.

It has been realized by many authors that sometimes a variation of the classical notion of amenability is better suited for the study of particular classes of Banach algebras. Over the years, many different variations of amenability have been introduced, among which one can mention: weak amenability by Bade, Curtis and Dales [**2**], approximate amenability by Ghahramani and Loy [**16**] , operator amenability by Z.J. Ruan [**28**] , Connes amenability by V. Runde [**30**], and more recently character amenability by Kaniuth-Lau-Pym [**25**] and Sangani Monfared [**31**]. Each of these variations either show greater flexibility for particular types of Banach algebras, or have properties not shared by classical amenability. The book by V. Runde [**29**] is a good survey of these various types of amenability. The purpose of this thesis is a study of character amenable Banach algebras.

Character amenability is weaker than the classical amenability introduced by B.E. Johnson. The definition requires continuous derivations from A into dual Banach A-bimodules to be inner, but only those modules are considered where either of the left or right module action is defined by a character of A. In chapter 1, we introduce some basic definitions about character amenability of Banach algebras. In chapter 2, we characterize character amenability in terms of bounded approximate identities and certain topological invariant elements of the second dual. In theorem 2.2.6, we prove that left φ-amenability is equivalent to the existence of a bounded left φ-approximate diagonal, which in turn is equivalent to the existence of a left φ-virtual diagonal. In theorem 2.2.17, we show that the character amenability for each of the Banach algebras $L^1(G)$ and $A_p(G)$ is equivalent to the amenability of G.

In chapter 3, we discuss main hereditary properties of character amenability. The connection of left φ-amenability with the existence of bounded left approximate identities is also studied.

In chapter 4, we study character amenability of Banach function algebras. In theorem 4.2.2, we show that if a unital Banach function algebra A on a compact space X is character amenable, then the Choquet boundary of A must coincide with X. In the case of uniform algebras we obtain complete characterization of character amenability in term of the Choquet boundary of the underlying space (Corollary 4.2.4).

In chapter 5, we introduce character amenable version of the reduction of order formula. We also discuss splitting properties of modules over character amenable Banach algebras. In theorem 5.1.17, we show triviality of cohomological groups with coefficients in finite-dimensional Banach modules over character amenable commutative Banach algebras. As a consequence we conclude that all finite-dimensional extensions of commutative character amenable Banach algebra splits strongly. The section ends with another splitting property of short exact sequences over character amenable Banach algebras.

1.2. Basic definitions

Definition 1.2.1. A Banach algebra A is a Banach space with an algebra structure for which the product is continuous, that is, for all x,y in A, $\|xy\| \leqslant \|x\| \, \|y\|$.

DEFINITION 1.2.2. Let A be a Banach algebra. By a left Banach A-module E, we mean a Banach space E together with a continuous bilinear map: $A \times E \to E$, such that for all $a, b \in A, x \in E, \alpha, \beta \in \mathbb{C}$,

(i) $a \cdot (\alpha x + \beta y) = \alpha(a \cdot x) + \beta(a \cdot y)$;

(ii) $(\alpha a + \beta b) \cdot x = \alpha(a \cdot x) + \beta(b \cdot x)$;

(iii) $a \cdot (b \cdot x) = (ab) \cdot x$.

REMARK 1.2.3. Continuity of a bilinear map is equivalent to the existence of $M \geq 0$ such that $\|a \cdot x\| \leq M\|a\| \|x\|$. By renorming E, we may suppose that $M = 1$.

A right Banach A-module is defined analogously. A Banach A-bimodule E is a Banach space which is both left and right Banach A-module and satisfies the following additional property:

(iv) $a \cdot (x \cdot b) = (a \cdot x) \cdot b, \qquad (a, b \in A, x \in E)$.

DEFINITION 1.2.4. Let E be a Banach left A-module. The dual space E^* of E has a canonical Banach right A-module structure defined by

$$\langle f \cdot a, x \rangle_{E^*,E} = \langle f, a \cdot x \rangle_{E^*,E}, \quad (x \in E, f \in E^*, a \in A).$$

Similarly if E is a Banach right A-module, the dual space E^* of E will have a natural Banach left A-module structure defined by

$$\langle a \cdot f, x \rangle_{E^*,E} = \langle f, x \cdot a \rangle_{E^*,E}, \quad (x \in E, f \in E^*, a \in A).$$

DEFINITION 1.2.5. Let A be a Banach algebra and E be a Banach A-bimodule. A linear map $d : A \to E$ is called a derivation if

$$d(ab) = a \cdot d(b) + d(a) \cdot b, \quad (a, b \in A).$$

If $x \in E$ is fixed, then the linear map $\delta_x : A \to E$ defined by

$$\delta_x(a) = a \cdot x - x \cdot a$$

is called the inner derivation at x. Note that inner derivations are automatically continuous linear maps.

DEFINITION 1.2.6. Let $\mathcal{Z}^1(A, E)$ denote the space of all continuous derivations from A into E and let $\mathcal{N}^1(A, E)$ denote the space of all inner derivations from A into E. Then the first Hochschild cohomology group of A with coefficients in E is the quotient vector space:

$$\mathcal{H}^1(A, E) = \mathcal{Z}^1(A, E)/\mathcal{N}^1(A, E).$$

The following is the definition of amenability first introduced by B.E. Johnson [**24**].

DEFINITION 1.2.7. A Banach algebra A is called amenable if $\mathcal{H}^1(A, E^*) = \{0\}$ for every Banach A-bimodule E. In other words, A is amenable if every continuous derivation from A into any dual Banach A-bimodule is an inner derivation.

EXAMPLE 1.2.8. Let $\mathbb{D}$ be the closed unit disk $\{z \in \mathbb{C},\ |z| \leq 1\}$ and $A(\mathbb{D})$ be the Banach algebra (under the supremum norm) of complex-valued functions which are analytic on the open unit disk and continuous up to the boundary. Suppose $x_0 \in \mathbb{D}$ and the module actions of $A(\mathbb{D})$ on $\mathbb{C}$ are given by

$$f \cdot z = z \cdot f := f(x_0)z, \quad (f \in A(\mathbb{D}), z \in \mathbb{C}).$$

Define

$$d : A(\mathbb{D}) \to \mathbb{C}, \quad f \mapsto f'(x_0).$$

Then d is a continuous derivation since for every $f, g \in A(\mathbb{D})$,

$$\begin{aligned} d(fg) &= (fg)'(x_0) = (f'g + fg')(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) \\ &= d(f)g(x_0) + f(x_0)d(g) = d(f) \cdot g + f \cdot d(g). \end{aligned}$$

But every inner derivation δ_z at z is zero since $\delta_z(f) = f \cdot z - z \cdot f = f(x_0)z - f(x_0)z = 0$. Thus $A(\mathbb{D})$ is not amenable.

Next we introduce the subject of amenability for groups and explains why the same terminology is used in both cases, even though the connection is not obvious from the given definitions.

Let G be a locally compact group. For $x \in G$ and $\varphi \in L^\infty(G)$, we define ${}_x\varphi$ to be the left translation by x, so ${}_x\varphi(t) = \varphi(xt)$ for any $t \in G$.

DEFINITION 1.2.9. A mean on $L^\infty(G)$ is a linear functional $\Phi \in L^\infty(G)^*$ such that $\Phi(1) = \|\Phi\| = 1$.

The above definition of mean corresponds to the definition of a state in a C^*-algebra context.

DEFINITION 1.2.10. Let G be a locally compact group. Then a mean Φ on $L^\infty(G)$ is called left invariant if

$$\langle \Phi, {}_x\varphi \rangle = \langle \Phi, \varphi \rangle, \quad (\varphi \in L^\infty(G),\ x \in G).$$

DEFINITION 1.2.11. A locally compact group G is called amenable if there is a left invariant mean on $L^\infty(G)$.

A natural question is to ask whether amenable group also possesses a right translation mean. The following lemma is proved in Runde [**29**, Theorem 1.1.11, p. 22].

LEMMA 1.2.12. *For a locally compact group G the following are equivalent:*

(i) *G is amenable.*

(ii) *There is a right invariant mean on $L^\infty(G)$.*

(iii) *There is a two-sided invariant mean on $L^\infty(G)$.*

There seems to be no obvious connection between invariant means on a locally compact group G and derivations on the Banach algebra $L^1(G)$. However, Johnson [**24**, theorem 2.5, p. 32] proved a remarkable relation between them.

THEOREM 1.2.13. *Let G be a locally compact group. Then G is amenable if and only if $L^1(G)$ is amenable as a Banach algebra.*

There are many characterizations of the notion of amenability, of which we note the following.

Let A be a Banach algebra. We denote the projective tensor product of A with itself by $A \widehat{\otimes} A$ (see section 1.3). Then diagonal operator on $A \widehat{\otimes} A$ is defined by

$$\pi_A : A \widehat{\otimes} A \to A, \quad a \otimes b \mapsto ab.$$

It is easy to check the projective tensor product $A\widehat{\otimes}A$ becomes a Banach A-bimodule by the following module actions:

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca, \quad (a,\, b,\, c \in A),$$

and π_A is a bimodule homomorphism with respect to this module structure on $A\widehat{\otimes}A$.

DEFINITION 1.2.14. Let A be a Banach algebra.

(i) An element M in $A\widehat{\otimes}A$ is called a diagonal for A if $a \cdot M - M \cdot a = 0$ and $a\,\pi_A(M) = a$ for all $a \in A$.

(ii) An elememt M in $(A\widehat{\otimes}A)^{**}$ is called a virtual diagonal for A if $a \cdot M = M \cdot a$ and $a \cdot \pi_A^{**}(M) = a$ for all $a \in A$.

(iii) A bounded net $(m_\alpha)_\alpha$ in $A\widehat{\otimes}A$ is called an approximate diagonal for A if $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $a\,\pi_A(m_\alpha) \to a$ for all $a \in A$.

The following characterization of amenability is due to Johnson [**23**, Theorem 1.3, p. 688].

THEOREM 1.2.15. *For a Banach algebra A, the following are equivalent:*

(i) *A is amenable.*

(ii) *There is an approximate diagonal for A.*

(iii) *There is a virtual diagonal for A.*

DEFINITION 1.2.16. Let $(A,\, \|\cdot\|)$ be a normed algebra. A left (right) approximate identity for A is a net $(e_\alpha)_\alpha$ in A such that $\lim_\alpha e_\alpha a = a$ ($\lim_\alpha a e_\alpha = a$) for each $a \in A$. A two-sided approximate identity for A is a net $(e_\alpha)_\alpha$ which is both a left and a right approximate identity. The approximate identity is called bounded if $\sup_\alpha \|e_\alpha\| < \infty$.

The following theorem is a stronger version of Cohen's factorization theorem shown in [**8**, Corollary 2.9.26, p. 314].

THEOREM 1.2.17. *Let A be a Banach algebra with a left approximate identity bounded by M and E be a left Banach A-module. Then $\overline{AE} = A \cdot E$ is a closed* *submodule of E. Moreover, for each* $x \in \overline{AE}$ *with* $\|x\| < 1$ *and each* $\epsilon > 0$*, there exists* $a \in A$ *and* $y \in \overline{A \cdot x}$ *such that* $x = a \cdot y$*,* $\|x - y\| < \epsilon$*, and* $\|a\|\|y\| < M$*.*

1.3. Tensor products

This section gives a brief introduction to tensor products. For a detailed study of tensor products, we refer to T. Palmer [**26**].

DEFINITION 1.3.1. Let A, B be linear spaces. A tensor product of A and B is a pair (M, τ), where M is a linear space and

$$\tau : A \times B \to M$$

is a bilinear map with the following property: for each linear space F and for each bilinear map $V : A \times B \to F$, there exists a unique linear map $\widetilde{V} : M \to F$ such that $V = \widetilde{V} \circ \tau$.

It can be shown that given any two linear spaces A and B, the tensor product of A and B always exists and is unique up to isomorphism. Given linear spaces A, B and their tensor product (M, τ), we write $A \otimes B$ for M and define

$$a \otimes b = \tau(a, b), \quad (a \in A,\ b \in B). \tag{I}$$

Elements of $A \otimes B$ are called tensors and elements of the form (I) are called elementary tensors. Every element of $A \otimes B$ can be represented as a finite sum of elementary tensors.

We can form tensor products in various categories: vector spaces, modules, algebras, Banach spaces, and Banach algebras. In each case, the product $A \otimes B$ will inherit the structural property of A and B. For example, if A and B are algebras, then the product on $A \otimes B$ is defined by $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2,$ where $a_1, a_2 \in A, b_1, b_2 \in B$.

Let A, B be normed spaces. For $x \in A \otimes B$, let us define

$$\|x\|_\pi = \inf \left\{ \sum_{i=1}^{m} \|a_i\|\ \|b_i\| :\ x = \sum_{i=1}^{m} a_i \otimes b_i,\ m \in \mathbb{N} \right\}$$

It can be shown that $\|\cdot\|_\pi$ defines a norm, called the projective norm on $A \otimes B$. It follows from the definition that $\|a \otimes b\| \leq \|a\|\,\|b\|$ for every elementary tensor $a \otimes b$.

DEFINITION 1.3.2. Let A, B be Banach spaces. Then the projective tensor product $A\widehat{\otimes}B$ is the completion of $A \otimes B$ with respect to $\|\cdot\|_\pi$.

THEOREM 1.3.3. *Let A, B be Banach spaces. For every $x \in A\widehat{\otimes}B$, there are sequences $(a_i)_{i=1}^\infty$ in A, $(b_i)_{i=1}^\infty$ in B, such that*

$$\sum_{i=1}^{\infty} \|a_i\|\,\|b_i\| < \infty \quad \textit{and} \quad x = \sum_{i=1}^{\infty} a_i \otimes b_i.$$

Furthermore,

$$\|x\|_\pi = \inf\left\{\sum_{i=1}^{\infty} \|a_i\|\,\|b_i\| < \infty, \quad x = \sum_{i=1}^{\infty} a_i \otimes b_i\right\}.$$

It can be shown that if A and B are Banach algebras, then $\|\cdot\|_\pi$ is submultiplicative with respect to the product on $A\widehat{\otimes}B$, so it turns $A\widehat{\otimes}B$ into a Banach algebra.

If A is a Banach algebra, then $A\widehat{\otimes}A$ has a canonical Banach A-bimodule action defined by

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca, \quad (a, b, c \in A).$$

This induces a canonical Banach A-bimodule structure on $(A\widehat{\otimes}A)^*$ and $(A\widehat{\otimes}A)^{**}$. If we use the identification $(A\widehat{\otimes}A)^* \cong \mathcal{B}(A, A^*)$, then the canonical Banach A-bimodule action on $(A\widehat{\otimes}A)^*$ takes the following form:

$$(a \cdot T)(b) = a \cdot (Tb), \quad (T \cdot a)(b) = T(ab), \quad (a, b \in A, T \in \mathcal{B}(A, A^*)).$$

To prove these, we note that for all $c \in A$,

$$\begin{aligned} \langle (a \cdot T)(b), c\rangle :&= \langle a \cdot T, b \otimes c\rangle_{(A\widehat{\otimes}A)^*, A\widehat{\otimes}A} = \langle T, (b \otimes c) \cdot a\rangle \\ &= \langle T, b \otimes ca\rangle = \langle Tb, ca\rangle_{A^*,A} = \langle a \cdot (Tb), c\rangle_{A^*,A}. \end{aligned}$$

Thus $(a \cdot T)(b) = a \cdot (Tb)$ and similarly we have $(T \cdot a)(b) = T(ab)$.

In the particular case that $T \in \mathcal{B}(A, A^*)$ is given by $T = g \otimes f$, $f, g \in A^*$, where $T(a) = (g \otimes f)(a) := g(a)f \in A^*$, it is easy to check that

$$a \cdot (g \otimes f) = g \otimes (a \cdot f), \quad (g \otimes f) \cdot a = (g \cdot a) \otimes f,$$

since for every $b \in A$,

$$[a \cdot (g \otimes f)](b) = a \cdot ((g \otimes f)(b)) = a \cdot (g(b)f) = g(b)(a \cdot f) = [g \otimes a \cdot f](b),$$

$$[(g \otimes f) \cdot a](b) = (g \otimes f)(ab) = g(ab)f = (g \cdot a)(b)f = [(g \cdot a) \otimes f](b).$$

CHAPTER 2

Character Amenability and its Properties

2.1. Character amenability

Let A be a Banach algebra and $\sigma(A)$ be the spectrum of A which is the set of all non-zero multiplicative linear functionals on A. Given $\varphi \in \sigma(A) \cup \{0\}$, we denote by $\mathcal{M}_{\varphi}^{A}$ the set of all Banach A-bimodules E for which the right module action is given by $x \cdot a = \varphi(a)x,\ a \in A,\ x \in E$. Similarly, we denote by ${}_{\varphi}\mathcal{M}^{A}$ the set of all Banach A-bimodules E for which the left module action is given by $a \cdot x = \varphi(a)x$, $a \in A,\ x \in E$.

Definition 2.1.1. Let A be a Banach algebra and $\varphi \in \sigma(A) \cup \{0\}$. We call A left φ-amenable if every continuous derivation $d : A \to E^*$ is inner for all $E \in \mathcal{M}_{\varphi}^{A}$. Moreover, A is called left character amenable if it is left φ-amenable for every $\varphi \in \sigma(A) \cup \{0\}$.

Right character amenability is defined analogously by considering $E \in {}_{\varphi}\mathcal{M}^{A}$. We call A character amenable if it is both left and right character amenable.

Remark 2.1.2. The above definitions show that character amenability is weaker than amenability. In other words, all amenable Banach algebras are automatically character amenable.

A natural question is under what conditions a Banach algebra is character amenable. We answer this question in proposition 2.2.2 and theorem 2.2.6.

Definition 2.1.3. Let A be a Banach algebra, $\varphi \in \sigma(A) \cup \{0\}$, and $\Phi \in A^{**}$.

(i) Φ is called φ-topologically left invariant element (φ-TLIE) if

$$\langle \Phi, a \cdot f \rangle = \varphi(a)\langle \Phi, f \rangle, \quad (a \in A,\ f \in A^*).$$

Equivalently, Φ is φ-TLIE if $\Phi \cdot a = \varphi(a)\Phi, \quad (a \in A,\ f \in A^*)$.

(ii) Φ is called φ-topologically right invariant element (φ-TRIE) if

$$\langle \Phi, f \cdot a\rangle = \varphi(a)\langle \Phi, f\rangle, \quad (a \in A,\ f \in A^*).$$

Equivalently, Φ is φ-TRIE if $a \cdot \Phi = \varphi(a)\Phi, \quad (a \in A,\ f \in A^*)$.

2.2. Basic properties

LEMMA 2.2.1. *Let A be a Banach algebra. Then A is left 0-amenable if and only if A has a bounded left approximate identity.*

PROOF. Consider A^* equipped with its canonical left A-module action, and with the trivial right A-module action defined by:

$$f \cdot a = 0, \quad (a \in A, f \in A^*).$$

Then $A^* \in \mathcal{M}_0^A$ and $A^{**} \in {}_0\mathcal{M}^A$. Let τ be the canonical embedding of A into its second dual A^{**}. Of course, τ is linear and continuous, i.e., $\tau \in \mathcal{B}(A, A^{**})$. Moreover, for all $a, b \in A, f \in A^*$,

$$\begin{aligned}\langle a \cdot \tau(b) + \tau(a) \cdot b, f\rangle &= \langle \tau(a) \cdot b, f\rangle = \langle \tau(a), b \cdot f\rangle \\ &= \langle b \cdot f, a\rangle = \langle f, ab\rangle \\ &= \langle \tau(ab), f\rangle.\end{aligned}$$

So $\tau \in \mathcal{Z}^1(A, A^{**})$. By the assumption of left 0-amenability of A, there exists $\mu \in A^{**}$ such that for all $a \in A$,

$$\tau(a) = \delta_\mu(a) = a \cdot \mu - \mu \cdot a = -\mu \cdot a.$$

In other words, for every $f \in E^*$, $\langle \tau(a), f\rangle = \langle -\mu \cdot a, f\rangle$, which implies $\langle f, a\rangle = \langle \mu, -a \cdot f\rangle$. Since $A \hookrightarrow A^{**}$ is w^*-dense in A^{**} by Goldstine's theorem [**10**, V.4.6 Theorem 5, p. 424], it follows that for such $\mu \in A^{**}$, there exists a bounded net $(e_\alpha)_\alpha$ in A, with $\|e_\alpha\| \le \|\mu\|$ and $\mu = w^*\text{-}\lim_\alpha \tau(e_\alpha)$. So we have

$$\langle \mu, -a \cdot f\rangle = \lim_\alpha \langle \tau(e_\alpha), -a \cdot f\rangle = \lim_\alpha \langle -a \cdot f, e_\alpha\rangle = \lim_\alpha \langle f, -e_\alpha a\rangle.$$

Therefore $\langle f, a\rangle = \lim_\alpha \langle f, -e_\alpha a\rangle$, i.e., A has a bounded weak left approximate identity $(-e_\alpha)_\alpha$. It is well-known that if A has a bounded weak left approximate identity, then A has a bounded left approximate identity [**9**, Proposition 33.2, p. 223].

For the converse, let $E \in \mathcal{M}_0^A$. It is clear that $E^* \in {}_0\mathcal{M}^A$ and $A \cdot E^* = \{0\}$. Let $d \in \mathcal{Z}^1(A, E^*)$. Then for $a, b \in A$,

$$d(ab) = a \cdot d(b) + d(a) \cdot b = d(a) \cdot b.$$

Let $(e_\alpha)_\alpha$ be a bounded left approximate identity for A. Then $d(e_\alpha) \in E^*$, for each α. By Anaoglu's theorem [**10**, V.4.6 Theorem 2, p. 424], there exists $f \in E^*$ such that f is the w^*-cluster point of $d(e_\alpha)$. By passing to a subnet of $(e_\alpha)_\alpha$ if necessary, we may assume $f = w^*\text{-}\lim_\alpha d(e_\alpha)$. Then for $a \in A$,

$$d(a) = \|\cdot\|\text{-}\lim_\alpha d(e_\alpha a) = w^*\text{-}\lim_\alpha d(e_\alpha) \cdot a = f \cdot a.$$

The last identity holds since for every $y \in E$,

$$\lim_\alpha \langle d(e_\alpha) \cdot a, y\rangle = \lim_\alpha \langle d(e_\alpha), a \cdot y\rangle = \langle f, a \cdot y\rangle = \langle f \cdot a, y\rangle.$$

Put $f' = -f \in E^*$. Then for every $a \in A$,

$$d(a) = -f' \cdot a = a \cdot f' - f' \cdot a = \delta_{f'}(a).$$

Hence A is left 0-amenable. □

The characterization of left character amenability was shown in [**31**, Theorem 2.3, p. 699].

PROPOSITION 2.2.2. *Let A be a Banach algebra. Then A is left character amenable if and only if A has a bounded left approximate identity and for every $\varphi \in \sigma(A)$ there exists a φ-topologically left invariant element $\Phi \in A^{**}$ such that $\Phi(\varphi) \neq 0$.*

Similar statement holds for right character amenability.

PROOF. By lemma 2.2.1 it remains to show that left φ-amenability is equivalent to the existence of φ-topologically left invariant element for every $\varphi \in \sigma(A)$.

Let $\varphi \in \sigma(A)$, consider A^*, the dual of A equipped with the usual left A-module action and the right action defined by

$$f \cdot a = \varphi(a) f, \quad (a \in A,\ f \in A^*).$$

Then A^{**} will have the usual right A-module action and the left action given by

$$a \cdot \Phi = \varphi(a)\Phi, \quad (a \in A,\ \Phi \in A^{**}).$$

Since φ is multiplicative on A, it follows that for every $b \in A$,

$$\langle a \cdot \varphi, b \rangle = \langle \varphi, ba \rangle = \varphi(b)\varphi(a) = \langle \varphi(a)\varphi, b \rangle.$$

Hence $a \cdot \varphi = \varphi(a)\varphi$ and similarly $\varphi \cdot a = \varphi(a)\varphi$. So φ generates one-dimensional submodule of A^*. We define

$$E = A^*/\mathbb{C}\varphi$$

to be the Banach quotient A-bimodule, and let

$$P : A^* \to E$$

be the canonical A-bimodule quotient map. Identifying E^* with the closed subspace of A^{**} vanishing on φ, the adjoint map

$$P^* : E^* \to A^{**}$$

will be the natural inclusion of E^* into A^{**}.

By the Hahn-Banach theorem, there exists $\Phi_0 \in A^{**}$ with $\Phi_0(\varphi) = 1$. Define

$$\delta_{\Phi_0} : A \to A^{**}, \quad \delta_{\Phi_0}(a) = a \cdot \Phi_0 - \Phi_0 \cdot a = \varphi(a)\Phi_0 - \Phi_0 \cdot a,$$

to be the inner derivation at Φ_0. It is easy to check that $\delta_{\Phi_0}(a) \in E^* \cong (\mathbb{C}\varphi)^{\perp}$ for all $a \in A$, in fact for any $\varphi \in \sigma(A)$,

$$\langle \delta_{\Phi_0}(a), \varphi \rangle = \langle \varphi(a)\Phi_0 - \Phi_0 \cdot a, \varphi \rangle = \langle \Phi_0, \varphi(a)\varphi \rangle - \langle \Phi_0, a \cdot \varphi \rangle = \langle \Phi_0, \varphi(a)\varphi \rangle - \langle \Phi_0, \varphi(a)\varphi \rangle = 0.$$

Thus the map

$$d : A \to E^*, \quad d(a) = \delta_{\Phi_0}(a) \in E^*$$

is a continuous derivation. But d is not an inner derivation by Φ_0 since $\Phi_0 \notin E^*$. Moreover, for every $f \in A^*, a \in A$,

$$(f + \mathbb{C}\varphi) \cdot a = f \cdot a + \mathbb{C}\varphi = \varphi(a)f + \mathbb{C}\varphi = \varphi(a)(f + \mathbb{C}\varphi),$$

that is $E \in \mathcal{M}_\varphi^A$. By our assumption of left character amenability of A, d must be inner and hence there exists $\Phi_1 \in E^*$ such that $d = \delta_{\Phi_1}$. We show that $\Phi := \Phi_0 - \Phi_1$ is the required φ-TLIE of A^{**}. In fact, firstly

$$\langle \Phi, \varphi \rangle = \langle \Phi_0 - \Phi_1, \varphi \rangle = 1 - 0 = 1.$$

Next, for every $a \in A$, we have

$$\begin{aligned}
\delta_{\Phi_0}(a) = \delta_{\Phi_1}(a) &\Longrightarrow a \cdot \Phi_0 - \Phi_0 \cdot a = a \cdot \Phi_1 - \Phi_1 \cdot a \\
&\Longrightarrow \varphi(a)\Phi_0 - \Phi_0 \cdot a = \varphi(a)\Phi_1 - \Phi_1 \cdot a \\
&\Longrightarrow (\Phi_0 - \Phi_1) \cdot a = \varphi(a)(\Phi_0 - \Phi_1) \\
&\Longrightarrow \Phi \cdot a = \varphi(a)\Phi.
\end{aligned}$$

Hence Φ is φ-TLIE with $\langle \Phi, \varphi \rangle \neq 0$.

It remains to prove the sufficiency part of the theorem. Let $\varphi \in \sigma(A)$ and $\Phi \in A^{**}$ be a φ-TLIE such that $\Phi(\varphi) \neq 0$. Suppose d is a continuous derivation from A into E^*, where $E \in \mathcal{M}_\varphi^A$. It suffices to show there exists $g \in E^*$, such that $d(a) = \delta_g(a) = a \cdot g - g \cdot a$. Let

$$d^* : E^{**} \to A^* \quad \text{and } d^{**} : A^{**} \to E^{***}$$

be the adjoint and double adjoint of d, respectively. Identifying E with its canonical image in E^{**}, we define

$$f = d^{**}(\Phi)|_E \in E^*.$$

We claim that $d = \delta_{f/\Phi(\varphi)}$. We observe that elements of A with $\varphi(a) = 1$ linearly span the entire A. In fact, if $\varphi(a) \neq 0$, then a can be written as the form of $\varphi(a)\frac{a}{\varphi(a)}$. Otherwise, there exists some $b \in A$ such that $\varphi(b) = 1$, so $a = \frac{b+a}{2} - \frac{b-a}{2}$.

Therefore it suffices to show that

$$d(a) = \delta_{f/\Phi(\varphi)}(a), \quad (a \in A \ \text{ with } \varphi(a) = 1).$$

In fact, for $x \in E$,

$$\begin{aligned}
\langle \delta_f(a), x\rangle &= \langle a \cdot f - f \cdot a, x\rangle \\
&= \langle f, x \cdot a\rangle - \langle f, a \cdot x\rangle \\
&= \langle f, x \cdot a - a \cdot x\rangle \\
&= \langle f, x - a \cdot x\rangle \qquad (\text{since } x \cdot a = \varphi(a)x = x) \\
&= \langle d^{**}(\Phi), x - a \cdot x\rangle \\
&= \langle \Phi, d^*(x) - d^*(a \cdot x)\rangle. \qquad\qquad (*)
\end{aligned}$$

But for every $b \in A$,

$$\begin{aligned}
\langle d^*(a \cdot x), b\rangle_{A^*,A} &= \langle a \cdot x, d(b)\rangle_{E,E^*} = \langle x, d(b) \cdot a\rangle_{E,E^*} \\
&= \langle x, d(ba) - b \cdot d(a)\rangle_{E,E^*} \\
&= \langle d^*(x), ba\rangle_{A^*,A} - \langle x \cdot b, d(a)\rangle_{E,E^*} \\
&= \langle a \cdot d^*(x), b\rangle_{A^*,A} - \langle \varphi(b)x, d(a)\rangle_{E,E^*} \\
&= \langle a \cdot d^*(x), b\rangle_{A^*,A} - \langle d(a), x\rangle_{E^*,E}\langle \varphi, b\rangle_{A^*,A} \\
&= \langle a \cdot d^*(x) - \langle d(a), x\rangle \varphi, b\rangle_{A^*,A}.
\end{aligned}$$

So $d^*(a \cdot x) = a \cdot d^*(x) - \langle d(a), x\rangle\varphi$.

Thus we can rewrite the equation $(*)$ as

$$\begin{aligned}
\langle \delta_f(a), x\rangle &= \langle \Phi, d^*(x) - d^*(a \cdot x)\rangle \\
&= \langle \Phi, d^*(x) - a \cdot d^*(x) + \langle d(a), x\rangle\varphi\rangle \\
&= \langle \Phi, d^*(x)\rangle - \langle \Phi, \varphi(a)d^*(x)\rangle + \Phi(\varphi)\langle d(a), x\rangle \ (\text{since } \Phi \text{ is a } \varphi\text{-TLIE}) \\
&= \langle \Phi, d^*(x)\rangle - \langle \Phi, d^*(x)\rangle + \Phi(\varphi)\langle d(a), x\rangle \quad (\text{since } \varphi(a) = 1) \\
&= \langle \Phi(\varphi)d(a), x\rangle.
\end{aligned}$$

Therefore $\delta_f(a) = \Phi(\varphi)d(a)$, or equivalently $d = \delta_{f/(\Phi(\varphi))}$. Hence A is left φ-amenable. $\square$

Corollary 2.2.3. *If A is a commutative Banach algebra, then A is left character amenable if and only if it is right character amenable.*

Proof. Let $\varphi \in \sigma(A)$. Suppose A is left character amenable, then A has a bounded left approximate identity $(e_\alpha)_\alpha$ and a φ-TLIE Φ in A^{**} such that $\Phi(\varphi) \neq 0$. Since A is commutative, $(e_\alpha)_\alpha$ is also a bounded right approximate identity for A. Moreover, for every $a, b \in A$ and $f \in A^*$,

$$\langle f \cdot a, b\rangle = \langle f, ab\rangle = \langle f, ba\rangle = \langle a \cdot f, b\rangle.$$

So

$$f \cdot a = a \cdot f \quad \text{and} \quad \langle \Phi, f \cdot a\rangle = \langle \Phi, a \cdot f\rangle = \varphi(a)\langle \Phi, f\rangle.$$

That is Φ is also a φ-TRIE for A. Hence A is right character amenable. The proof of the other direction is similar. $\square$

Definition 2.2.4. Let A be a Banach algebra and $\varphi \in \sigma(A)$. A left φ-approximate diagonal for A is a net $(m_\alpha)_\alpha$ in $A\widehat{\otimes}A$ such that $m_\alpha \cdot a - \varphi(a)m_\alpha \to 0$ in the norm topology of $A\widehat{\otimes}A$ for every $a \in A$ and $\langle \varphi \otimes \varphi, m_\alpha\rangle = \varphi(\pi(m_\alpha)) \to 1$. A right φ-approximate diagonal is defined similarly.

To justify the equality $\langle \varphi \otimes \varphi, m_\alpha\rangle = \varphi(\pi(m_\alpha))$, we argue as follows: if $m_\alpha = \sum_{i=1}^{\infty} a_i^\alpha \otimes b_i^\alpha$, where a_i^α , $b_i^\alpha \in A$, then using the absolute convergence of the sum, we have

$$\varphi(\pi(m_\alpha)) = \varphi(\sum_{i=1}^{\infty} a_i^\alpha b_i^\alpha) = \sum_{i=1}^{\infty} \varphi(a_i^\alpha)\varphi(b_i^\alpha).$$

Also,

$$\langle \varphi \otimes \varphi, m_\alpha\rangle = \langle \varphi \otimes \varphi, \sum_{i=1}^{\infty} a_i^\alpha \otimes b_i^\alpha\rangle = \sum_{i=1}^{\infty} \varphi(a_i^\alpha)\varphi(b_i^\alpha),$$

from which the equality in question follows.

Definition 2.2.5. An element M of $(A\widehat{\otimes}A)^{**}$ is called a left φ-virtual diagonal for A, if $M \cdot a = \varphi(a)M$ for every $a \in A$ and $\langle M, \varphi \otimes \varphi\rangle = \langle \pi^{**}(M), \varphi\rangle = 1$.

The equality $\langle M, \varphi\otimes\varphi\rangle = \langle \pi^{**}(M), \varphi\rangle$ in the above definition needs justification. In fact, for every x, y in A,

$$\langle \pi^*(\varphi), x\otimes y\rangle = \langle \varphi, \pi(x\otimes y)\rangle = \langle\varphi, xy\rangle = \varphi(xy) = \varphi(x)\varphi(y) = \langle \varphi\otimes\varphi, x\otimes y\rangle,$$

therefore $\pi^*(\varphi) = \varphi\otimes\varphi$. Hence $\langle \pi^{**}(M), \varphi\rangle = \langle M, \pi^*(\varphi)\rangle = \langle M, \varphi\otimes\varphi\rangle$.

The characterizations of left φ-amenability were shown in [**21**, Theorem 2.3, p. 56].

THEOREM 2.2.6. *Let A be a Banach algebra and $\varphi \in \sigma(A)$. Then the following are equivalent:*

(i) *A has a left φ-virtual diagnonal.*
(ii) *A has a bounded left φ-approximate diagonal.*
(iii) *A is left φ-amenable.*
(iv) *There exists a φ-TLIE, $\Phi \in A^{**}$ such that $\Phi(\varphi) = 1$.*
(v) *There exists a bounded net $(u_\alpha)_\alpha$ in A such that $u_\alpha \cdot a - \varphi(a)u_\alpha \to 0$ for all $a \in A$ and $\varphi(u_\alpha) = 1$ for all α.*

PROOF. (ii) $\Rightarrow$ (i) If A has a bounded left φ-approximate diagonal $(m_\alpha)_\alpha$, then by Alaogu's theorem [**10**, V.4.6 Theorem 2, p. 424] there exists $M \in (A\widehat{\otimes}A)^{**}$ such that M is a w^*-cluster point of the canonical image of m_α. Then by going to a subnet of (m_α) if necessary, we may assume $M = w^*\text{-}\lim_\alpha m_\alpha$. So

$$\langle M, \varphi\otimes\varphi\rangle = \lim_\alpha \langle m_\alpha, \varphi\otimes\varphi\rangle = \lim_\alpha \langle \varphi\otimes\varphi, m_\alpha\rangle = 1.$$

Note that for fixed $a \in A$,

$$R_a : (A\widehat{\otimes}A)^{**} \to (A\widehat{\otimes}A)^{**}, \quad M \mapsto M\cdot a,$$

is w^*-continuous, since R_a is the double adjoint map of the right module multiplication

$$r_a : A\widehat{\otimes}A \to A\widehat{\otimes}A, \quad (b\otimes c) \mapsto (b\otimes c)\cdot a = b\otimes ca,$$

which implies

$$M\cdot a = w^*\text{-}\lim_\alpha (m_\alpha \cdot a),$$

and

$$M\cdot a-\varphi(a)M = w^*\text{-}\lim_\alpha(m_\alpha\cdot a)-w^*\text{-}\lim_\alpha(\varphi(a)m_\alpha) = w^*\text{-}\lim_\alpha(m_\alpha\cdot a-\varphi(a)m_\alpha) = 0,$$

since by assumption $\|m_\alpha \cdot a - \varphi(a)m_\alpha\| \to 0$.
(i) $\Rightarrow$ (iv) Suppose A has a left φ-virtual diagonal $M \in (A\widehat{\otimes}A)^{**}$, define $\Phi \in A^{**}$ by

$$\Phi(f) := \langle M, \varphi \otimes f\rangle, \quad (f \in A^*).$$

Then

$$\Phi(\varphi) = \langle M, \varphi \otimes \varphi\rangle = \langle \pi^{**}(M), \varphi\rangle = 1.$$

Moreover, $M \cdot a = \varphi(a)M$, therefore for all $f \in A^*$,

$$\begin{aligned}\langle \Phi, a\cdot f\rangle &= \langle M, \varphi\otimes(a\cdot f)\rangle = \langle M, a\cdot(\varphi\otimes f)\rangle = \langle M\cdot a, \varphi\otimes f\rangle\\ &= \langle \varphi(a)M, \varphi\otimes f\rangle = \varphi(a)\langle M, \varphi\otimes f\rangle = \varphi(a)\langle\Phi, f\rangle.\end{aligned}$$

The equivalence of (iii) and (iv) was shown in the proof of proposition 2.2.2.
(iii) $\Rightarrow$ (i) Consider the Banach A-bimodule $A\widehat{\otimes}A$ with the module actions given by

$$a\cdot(b\otimes c) = \varphi(a)(b\otimes c), \quad (b\otimes c)\cdot a = b\otimes ca, \quad (a,\ b,\ c \in A).$$

Then the right module action on $(A\widehat{\otimes}A)^*$ is defined by

$$f\cdot a = \varphi(a)f, \quad (f \in (A\widehat{\otimes}A)^*, a \in A).$$

So $(A\widehat{\otimes}A)^{**}$ will have the canonical right A-module action, while the left action is given by

$$a\cdot\Phi = \varphi(a)\Phi, \quad (\Phi \in (A\widehat{\otimes}A)^{**}, a \in A).$$

Since φ is multiplicative on A, it follows that for every $a, b, c \in A$,

$$\langle a\cdot(\varphi\otimes\varphi), b\otimes c\rangle = \langle \varphi\otimes\varphi, (b\otimes c)\cdot a\rangle = \langle\varphi\otimes\varphi, b\otimes ca\rangle = \varphi(b)\varphi(ca) = \varphi(a)\langle\varphi\otimes\varphi, b\otimes c\rangle.$$

Hence $a\cdot(\varphi\otimes\varphi) = \varphi(a)(\varphi\otimes\varphi)$. Clearly $\varphi\otimes\varphi$ generates a 1-dimensional submodule $\mathbb{C}\cdot(\varphi\otimes\varphi)$ of $(A\widehat{\otimes}A)^*$. We define

$$E = (A\widehat{\otimes}A)^*/\mathbb{C}\cdot(\varphi\otimes\varphi)$$

to be the quotient Banach A-bimodule and

$$Q : (A\widehat{\otimes}A)^* \to E$$

to be the canonical A-bimodule quotient map. Since

$$E^* = ((A\widehat{\otimes}A)^*/\mathbb{C}\cdot(\varphi\otimes\varphi))^* \cong (\mathbb{C}\cdot(\varphi\otimes\varphi))^\perp \subset (A\widehat{\otimes}A)^{**},$$

we can identify E^* with closed subspace of $(A\widehat{\otimes}A)^{**}$ vanishing on $\varphi\otimes\varphi$. Then the adjoint of Q, that is,

$$Q^* : E^* \to (A\widehat{\otimes}A)^{**}$$

will be the canonical inclusion of E^* into $(A\widehat{\otimes}A)^{**}$. By the Hahn-Banach theorem, there exists $\Phi_0 \in (A\widehat{\otimes}A)^{**}$ such that $\langle \Phi_0, \varphi\otimes\varphi\rangle = 1$. Let

$$\delta_{\Phi_0} : A \to (A\otimes A)^{**}, \quad a \mapsto a\cdot\Phi_0 - \Phi_0\cdot a = \varphi(a)\Phi_0 - \Phi_0\cdot a,$$

be the inner derivation defined by Φ_0. It is routine to check that $\delta_{\Phi_0}(a) \in E^*$ for all $a \in A$. In fact,

$$\begin{aligned}\langle \delta_{\Phi_0}(a), \varphi\otimes\varphi\rangle &= \langle \varphi(a)\Phi_0 - \Phi_0\cdot a, \varphi\otimes\varphi\rangle \\ &= \langle \Phi_0, \varphi(a)(\varphi\otimes\varphi)\rangle - \langle \Phi_0, a\cdot(\varphi\otimes\varphi)\rangle \\ &= \langle \Phi_0, \varphi(a)(\varphi\otimes\varphi)\rangle - \langle \Phi_0, \varphi(a)(\varphi\otimes\varphi)\rangle = 0\end{aligned}$$

Define the map

$$d : A \to E^*, \quad a \mapsto \delta_{\Phi_0}(a).$$

Clearly d is a continuous derivation. Moreover, as we saw above, $E \in \mathcal{M}^A_\varphi$. By the assumption of left φ-amenability of A, d must be inner. Hence there exists $\Phi_1 \in E^* \cong (\mathbb{C}\cdot(\varphi\otimes\varphi))^\perp$ such that $d = \delta_{\Phi_0} = \delta_{\Phi_1}$. We claim that $M := \Phi_0 - \Phi_1$ is the required φ-virtual diagonal for A. In fact,

$$\langle M, \varphi\otimes\varphi\rangle = \langle \Phi_0, \varphi\otimes\varphi\rangle - \langle \Phi_1, \varphi\otimes\varphi\rangle = 1 - 0 = 1,$$

and for every $a \in A$,

$$\begin{aligned}\delta_{\Phi_0}(a) = \delta_{\Phi_1}(a) &\Longrightarrow a \cdot \Phi_0 - \Phi_0 \cdot a = a \cdot \Phi_1 - \Phi_1 \cdot a \\ &\Longrightarrow \varphi(a)\Phi_0 - \Phi_0 \cdot a = \varphi(a)\Phi_1 - \Phi_1 \cdot a \\ &\Longrightarrow (\Phi_0 - \Phi_1) \cdot a = \varphi(a)(\Phi_0 - \Phi_1). \\ &\Longrightarrow M \cdot a = \varphi(a)M.\end{aligned}$$

Hence M is a φ-virtual diagonal for A.

(i) $\Rightarrow$ (ii) If A has a left φ-virtual diagonal $M \in (A\widehat{\otimes}A)^{**}$, by Goldstine's theorem [**10**, V.4.6 Theorem 5, p. 424], there exists a bounded net $(e_\alpha)_\alpha$ in $A\widehat{\otimes}A$ such that $\|e_\alpha\| \leq \|M\|$ and $M = w^*\text{-}\lim_\alpha e_\alpha$. Then $e_\alpha \cdot a - \varphi(a)e_\alpha \to 0$ in the weak topology of $A\widehat{\otimes}A$ and $\varphi(\pi(e_\alpha)) = \langle \varphi \otimes \varphi, e_\alpha \rangle \to 0$. In fact, for all $f \in (A\widehat{\otimes}A)^*$,

$$\begin{aligned}\lim_\alpha \langle f, e_\alpha \cdot a - \varphi(a)e_\alpha \rangle &= \lim_\alpha \langle f, e_\alpha \cdot a \rangle - \lim_\alpha \langle f, \varphi(a)e_\alpha \rangle \\ &= \lim_\alpha \langle a \cdot f, e_\alpha \rangle - \lim_\alpha \langle \varphi(a)f, e_\alpha \rangle \\ &= \lim_\alpha \langle a \cdot f - \varphi(a)f, e_\alpha \rangle \\ &= \lim_\alpha \langle M, a \cdot f - \varphi(a)f \rangle \\ &= \lim_\alpha \langle M \cdot a, f \rangle - \lim_\alpha \langle \varphi(a)M, f \rangle \\ &= \lim_\alpha \langle M \cdot a - \varphi(a)M, f \rangle \\ &= \lim \langle 0, f \rangle \\ &= 0.\end{aligned}$$

Thus we have shown that $w\text{-}\lim_\alpha (e_\alpha \cdot a - \varphi(a)e_\alpha) = 0$, for all $a \in A$. Moreover for every $f \in (A\widehat{\otimes}A)^*$,

$$\langle M, f \rangle = \lim_\alpha \langle e_\alpha, f \rangle = \lim_\alpha \langle f, e_\alpha \rangle.$$

In particular, for $\varphi \otimes \varphi \in (A\widehat{\otimes}A)^*$, we have

$$\lim_\alpha \varphi(\pi(e_\alpha)) = \lim_\alpha \langle \varphi \otimes \varphi, e_\alpha \rangle = \langle M, \varphi \otimes \varphi \rangle = 1.$$

Of course, $a = w\text{-}\lim_\alpha \varphi(\pi(e_\alpha))a$. Now fix $F = \{a_1, a_2, \cdots, a_n\} \subset A$ and $\epsilon \geq 0$. The bounded net

$$V = \{(e_\alpha \cdot a_1 - \varphi(a_1)e_\alpha, \varphi(\pi(e_\alpha))a_1 - a_1, \cdots, e_\alpha \cdot a_n - \varphi(a_n)e_\alpha, \varphi(\pi(e_\alpha))a_n - a_n\}$$

converges to 0 in the product space $((A\widehat{\otimes}A) \times A)^n$ with respect to the weak topology (which is equal to the product of weak topologies). By Mazur's theorem [**8**, Theorem A.3.29, p. 818] in a Banach space, each convex set has the same closure in the norm and in the weak topologies. It follows that there is a convex combination $f_{F,\epsilon} = \sum \lambda_j e_{\alpha j}$ with $\sum_{j=1}^m \lambda_j = 1$, such that

$$\|(f_{F,\epsilon}\cdot a_1 - \varphi(a_1)f_{F,\epsilon}, \varphi(\pi(f_{F,\epsilon}))a_1 - a_1, \cdots, f_{F,\epsilon}\cdot a_n - \varphi(a_n)f_{F,\epsilon}, \varphi(\pi(f_{F,\epsilon}))a_n - a_n)\| < \epsilon.$$

Thus we have

$$\|f_{F,\epsilon} \cdot a_i - \varphi(a_i)f_{F,\epsilon}\| < \epsilon, \quad \|\varphi(\pi(f_{F,\epsilon}))a_i - a_i\| < \epsilon, \quad i = 1, 2, \cdots n.$$

If we take $(F, \epsilon) > (F', \epsilon')$ to mean $F \supset F', \epsilon < \epsilon'$, then $\{f_{F,\epsilon}\}$ is a bounded left φ-approximate diagonal.

Finally, the equivalence of (iii) and (v) has been shown in [**25**, Theorem 1.4, p. 88]. □

Corollary 2.2.7. *Let A be a Banach algebra. Then the following are equivalent:*

(i) *A is left character amenable.*
(ii) *A has a bounded left approximate identity and there exists a φ-TLIE $\Phi \in A^{**}$ such that $\Phi(\varphi) \neq 0$ for every $\varphi \in \sigma(A)$.*
(iii) *A has a bounded left approximate identity and has a bounded left φ-approximate diagonal for every $\varphi \in \sigma(A)$.*
(iv) *A has a bounded left approximate identity and has a left φ-virtual diagonal for every $\varphi \in \sigma(A)$.*

Let G be a locally compact group with a fixed left Haar measure.

Recall if $1 \leq p < \infty$,

$$L^p(G) = \left\{ f : G \to \mathbb{C}, \quad f \text{ is measurable }, \ \left(\int_G |f|^p dx \right)^{\frac{1}{p}} < \infty \right\}.$$

In the case $p = 1$, it can be shown that $L^1(G)$ is an algebra with respect to the following convolution product

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) dy, \quad (f, g \in L^1(G), x, y \in G).$$

$L^1(G)$ is called the group algebra of G. We let $A_p(G)$ be the subspace of $\mathcal{C}_0(G)$ consisting of functions of the form

$$u = \sum_{i=1}^{\infty} g_i * \check{f}_i, \quad \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty,$$

where $f_i \in L^p(G), g_i \in L^q(G), \frac{1}{p} + \frac{1}{q} = 1, \check{f}_i(x) = f_i(x^{-1})$. Moreover we let

$$\|u\|_{A_p} = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q, \quad u = \sum_{i=1}^{\infty} g_i * \check{f}_i, \quad f_i \in L^p(G), g_i \in L^q(G) \right\}.$$

It is well known that with the norm $\|u\|_{A_p}$ and usual pointwise operations, $A_p(G)$ becomes a commutative Banach algebra called the Herz–Figà-Talamanca algebra of G [**18**]. In the case $p = 2$, we simply write $A(G)$ for $A_2(G)$. $A(G)$ is called the Fourier algebra of G, introduced by P. Eymard [**11**]. The dual of $A_p(G)$ is the space of $PM_p(G)$, which is the w^*-closure of the set $\{\lambda_p(f) : f \in L^1(G)\}$ in $\mathcal{B}(L^p(G))$. Here $\lambda_p(f)$ is the convolution operator on $L^p(G)$ defined by $\lambda_p(f)(g) = f * g \ (g \in L^p(G))$. In the case of $p = 2$, the dual of $A(G)$ is the von Neumann algebra VN(G) generated by the left translation operators acting on the Hilbert space $L^2(G)$.

$A(G)$ has a close connection with $L^1(G)$. In fact, when G is abelian, $A(G)$ is isometrically isomorphic to $L^1(\widehat{G})$ via the Fourier transform, where $\widehat{G}$ is the dual group of G, consisting of all continuous homomorphisms $\chi : G \to \mathbb{T}$.

Our main goal in the remaining of this section is to show that the character amenability of $L^1(G)$ and $A_p(G)$ are completely characterized by the amenability of their underlying group G. We firstly show two useful identities related to natural left (right) $L^1(G)$-module action on its dual $L^\infty(G)$.

LEMMA 2.2.8. *Let G be a locally compact group. If $f \cdot \varphi$ and $\varphi \cdot f$ denote the canonical module operations of $L^1(G)$ on its dual $L^\infty(G)$, then $f \cdot \varphi = \varphi * \check{f}$ and $\varphi \cdot f = \bar{f}^* * \varphi$ for every $f \in L^1(G), \varphi \in L^\infty(G)$.*

PROOF. For every $g \in L^1(G)$,

$$\begin{aligned}
\langle f \cdot \varphi, g\rangle_{L^\infty, L^1} &= \langle \varphi, g * f\rangle_{L^\infty, L^1} \\
&= \int_G \varphi(x)(g * f)(x)dx \\
&= \int_G \varphi(x)(\int_G g(y)f(y^{-1}x)dy)dx \\
&= \int_G g(y)(\int_G \varphi(x)f(y^{-1}x)dx)dy \\
&= \int_G g(y)(\int_G \varphi(x)\check{f}(x^{-1}y)dx)dy \\
&= \int_G (\varphi * \check{f})(y)g(y)dy \\
&= \langle \varphi * \check{f}, g\rangle_{L^\infty, L^1}.
\end{aligned}$$

Thus $f \cdot \varphi = \varphi * \check{f}$. Similarly we can conclude

$$\langle \varphi \cdot f, g\rangle_{L^\infty, L^1} = \langle \varphi, f * g\rangle_{L^\infty, L^1} = \langle \bar{f}^* * \varphi, g\rangle_{L^\infty, L^1}.$$

Thus $\varphi \cdot f = \bar{f}^* * \varphi$. □

It is well known that the spectrum of $L^1(G)$ is completely characterized by

$$\widehat{G} = \{\chi : G \to \mathbb{T}, \quad \chi \text{ is a continuous homomorphism}\}.$$

In fact, for every $\chi \in \widehat{G}$, the corresponding character on $L^1(G)$, which we denote by Φ_χ, is given by

$$\Phi_\chi(f) = \int_G f(x)\overline{\chi(x)}dx, \quad (f \in L^1(G)),$$

[**19**, Corollary 23.7, p. 358]. In the particular case that $1_G : G \to \mathbb{T}$ is a constant function, we obtain the character

$$1_G : L^1(G) \to \mathbb{C}, \quad 1_G(f) = \int_G f(x)dx.$$

DEFINITION 2.2.9. Let G be a locally compact group. We call an element $\Phi \in L^\infty(G)^*$ a TLIE (respectively TRIE) if it is 1_G-TLIE (respectively 1_G-TRIE. Furthermore, a topological left (right) invariant mean on $L^\infty(G)$ is a TLIE (TRIE) Φ such that $\|\Phi\| = \Phi(1) = 1$. If Φ is both a left and a right topological invariant element (respectively, mean), then Φ is called a topological invariant element (respectively, mean).

REMARK 2.2.10. Our convention of 'left' and 'right' in the above definition is opposite to the one usually used in the literature, but is consistent with our own convention in definition 2.1.3.

Let us define

$$P(G) := \left\{ f \in L^1(G),\ f \geq 0,\ \|f\|_1 = \int |f(x)|dx = 1 \right\}.$$

The following result follows from lemma 2.2.8 and the fact that $P(G)$ linearly spans $L^1(G)$.

THEOREM 2.2.11. *(a) For an element $\Phi \in L^\infty(G)^*$, the following are equivalent:*

(i) Φ *is a TLIE.*
(ii) $\langle \Phi, \varphi * \check{f} \rangle = (\int f(x)dx)\langle \Phi, \varphi \rangle$ *for every* $f \in L^1(G), \varphi \in L^\infty(G)$.
(iii) $\Phi \cdot f = \Phi$ *for every* $f \in P(G)$.
(iv) $\Phi \cdot f = (\int f(x)dx)\Phi$ *for every* $f \in L^1(G)$.

(b) For an element $\Phi \in L^\infty(G)^*$, then the following are equivalent:

(i) Φ *is a TRIE.*
(ii) $\langle \Phi, f * \varphi \rangle = (\int f(x)dx)\langle \Phi, \varphi \rangle$ *for every* $f \in L^1(G), \varphi \in L^\infty(G)$.
(iii) $f \cdot \Phi = \Phi$ *for every* $f \in P(G)$.
(iv) $f \cdot \Phi = (\int f(x)dx)\Phi$ *for every* $f \in L^1(G)$.

LEMMA 2.2.12. *The existence of TLIE on $L^\infty(G)$ implies the existence of TLIM on $L^\infty(G)$. The existence of TRIE on $L^\infty(G)$ implies the existence of TRIM on $L^\infty(G)$.*

PROOF. Let Φ be a TLIE. By the definition of TLIE, we have $\Phi \cdot f = \Phi$, for every $f \in P(G)$. Then for $f \in P(G)$,

$$\Phi^* \cdot f = \Phi^*, \qquad \text{where } \langle \Phi^*, \varphi \rangle := \overline{\langle \Phi, \bar{\varphi} \rangle} \quad (\varphi \in L^\infty(G)).$$

Indeed, since $\varphi \in L^\infty(G)$,

$$\begin{aligned}
\langle \Phi^* \cdot f, \varphi \rangle &= \langle \Phi^*, f \cdot \varphi \rangle \\
&= \langle \Phi^*, \varphi * \check{f} \rangle \\
&= \overline{\langle \Phi, \overline{\varphi * \check{f}} \rangle} \\
&= \overline{\langle \Phi, \bar{\varphi} * \check{f} \rangle} \quad \text{since } f \geq 0,\ \bar{f} = f \\
&= \overline{\langle \Phi, f \cdot \bar{\varphi} \rangle} \\
&= \langle (\Phi \cdot f)^*, \varphi \rangle \\
&= \langle \Phi^*, \varphi \rangle \quad \text{since } \Phi \cdot f = \Phi.
\end{aligned}$$

Thus replacing Φ_0 with $\Phi_0 + {\Phi_0}^*$ if necessary, we may assume Φ_0 is self-adjoint. By the Jordan decomposition theorem [**32**, Proposition III. 2.1, p. 120], there is a unique decomposition of Φ_0 such that

$$\Phi_0 = \Phi_0^+ - \Phi_0^-, \quad \text{where } \Phi_0^\pm \in (L^\infty(G))_+^* \quad \text{and } \|\Phi_0\| = \|\Phi_0^+\| + \|\Phi_0^-\|.$$

So if $f \in P(G)$, we have

$$\Phi_0 \cdot f = \Phi_0^+ \cdot f - \Phi_0^- \cdot f, \quad \text{where } \Phi_0^\pm \cdot f \in (L^\infty(G))_+^* \quad \text{and } \|\Phi_0^+ \cdot f\| + \|\Phi_0^- \cdot f\| = \|\Phi_0\|.$$

In fact, for every $\varphi \in L^\infty(G)$,

$$\begin{aligned}
\langle \Phi_0^\pm \cdot f, \varphi\bar{\varphi} \rangle &= \langle \Phi_0^\pm \cdot f, |\varphi|^2 \rangle \\
&= \langle \Phi_0^\pm, f \cdot |\varphi|^2 \rangle \\
&= \langle \Phi_0^\pm, |\varphi|^2 * \check{f} \rangle
\end{aligned}$$

Since $|\varphi|^2 * \check{f} \geq 0$, it has a positive square root in $L^\infty(G)_+$, say $\psi \geq 0$, so

$$\langle \Phi_0^\pm \cdot f, \varphi\bar{\varphi} \rangle = \langle \Phi_0^\pm, \psi^2 \rangle = \langle \Phi_0^\pm, \psi\bar{\psi} \rangle \geq 0.$$

Thus $\Phi_0^{\pm} \cdot f \in L^\infty(G)^*_+$.

Also note that for any positive linear functional w on a unital involutive Banach algebra, $\|w\| = w(1)$ [**32**, Lemma I.9.9, p. 38]. So in our case,

$$\begin{aligned}\|\Phi_0^+ \cdot f\| + \|\Phi_0^- \cdot f\| &= \langle \Phi_0^+ \cdot f, 1\rangle + \langle \Phi_0^- \cdot f, 1\rangle \\ &= \langle \Phi_0^+, f \cdot 1\rangle + \langle \Phi_0^-, f \cdot 1\rangle.\end{aligned}$$

Since for every $x \in G$,

$$(f \cdot 1)(x) = (1 * \check{f})(x) = \int 1(y)\check{f}(y^{-1}x)dy = \int f(x^{-1}y)dy = \int f(y)dy = 1,$$

it follows that $\|\Phi_0^+ \cdot f\| + \|\Phi^- \cdot f\| = \langle \Phi_0^+, 1\rangle + \langle \Phi_0^-, 1\rangle = \|\Phi_0^+\| + \|\Phi_0^-\| = \|\Phi_0\|$. By the uniqueness of the Jordan decomposition [**32**, Theorem III.4.2(ii), p. 140], we have

$$\Phi_0^+ \cdot f = \Phi_0^+, \quad \Phi_0^- \cdot f = \Phi_0^- \quad \text{for every } f \in P(G).$$

Therefore if, say, $\Phi_0^+ \neq 0$, $\Psi := \Phi_0^+/\Phi_0^+(1)$ is the required TLIM on $L^\infty(G)$.

The proof for the case that Φ_0 is a TRIE is similar. Let Φ be a TRIE. By the definition of TRIE, we have $f \cdot \Phi = \Phi$, for every $f \in P(G)$. Then for $f \in P(G)$,

$$f \cdot \Phi^* = \Phi^*, \qquad \text{where } \langle \Phi^*, \varphi\rangle := \overline{\langle \Phi, \bar{\varphi}\rangle} \quad (\varphi \in L^\infty(G)).$$

Indeed, since $\varphi \in L^\infty(G)$,

$$\begin{aligned}\langle f \cdot \Phi^*, \varphi\rangle &= \langle \Phi^*, \varphi \cdot f\rangle_{L^\infty(G)^*, L^\infty(G)} \\ &= \langle \Phi^*, \bar{f}^* * \varphi\rangle_{L^\infty(G)^*, L^\infty(G)} \\ &= \overline{\langle \Phi, f^* * \bar{\varphi}\rangle} \\ &= \overline{\langle \Phi, \bar{f}^* * \bar{\varphi}\rangle} \quad \text{since } f \geq 0,\ \bar{f} = f \\ &= \overline{\langle \Phi, \bar{\varphi} \cdot f\rangle} \\ &= \overline{\langle f \cdot \Phi, \bar{\varphi}\rangle} \\ &= \langle \Phi^*, \varphi\rangle \quad \text{since } f \cdot \Phi = \Phi.\end{aligned}$$

Thus replacing Φ_0 with $\Phi_0 + {\Phi_0}^*$ if necessary, we may assume Φ_0 is self-adjoint. Using the Jordan decomposition theorem, there is a unique decomposition of Φ_0

such that

$$\Phi_0 = \Phi_0^+ - \Phi_0^- \quad \text{where } \Phi_0^\pm \in (L^\infty(G))_+^* \quad \text{ and } \|\Phi_0\| = \|\Phi_0^+\| + \|\Phi_0^-\|.$$

So if $f \in P(G)$, we have

$$f\cdot\Phi_0 = f\cdot\Phi_0^+ - f\cdot\Phi_0^-, \quad \text{where } f\cdot\Phi_0^\pm \in L^\infty(G)_+^* \quad \text{and } \|f\cdot\Phi_0^+\| + \|f\cdot\Phi_0^-\| = \|\Phi_0\|.$$

In fact, for every $\varphi \in L^\infty(G)$,

$$\begin{aligned}
\langle f \cdot \Phi_0^\pm, \varphi\bar{\varphi}\rangle &= \langle f \cdot \Phi_0^\pm, |\varphi|^2\rangle \\
&= \langle \Phi_0^\pm, |\varphi|^2 \cdot f\rangle \\
&= \langle \Phi_0^\pm, \bar{f}^* * |\varphi|^2\rangle \\
&= \langle \Phi_0^\pm, \psi^2\rangle \quad \text{for some } \psi \in L^\infty(G)_+ \\
&= \langle \Phi_0^\pm, \psi\bar{\psi}\rangle \geq 0.
\end{aligned}$$

Then

$$\begin{aligned}
\|f \cdot \Phi_0^+\| + \|f \cdot \Phi_0^-\| &= \langle f \cdot \Phi_0^+, 1\rangle + \langle f \cdot \Phi_0^-, 1\rangle \\
&= \langle \Phi_0^+, 1 \cdot f\rangle + \langle \Phi_0^-, 1 \cdot f\rangle.
\end{aligned}$$

Since for every $x \in G$,

$$(1 \cdot f)(x) = (\bar{f}^* * 1)(x) = \int \Delta(y^{-1}) f(y^{-1}) 1(y^{-1}x)dy = \int f(y)dy = 1,$$

it follows that $\|f \cdot \Phi_0^+\| + \|f \cdot \Phi^-\| = \langle \Phi_0^+, 1\rangle + \langle \Phi_0^-, 1\rangle = \|\Phi_0^+\| + \|\Phi_0^-\| = \|\Phi_0\|$. Applying the uniqueness of the Jordan decomposition , we have

$$f \cdot \Phi_0^+ = \Phi_0^+, \quad f \cdot \Phi_0^- = \Phi_0^- \quad \text{ for every } f \in P(G).$$

Consequently if, say, $\Phi_0^+ \neq 0$, $\Psi := \Phi_0^+/\Phi_0^+(1)$ is the required TRIM on $L^\infty(G)$. $\square$

LEMMA 2.2.13. *Let G be a locally compact group and $x \in G$. Let $M_x = \{\Phi \in A_p(G)^{**},\ \|\Phi\| = \Phi(\varphi_x) = 1\}$. Then M_x is a w^*-compact, nonempty subset of $A_p(G)^{**}$.*

PROOF. Since M_x is a subset of the unit ball of $A_p(G)^{**}$, M_x will be a w^*-compact if we can show it is w^*-closed in $A_p(G)^{**}$. In fact, let $(\Phi_\alpha)_\alpha$ be a net in M_x such that $\Phi_\alpha \xrightarrow{w^*} \Phi \in A_p(G)^{**}$ and $\|\Phi\| \leq 1$. Then $\langle \Phi, \varphi_x \rangle = \lim_\alpha \langle \Phi_\alpha, \varphi_x \rangle = \lim_\alpha 1 = 1$, which implies $\|\Phi\| \geq 1$. Therefore $\Phi \in M_p$. To show M_x is nonempty, let U be a compact symmetric neighborhood of $x \in G$ and define

$$u_U = |U|^{-1} 1_U * \check{1}_{x^{-1}U} \in A_p(G).$$

Then

$$\begin{aligned}\|u_U\|_{A_p} &= |U|^{-1} \|1_U * \check{1}_{x^{-1}U}\|_{A_p} \leq |U|^{-1} \|\check{1}_{x^{-1}U}\|_p \|1_U\|_q \\ &= |U|^{-1} |U_x|^{\frac{1}{p}} |U|^{\frac{1}{q}} = |U|^{-1} |U|^{\frac{1}{p}} |U|^{\frac{1}{q}} = 1,\end{aligned}$$

and

$$\begin{aligned}u_U(x) &= |U|^{-1} 1_U * \check{1}_{x^{-1}U}(x) = |U|^{-1} \int_U 1_U(y) \check{1}_{x^{-1}U}(y^{-1}x) dy \\ &= |U|^{-1} \int_U 1_{x^{-1}U}(x^{-1}y) dy = |U|^{-1} \int_U 1 dy = |U|^{-1}|U| = 1\end{aligned}$$

Hence the canonical image of u_U in $A_p(G)^{**}$ belongs to M_x, so M_x is not empty. $\square$

For $x \in G$, define

$$\varphi_x : A_p(G) \to \mathbb{C}, \quad u \mapsto \varphi_x(u) = u(x),$$

to be the evaluation functional at x. Then it is well-known that the spectrum of $A_p(G)$ consists of all evaluation functionals at every $x \in G$ (see [**18**, Theorem 3, p. 102]).

Let us define the multipler algebra $B_p = B_p(G)$ of $A_p(G)$ by

$$B_p(G) = B_p = \{u \in \mathcal{C}_0(G),\ uv \in A_p(G) \text{ for all } v \in A_p(G)\}$$

with the norm $\|u\|_{B_p} = \sup\{\|uv\|_{A_p},\ v \in A_p, \|v\|_{A_p} = 1\}$. Let $S_p(G) = S_p = \{u \in B_p,\ \|u\|_{B_p} = u(e) = 1\}$.

LEMMA 2.2.14. *Let $\Phi \in M_e$ be such that $u \cdot \Phi = \Phi$ for every $u \in S_p$. If $u \in B_p$ such that $u = 1$ on some neighborhood V of e, then $u \cdot \Phi = \Phi$. Similarly, if $u \in B_p$ is such that $u = 0$ on some neighborhood V of e, then $u \cdot \Phi = 0$.*

PROOF. Assume that $u = 1$ on V and let U be open such that $U = U^{-1}$ and $U^2 \subset V$. Then the function $\varphi_U = |U|^{-1} 1_U * \check{1}_U \in S_p$ and

$$\{x : \ \varphi_U(x) \neq 0\} \subset \{x : \ x^{-1}U \cap U \neq 0\} \subset U^2 \subset V.$$

So $u(x)\varphi_U(x) = \varphi_U(x)$ for all x. Then

$$u \cdot \Phi = u \cdot (\varphi_U \cdot \Phi) = (u\varphi_U) \cdot \Phi = \varphi_U \cdot \Phi = \Phi.$$

Suppose that $u = 0$ on V and $u \in B_p$. Then $1 - u \in B_p$ and $1 - u = 0$ on V. Applying the first part, we have for every $\Phi \in M_e$, $\Phi = (1 - u) \cdot \Phi = \Phi - u \cdot \Phi$, i.e., $u \cdot \Phi = 0$. $\square$

LEMMA 2.2.15. *Let $\Phi \in M_e$ be such that $u \cdot \Phi = \Phi$ for every $u \in S_p$. Then $u \cdot \Phi = u(e)\Phi$ for every $u \in B_p$.*

PROOF. Suppose that $v \in A_p(G)$ with $v(e) = 0$. Then the set $\{e\}$ is a set of spectral synthesis for the algebra $A_p(G)$ [**18**, Theorem B, p. 91]. Hence there exists a sequence v_n in $A_p(G)$ such that $v_n = 0$ on some neighborhood V_n of e, v_n has compact support and $\|v_n - v\|_{A_p} \to 0$. Applying lemma 2.2.14, we have $v_n \cdot \Phi = 0$. Furthermore $\|v \cdot \Phi\| = \|(v_n - v) \cdot \Phi\| \leq \|v_n - v\|_{A_p} \|\Phi\| \to 0$. Hence if $v \in A_p(G)$ with $v(e) = 0$, then $v \cdot \Phi = 0$.

Now assume $u \in A_p(G)$ with $u(e) = 1$ and let $v \in A_p(G)$ such that $v = 1$ on some neighborhood V of e. Then $(u - v) \cdot \Phi = 0$ by the above argument and $v \cdot \Phi = v$ by lemma 2.2.14. Thus if $u(e) = 1$, then $u \cdot \Phi = v \cdot \Phi = \Phi = u(e)\Phi$.

Let u be an arbitrary element of B_p and $v \in A_p(G)$ with $v(e) = 1$. Then $uv \in A_p(G)$ and $(uv)(e) = u(e)$. By the above argument, we have $u \cdot \Phi = u \cdot (v \cdot \Phi) = (uv) \cdot \Phi = u(e)\Phi$, as required. $\square$

THEOREM 2.2.16. *Let G be a locally compact group and $x \in G$. Then $A_p(G)$ has a φ_x-topological invariant mean.*

PROOF. First of all, S_p is a nonempty subset of B_p since u_U constructed in lemma 2.2.13 belongs to S_p. Put $u \in S_p, \varphi_e \in \sigma(A_p(G)), \Phi \in M_e$. Then

$$\langle u \cdot \Phi, \varphi_e \rangle = \langle \Phi, \varphi_e \cdot u \rangle = \langle \Phi, \varphi_e(u)\varphi_e \rangle = \langle \Phi, u(e)\varphi_e \rangle = \langle \Phi, \varphi_e \rangle = 1.$$

So $u \cdot \Phi \in M_e$ and consequently we can define $\mathfrak{F} = \{T_u : \ u \in S_p\}$, where T_u is defined by

$$T_u : M_e \to M_e, \quad \Phi \mapsto u \cdot \Phi.$$

Since $A_p(G)$ is abelian and

$$T_u T_v \Phi = T_u(v \cdot \Phi) = u \cdot (v \cdot \Phi) = (uv) \cdot \Phi = (vu) \cdot \Phi = v \cdot (T_u \Phi) = T_v T_u \Phi,$$

it follows that T_u and T_v commute. Moreover, the multiplication by u is a linear map so that T_u is affine on M_e and is w^*-continuous. Therefore $\mathfrak{F}$ is a family of abelian continuous affine maps of M_e into itself. By the Markov-Kakutani fixed point theorem [**6**, Theorem 10.1, p. 151], $\{T_u\}$ has a fixed point in M_e, so there exists an element $\Phi \in M_e$ such that $T_u(\Phi) = \Phi$ for all $u \in S_p$. In other words, $u \cdot \Phi = \Phi = u(e)\Phi = \varphi_e(u)\Phi$ for every $u \in S_p$. By lemma 2.2.15, $A_p(G)$ has a φ_e-TIM Φ. Now if $x \in G$ is arbitrary, and L_x is the left translation operator on $A_p(G)$, then it is easy to check that $\Phi_x := L_x^{**}\Phi$ is the required φ_x-TIM. $\square$

The next theorem in [**31**, Corollary 2.4, p. 699] shows that the character amenability of $L^1(G)$ and $A_p(G)$ are completely characterized by the amenability of their underlying group G.

THEOREM 2.2.17. *Let $1 < p < +\infty$ and G be a locally compact group. Then the following are equivalent:*

(i) *G is amenable.*
(ii) *$L^1(G)$ is character amenable.*
(iii) *$A_p(G)$ is character amenable.*

PROOF. (i) $\Rightarrow$ (ii) By Johnson's theorem 1.2.13, if G is amenable then $L^1(G)$ is amenable, and hence left character amenable or right character amenable.

On the other hand, if $L^1(G)$ is left character amenable, then for the character $1_G \in \sigma(L^1(G))$ there exists 1_G-TLIE $\Phi \in (L^1(G))^{**}$ such that $\Phi(1_G) \neq 0$. So

$$\langle \Phi, f \cdot \varphi \rangle = 1_G(f)\langle \Phi, \varphi \rangle = (\int f(x)dx)\langle \Phi, \varphi \rangle, \quad (f \in L^1(G), \varphi \in L^\infty(G)).$$

That is $\Phi \cdot f = (\int f(x)dx)\Phi$ which coincides the definition of TLIE. By the lemma 2.2.12, it follows that there exists a TLIM $\Phi' \in (L^\infty(G))^*$. But Φ' is also a left invariant mean as defined in definition 1.2.10 [**29**, Lemma 1.1.7, p.20]. Thus G is amenable. Similarly if $L_1(G)$ is right character amenable, then G must be amenable.

(iii) $\Leftrightarrow$ (i) By Herz [**18**, Theorem 6, p. 120] if G is an amenable locally compact group, then $A_p(G)$ has an approximate identity of bound 1 for all p. Conversely if $A_p(G)$ has a bounded approximate identity for any p, then G is amenable. In other words, $A_p(G)$ has a bounded approximate identity if and only if G is amenable. So the implication (iii) $\Rightarrow$ (i) holds.

To complete the proof, we only need to find a φ-TLIE $\Phi \in (A_p(G))^{**}$ such that $\Phi(\varphi) \neq 0$ for every $\varphi \in \sigma(A_p(G))$. But the existence has been shown in theorem 2.2.16. $\square$

CHAPTER 3

Additional Properties of Character Amenability

3.1. Hereditary properties of character amenability

The following result shown in [**31**, Theorem 2.6, p. 700] summarizes the main hereditary properties of left character amenability. A similar result holds for right character amenability.

THEOREM 3.1.1. *Let A, B be Banach algebras and I be a closed two-sided ideal of A.*

(i) *If A is left character amenable, and $u : A \to B$ is a continuous homomorphism with dense range (i.e. $\overline{u(A)} = B$), then B is also left character amenable. In particular, if A is left character amenable, then A/I is character amenable for every closed ideal I of A.*

(ii) *If A is left character amenable, then I is left character amenable if and only if I has a bounded left approximate identity.*

(iii) *If both I and A/I are left character amenable then A is also left character amenable.*

(iv) *The unitization algebra $A^\sharp$ is left character amenable if and only if A is left character amenable.*

(v) *$A \times B$ is left character amenable if and only if both A and B are left character amenable.*

PROOF. (i) It suffices to show that B is left φ-amenable for every $\varphi \in \sigma(B) \cup \{0\}$. Let $\varphi \in \sigma(B) \cup \{0\}$ and E be a Banach B-bimodule for which the right module action is given by $x \cdot b = \varphi(b)x$, where $b \in B, x \in E$. Clearly if $\varphi \neq 0$, then $\varphi \circ u \neq 0$ and for every $a, b \in A$,

$$\varphi \circ u(ab) = \varphi(u(a)u(b)) = \varphi \circ u(a)\varphi \circ u(b).$$

So $\varphi \circ u \in \sigma(A) \cup \{0\}$. Then E can be identified as a Banach A-bimodule with the following operations:

$$x \cdot a = x \cdot u(a) = \varphi(u(a))x, \quad a \cdot x = u(a) \cdot x, \quad (a \in A, x \in E).$$

Now take $d \in \mathcal{Z}^1(B, E^*)$. It is routine to check $d \circ u \in \mathcal{Z}^1(A, E^*)$. In fact, for $a, a' \in A$,

$$d \circ u(aa') = d(u(a)u(a')) = d(u(a)) \cdot u(a') + u(a) \cdot d(u(a')) = d \circ u(a) \cdot a' + a \cdot d \circ u(a').$$

Since A is left character amenable, it follows that there exists $f \in E^*$ such that for every $a \in A$,

$$d \circ u(a) = \delta_f(a) = a \cdot f - f \cdot a.$$

Since $\overline{u(A)} = B$, for every $b \in B$, there exists a sequence $(a_n)_n \subset A$ such that $u(a_n) \to b$. Using the continuity of module actions, we have

$$\begin{aligned} d(b) &= d(\lim_n u(a_n)) = \lim_n d \circ u(a_n) = \lim_n \delta_f(a_n) = \lim_n (a_n \cdot f - f \cdot a_n) \\ &= \lim_n (u(a_n) \cdot f - f \cdot u(a_n)) = b \cdot f - f \cdot b = \delta_f(b). \end{aligned}$$

Thus there exists $f \in E^*$ such that $d(b) = \delta_f(b)$. Hence B is left φ-amenable for every $\varphi \in \sigma(B) \cup \{0\}$ and hence is left character amenable.

(ii) If I is left character amenable, then I has a bounded left approximate identity by proposition 2.2.2. For the converse, let $I \lhd A$, $\varphi \in \sigma(I)$. Let $u_0 \in I$ be such that $\varphi(u_0) = 1$ and let $a \in A$. We first show $\tilde{\varphi}(a) = \varphi(au_0)$ defines an element of $\sigma(A)$ extending φ on A. Indeed, put $J = \ker \varphi$. If $a \in A, b \in J$, then $ab \in I$ and

$$\varphi(ab) = \varphi(u_0)\varphi(ab) = \varphi(u_0 ab) = \varphi(u_0 a)\varphi(b) = 0.$$

So J is a closed ideal of A. Obviously, $\tilde{\varphi}$ is a linear functional. For the multiplicative property of $\tilde{\varphi}$, given every $a, b \in A$,

$$\tilde{\varphi}(ab) = \varphi(abu_0) = \varphi(au_0bu_0) = \varphi(au_0)\varphi(bu_0) = \tilde{\varphi}(a)\tilde{\varphi}(b),$$

since $au_0bu_0 - abu_0 = a(u_0bu_0 - bu_0) \subset J$. Hence $\tilde{\varphi} \in \sigma(A)$ and $\tilde{\varphi}|_I = \varphi$. By the assumption of left character amenability of A, there exists $\tilde{\varphi}$-TLIE $\tilde{\Phi} \in A^{**}$ such that $\tilde{\Phi}(\tilde{\varphi}) \neq 0$.

Since $I^{**} = (I^*)^* = (A^*/I^\perp)^*$, if we can show $\tilde{\Phi}(I^\perp) = \{0\}$, then $\Phi(f + I^\perp) := \tilde{\Phi}(f)$ where $f \in A^*$ will be a well defined element of I^{**}. Take $\lambda \in I^\perp$, then

$$\langle \tilde{\Phi}, \lambda \rangle = \tilde{\varphi}(u_0)\langle \tilde{\Phi}, \lambda \rangle = \langle \tilde{\Phi}, u_0 \cdot \lambda \rangle = \langle \tilde{\Phi}, 0 \rangle = 0,$$

since $\langle u_0 \cdot \lambda, a \rangle = \langle \lambda, au_0 \rangle = 0$, for every $a \in A$. Moreover, for $f \in A^*$ and $a \in I$,

$$\langle \Phi, a \cdot (f + I^\perp) \rangle = \langle \Phi, a \cdot f + I^\perp \rangle = \langle \tilde{\Phi}, a \cdot f \rangle = \tilde{\varphi}(a)\langle \tilde{\Phi}, f \rangle = \varphi(a)\langle \Phi, f + I^\perp \rangle,$$

and

$$\langle \Phi, \varphi \rangle = \langle \Phi, \tilde{\varphi} + I^\perp \rangle = \langle \tilde{\Phi}, \tilde{\varphi} \rangle \neq 0.$$

Thus Φ is the required φ-TLIE, so I is also left character amenable.

(iii) Suppose $\varphi \in \sigma(A)$ and $d \in \mathcal{Z}^1(A, E^*)$, where $E \in \mathcal{M}_\varphi^A$. Since E can be identified with a Banach I-bimodule such that $E \in \mathcal{M}_\varphi^I$, it follows that $d|_I \in \mathcal{Z}^1(I, E^*)$. By the left character amenability of I, there exists $f \in E^*$ such that

$$d|_I = \delta'_f,$$

where $\delta'_f : I \to E^*$ is the inner derivation by f, in other words, $d(a) = \delta'_f(a) = a \cdot f - f \cdot a$, for every $a \in I$. If we denote δ_f to be the natural extension of δ'_f on A, then for $a \in A, b \in I$,

$$0 = (d - \delta_f)(ab) = a \cdot (d - \delta_f)(b) + (d - \delta_f)(a) \cdot b,$$

so $(d - \delta_f)(a) \cdot b = 0$. Similarly, we have

$$0 = (d - \delta_f)(ba) = b \cdot (d - \delta_f)(a) + (d - \delta_f)(b) \cdot a,$$

so $b \cdot (d - \delta_f)(a) = 0$. Thus for $x \in E$,

$$\langle (d - \delta_f)(a), b \cdot x \rangle = \langle (d - \delta_f)(a) \cdot b, x \rangle = 0,$$

and

$$\langle (d - \delta_f)(a), x \cdot b \rangle = \langle b \cdot (d - \delta_f)(a), x \rangle = 0.$$

Let E_I be a closed linear span of $I \cdot E \cup E \cdot I$. The above two identities imply that $d - \delta_f$ maps A into $E_I^\perp$, and $d - \delta_f \in \mathcal{Z}^1(A, E_I^\perp) = \mathcal{Z}^1(A, (E/E_I)^*)$. We already know that E/E_I is a Banach A-bimodule since

$$a \cdot (x + E_I) = a \cdot x + E_I, \quad (x + E_I) \cdot a = x \cdot a + E_I = \varphi(a)x + E_I, \quad (a \in A, x \in E)$$

are both well-defined. We distinguish two cases.

Case I: If $I \subset \ker \varphi$, we can define a character $\widehat{\varphi}$ of A/I by $\widehat{\varphi}(a + I) = \varphi(a)$. We will construct a derivation $\widetilde{d - \delta_f} \in \mathcal{Z}^1(A/I, (E/E_I)^*)$ induced by $d - \delta_f$. Indeed, the module operations from I to E give rise to trivial operators on E/E_I, so E/E_I is an A/I-bimodule in which the module structures on E/E_I are defined by:

$$(x + E_I) \cdot (a + I) := (x + E_I) \cdot a = \varphi(a)x + E_I = \widehat{\varphi}(a + I)x + E_I = \widehat{\varphi}(a + I)(x + E_I),$$

$$(a + I) \cdot (x + E_I) := a \cdot (x + E_I) = a \cdot x + E_I, \quad (a \in A, x \in E).$$

Therefore $E/E_I \in \mathcal{M}_{\widehat{\varphi}}^{A/I}$. Furthermore, $d - \delta_f = 0$ on I and $d - \delta_f \in \mathcal{Z}^1(A, E_I^{\perp}) = \mathcal{Z}^1(A, (E/E_I)^*)$. Define $\widetilde{d - \delta_f}(a + I) := (d - \delta_f)(a)$ for $a \in A$. We will show that $\widetilde{d - \delta_f} \in \mathcal{Z}^1(A/I, (E/E_I)^*)$. Firstly since $d - \delta_f = 0$ on I, the map $\widetilde{d - \delta_f}$ is well defined and also continuous. For the derivation property, for $a + I, a' + I \in A/I$, then

$$\begin{aligned}
\widetilde{d - \delta_f}((a + I)(a' + I)) &= \widetilde{d - \delta_f}(aa' + I) = (d - \delta_f)(aa') \\
&= (d - \delta_f)(a) \cdot a' + a \cdot (d - \delta_f)(a') \\
&= (d - \delta_f)(a) \cdot (a' + I) + (a + I) \cdot (d - \delta_f)(a') \\
&= (\widetilde{d - \delta_f})(a + I) \cdot (a' + I) + (a + I) \cdot (\widetilde{d - \delta_f})(a' + I).
\end{aligned}$$

By the left character amenability of A/I, there exists $g \in (E/E_I)^* = E_I^{\perp} \subset E^*$ such that $\widetilde{d - \delta_f} = \delta_g$. So for every $a \in A$,

$$(d - \delta_f)(a) = \widetilde{d - \delta_f}(a + I) = \delta_g(a + I) = (a + I) \cdot g - g \cdot (a + I) = a \cdot g - g \cdot a.$$

Thus $d = \delta_f + \delta_g = \delta_{f+g}$, as was to be shown.

Case II: If $I \not\subseteq \ker \varphi$ then there exists $b_0 \in I$ such that $\varphi(b_0) \neq 0$. Then $\tilde{d} := d - \delta_f = 0$ on A. Indeed, for every $a \in A$, $b_0 \cdot \tilde{d}(a) = \tilde{d}(b_0 \cdot a) - \tilde{d}(b_0) \cdot a = 0$. Thus $b_0 \cdot \tilde{d}(a) = 0$ for every $a \in A$. By the definition of module action on E^*, we have $b_0 \cdot \tilde{d}(a) = \varphi(b_0)\tilde{d}(a)$. So $\tilde{d}(a) = 0$ for every $a \in A$. Then $\tilde{d} = d - \delta_f = 0$. In other words, $d = \delta_f$, as required.

Note: for the case of $I \nsubseteq \ker\varphi$, we do not have to use character amenability of A/I. Instead, character amenability of I will be sufficient to show character amenability of A.

(iv) If A is left character amenable, then since A is a closed ideal of $A^\sharp = A \oplus \mathbb{C}$, and $\mathbb{C}$ is left character amenable, it follows that $A^\sharp$ is left character amenable by (iii).

Conversely, suppose $A^\sharp$ is left character amenable, and let $\varphi \in \sigma(A)$, $E \in \mathcal{M}_\varphi^A$ and $d \in \mathcal{Z}^1(A, E^*)$. We construct $\tilde{\varphi} \in \sigma(A^\sharp)$ by

$$\tilde{\varphi} : A^\sharp \to \mathbb{C} \qquad (a, \alpha) \mapsto \alpha + \varphi(a).$$

It is easy to check $\tilde{\varphi}$ is linear and $\tilde{\varphi} \in \sigma(A^\sharp)$. In fact for $(a, \alpha), (a', \alpha') \in A^\sharp$,

$$\begin{aligned}\tilde{\varphi}((a, \alpha)(a', \alpha')) &= \tilde{\varphi}(aa' + \alpha a' + \alpha' a, \alpha\alpha') = \alpha\alpha' + \varphi(aa') + \alpha\varphi(a') + \alpha'\varphi(a) \\ &= (\alpha + \varphi(a))(\alpha' + \varphi(a)) = \tilde{\varphi}(a, \alpha)\,\tilde{\varphi}(a', \alpha').\end{aligned}$$

Next E can be viewed as a Banach $A^\sharp$-bimodule by the following module actions:

$$(a, \alpha) \cdot x = \alpha x + a \cdot x, \quad x \cdot (a, \alpha) = \alpha x + x \cdot a = \tilde{\varphi}(a, \alpha)x, \quad ((a, \alpha) \in A^\sharp, x \in E).$$

Hence $E \in \mathcal{M}_{\tilde{\varphi}}^{A^\sharp}$. Define $\tilde{d} : A^\sharp \to E^*$ by $\tilde{d}(a, \alpha) = d(a)$, $(a, \alpha) \in A^\sharp$. Clearly $\tilde{d}$ is linear and continuous. It is also a derivation. Since for $(a, \alpha), (a', \alpha') \in A^\sharp$,

$$\begin{aligned}\tilde{d}((a, \alpha)(a', \alpha')) &= \tilde{d}(aa' + \alpha' a + \alpha a', \alpha\alpha') = d(aa' + \alpha' a + \alpha a') \\ &= \alpha d(a') + \alpha' d(a) + d(aa') = \alpha d(a') + \alpha' d(a) + a \cdot d(a') + d(a) \cdot a' \\ &= \alpha d(a') + \alpha' d(a) + \varphi(a) d(a') + d(a) \cdot a'.\end{aligned}$$

On the other hand,

$$\begin{aligned}(a, \alpha) \cdot \tilde{d}(a', \alpha') + \tilde{d}(a, \alpha) \cdot (a', \alpha') &= (a, \alpha) d(a') + d(a) \cdot (a', \alpha') \\ &= \tilde{\varphi}(a, \alpha) d(a') + \alpha' d(a) + d(a) \cdot a' \\ &= (\varphi(a) + \alpha) d(a') + \alpha' d(a) + d(a) \cdot a' \\ &= \varphi(a) d(a') + \alpha d(a') + \alpha' d(a) + d(a) \cdot a'.\end{aligned}$$

Therefore $\tilde{d} \in \mathcal{Z}^1(A^\sharp, E^*)$, so there exists $f \in E^*$ such that $\tilde{d} = \delta_f$. Then for every $a \in A, \alpha \in \mathbb{C}$,

$$d(a) = \tilde{d}(a,\alpha) = \delta_f(a,\alpha) = (a,\alpha)\cdot f - f\cdot(a,\alpha) = a\cdot f - f\cdot a.$$

Hence $d = \delta_f$ on A. Then A is also left character amenable.

(v) Let $\pi : A\times B \to A, \quad (a,b) \mapsto a$ be the projection map. Then π is a continuous surjective homomorphism. If $A \times B$ is left character amenable, then applying (i) we have A is left character amenable. Similarly B is also left character amenable.

Conversely, if both A and B are left character amenable, clearly $B \cong \{(0,b), b \in B\}$, $B \lhd A\times B$ and $(A\times B)/B = A$, so by (iii), we have $A \times B$ is left character amenable. □

For the rest of this section our aim is to give an alternative proof of theorem 3.1.1 (ii) just using the original definition of character amenability involving derivations. But we require an extra condition that I has a bounded two-sided approximate identity. More precisely, we will prove that if A is left character amenable and I has a bounded two-sided approximate identity then I is left character amenable.

Definition 3.1.2. Let A be a Banach algebra. A Banach A-bimodule E is called left pseudo-unital if

$$E = A\cdot E = \{a\cdot x : \quad a \in A, x \in E\}.$$

Similarly, one defines right pseudo-unital and pseudo-unital Banach modules. The next lemma is similar to lemma 2.2.1.

Lemma 3.1.3. *Let A be a Banach algebra with a bounded right approximate identity and let E be a Banach A-bimodule such that $A\cdot E = \{0\}$. Then $\mathcal{H}^1(A, E^*) = \{0\}$.*

Lemma 3.1.4. *Let A be a Banach algebra with a bounded two-sided approximate identity and $\varphi \in \sigma(A)$. The following are equivalent:*

(i) $\mathcal{H}^1(A, E^*) = \{0\}$ *for each* $E \in \mathcal{M}_\varphi^A$.

(ii) $\mathcal{H}^1(A, E^*) = \{0\}$ *for each left pseudo-unital* $E \in \mathcal{M}_\varphi^A$.

PROOF. That (i) implies (ii) is trivial. For the converse, let $E \in \mathcal{M}_{\varphi}^{A}$, and let $d \in \mathcal{Z}^1(A, E^*)$. Let

$$E_0 = A \cdot E = \{a \cdot x : \ a \in A, \ x \in E\}.$$

By Cohen's factorization theorem, E_0 is a closed bimodule of E. Define $\pi : E^* \to {E_0}^*$ to be the restriction map. It is routinely checked that π is a continuous module homomorphism. So $\pi \circ d \in \mathcal{Z}^1(A, {E_0}^*)$ and $E_0 \in \mathcal{M}_{\varphi}^{A}$. By our assumption, $\mathcal{H}^1(A, {E_0}^*) = \{0\}$. So there exists $f_0 \in {E_0}^*$ such that

$$\pi \circ d(a) = \delta_{f_0}(a) = a \cdot f_0 - f_0 \cdot a, \quad (a \in A). \tag{$*$}$$

By the Hahn-Banach theorem, there exists $f \in E^*$ such that $f|_{E_0} = f_0$. Define $\tilde{d} := d - \delta_f$. Then $\tilde{d}$ is a continuous derivation from A to E^*. We show that $\tilde{d} \in \mathcal{Z}^1(A, E_0^{\perp})$. Indeed, for every $a \in A, x \in E_0$,

$$\begin{aligned}
\langle (d - \delta_f)(a), x \rangle_{E^*, E_0} &= \langle d(a), x \rangle_{E^*, E_0} - \langle \delta_f(a), x \rangle_{E^*, E_0} \\
&= \langle d(a), x \rangle_{E^*, E_0} - \langle f, x \cdot a \rangle_{E^*, E_0} + \langle f, a \cdot x \rangle_{E^*, E_0} \\
&= \langle d(a)|_{E_0}, x \rangle_{E_0^*, E_0} - \langle f|_{E_0}, x \cdot a \rangle_{E_0^*, E_0} + \langle f|_{E_0}, a \cdot x \rangle_{E_0^*, E_0} \\
&= \langle (\pi \circ d)(a), x \rangle_{E_0^*, E_0} - \langle f_0, x \cdot a \rangle_{E_0^*, E_0} + \langle f_0, a \cdot x \rangle_{E_0^*, E_0} \\
&= \langle (\pi \circ d)(a) - a \cdot f_0 + f_0 \cdot a, x \rangle_{E_0^*, E_0} = 0,
\end{aligned}$$

where the last identity followed from $(*)$. The quotient space E/E_0 can be identified with a A-bimodule and from the definition of E_0, we have

$$A \cdot (E/E_0) = \{a \cdot (x + E_0) : \ a \in A, x \in E\} = \{a \cdot x + E_0 : \ a \in A, x \in E\} = \{0\}.$$

So $\mathcal{H}^1(A, (E/E_0)^*) = \mathcal{H}^1(A, {E_0}^{\perp}) = \{0\}$ by lemma 3.1.3. Hence, there is $\varphi \in {E_0}^{\perp} \subset E^*$ such that for every $a \in A$, $\tilde{d}(a) = \delta_{\varphi}(a) = a \cdot \varphi - \varphi \cdot a$. Then $d = \delta_f + \delta_{\varphi} = \delta_{f+\varphi}$, as required. $\square$

Alternative proof of Theorem 3.1.1 (ii): our objective is to give a direct proof that if A is left character amenable and I has a bounded two-sided approximate identity, then I is left character amenable.

By the result of lemma 3.1.4, it suffices to show that $\mathcal{H}^1(I, E^*) = \{0\}$ for each left pseudo-unital $E \in \mathcal{M}_\varphi^I$ provided that I has a bounded two-sided approximate identity.

In the proof of theorem 3.1.1 (ii) we already showed that every $\varphi \in \sigma(I)$ extends to some $\tilde{\varphi} \in \sigma(A)$. More precisely, we defined $\tilde{\varphi}(a) = \varphi(au_0)$ where $a \in A$ and $u_0 \in I$ with $\varphi(u_0) = 1$. Let $(e_\alpha)_\alpha$ be a bounded two-sided approximate identity for I with $\|e_\alpha\| \leq M$. Let $E \in \mathcal{M}_\varphi^I$ be left pseudo-unital and $d \in \mathcal{Z}^1(I, E^*)$. We will construct $E \in \mathcal{M}_{\tilde{\varphi}}^A$ from $E \in \mathcal{M}_\varphi^I$ and extend d to a continuous derivation $\tilde{d}$ on A.

The Banach A-bimodule structure on E extending its I-bimodule structure is defined as follows: for $a \in A, x \in E$ with $x = b \cdot y$, $b \in I$ and $y \in E$, we set

$$a \cdot x = a \cdot (b \cdot y) := (ab) \cdot y, \quad x \cdot a = \tilde{\varphi}(a)x.$$

We first show that the left action above does not depend on the particular representation of x as $b \cdot y$, that is, the left module action is well-defined. In fact, let $b', y' \in E$ with $x = b' \cdot y' = b \cdot y$, then using the fact that $(e_\alpha)_\alpha$ is a bounded two-sided approximate identity for I,

$$(ab') \cdot y' = a \cdot (b' \cdot y') = \lim_\alpha (ae_\alpha) \cdot (b' \cdot y') = \lim_\alpha (ae_\alpha) \cdot (b \cdot y) = \lim_\alpha (ae_\alpha b) \cdot y = (ab) \cdot y.$$

It is routinely checked that the two operations of A on E turn E into a Banach A-bimodule and $E \in \mathcal{M}_{\tilde{\varphi}}^A$.

Next, we extend d to a continuous derivation $\tilde{d}$ on A. Define

$$\tilde{d} : A \to E^*, \quad a \mapsto w^*\text{-}\lim_\alpha d(e_\alpha a).$$

To show that the limit exists, we verify that for $a \in A, x \in E$ and $x = b \cdot y$ with $b \in I, y \in E$, we have

$$\langle \tilde{d}(a), x \rangle_{E^*,E} := \lim_\alpha \langle d(e_\alpha a), x \rangle_{E^*,E} = \langle d(ab) - \tilde{\varphi}(a) d(b), y \rangle_{E^*,E}.$$

In fact,

$$\begin{aligned} \lim_\alpha \langle d(e_\alpha a), x \rangle_{E^*,E} &= \lim_\alpha \langle d(e_\alpha a), b \cdot y \rangle_{E^*,E} \\ &= \lim_\alpha \langle d(e_\alpha a) \cdot b, y \rangle_{E^*,E} \end{aligned}$$

Since d is a derivation, it follows that

$$\begin{aligned}
\lim_\alpha \langle d(e_\alpha a), x\rangle_{E^*,E} &= \lim_\alpha \langle d(e_\alpha (ab)) - (e_\alpha a)\cdot d(b), y\rangle_{E^*,E} \\
&= \lim_\alpha \langle d(e_\alpha (ab)), y\rangle_{E^*,E} - \lim_\alpha \langle (e_\alpha a)\cdot d(b), y\rangle_{E^*,E} \\
&= \lim_\alpha \langle d(e_\alpha (ab)), y\rangle_{E^*,E} - \lim_\alpha \varphi(e_\alpha a)\langle d(b), y\rangle_{E^*,E} && (3.1.1)\\
&= \langle d(ab), y\rangle_{E^*,E} - \varphi(au_0)\langle d(b), y\rangle_{E^*,E} && (3.1.2)\\
&= \langle d(ab) - \tilde{\varphi}(a)d(b), y\rangle_{E^*,E}.
\end{aligned}$$

The equation (3.1.1) holds since $E^* \in {}_\varphi\mathcal{M}^I$. The equation (3.1.2) holds since d is continuous and in addition

$$\begin{aligned}
\lim_\alpha \varphi(e_\alpha a) &= \lim_\alpha \varphi(e_\alpha a u_0) \quad \text{since } \varphi(u_0) = 1 \\
&= \lim_\alpha \varphi(e_\alpha u_0 a u_0) && (3.1.3)\\
&= \lim_\alpha \varphi(e_\alpha u_0)\varphi(a u_0) \\
&= \varphi(u_0)\varphi(au_0) = \varphi(au_0).
\end{aligned}$$

The equation (3.1.3) holds since $\ker\varphi$ is an ideal of I and $e_\alpha a u_0 - e_\alpha u_0 a u_0 = e_\alpha(au_0 - u_0 a u_0) \in \ker\varphi$.

Therefore $w^*\text{-}\lim_\alpha d(e_\alpha a)$ exists and $\tilde{d}$ is well-defined. Next we will show that $\tilde{d} \in \mathcal{Z}^1(A, E^*)$ when E is equipped with the above module actions.

Fix $a \in A$, the linearity of $\tilde{d}(a)$ and $\tilde{d}$ are both clear. For the continuity,

$$\begin{aligned}
\|\tilde{d}(a)\|_{E^*} &= \sup_{x\in E, \|x\|\le 1} |\langle \tilde{d}(a), x\rangle| = \sup_{x\in E, \|x\|\le 1} |\lim_\alpha \langle d(e_\alpha a), x\rangle| \\
&= \sup_{x\in E, \|x\|\le 1} \lim_\alpha |\langle d(e_\alpha a), x\rangle| \le \sup_{x\in E, \|x\|\le 1} \lim_\alpha \|d(e_\alpha a)\|\|x\| \\
&\le \lim_\alpha \|d(e_\alpha a)\| \le \|d\|\|M\|\|a\|.
\end{aligned}$$

So $\tilde{d}(a) \in E^*$. Moreover,

$$\|\tilde{d}\| = \sup_{a\in A, \|a\|\le 1} \|\tilde{d}(a)\| \le \sup_{a\in A, \|a\|\le 1} \|d\|\|M\|\|a\| \le \|d\|\|M\|.$$

Hence $\tilde{d} \in \mathcal{B}(A, E^*)$. Next we show that $\tilde{d}(ae_\alpha c) \xrightarrow{w^*} \tilde{d}(ac)$. In fact for each $x = b \cdot y \in E$ with $b \in I, y \in E$,

$$\langle \tilde{d}(a), x\rangle = \langle \tilde{d}(a), b \cdot y\rangle = \langle d(ab) - \tilde{\varphi}(a)d(b), y\rangle,$$

so

$$\begin{aligned}
\langle \tilde{d}(ae_\alpha c), x\rangle &= \langle \tilde{d}(ae_\alpha c), b \cdot y\rangle \\
&= \langle d(ae_\alpha cb) - \tilde{\varphi}(ae_\alpha c)d(b), y\rangle \\
&\to \langle d(acb) - \tilde{\varphi}(ac)d(b), y\rangle = \langle \tilde{d}(ac), b \cdot y\rangle = \langle \tilde{d}(ac), x\rangle.
\end{aligned}$$

It remains to show $\tilde{d}$ is also a derivation, i.e., $\tilde{d}(ac) = a \cdot \tilde{d}(c) + \tilde{d}(a) \cdot c$, for $a, c \in A$. Indeed,

$$\begin{aligned}
\tilde{d}(ac) &= w^*\text{-}\lim_\alpha \tilde{d}(ae_\alpha c) = w^*\text{-}\lim_\alpha [w^*\text{-}\lim_\beta d(e_\beta a e_\alpha c)] \\
&= w^*\text{-}\lim_\alpha [w^*\text{-}\lim_\beta d(e_\beta a) \cdot (e_\alpha c) + w^*\text{-}\lim_\beta (e_\beta a) \cdot d(e_\alpha c)] \\
&= w^*\text{-}\lim_\alpha [\tilde{d}(a) \cdot (e_\alpha c) + a \cdot d(e_\alpha c)] \qquad (3.1.4) \\
&= \tilde{d}(a) \cdot c + a \cdot \tilde{d}(c). \qquad (3.1.5)
\end{aligned}$$

The equation (3.1.4) holds since for every $x \in E$,

$$\begin{aligned}
\lim_\beta \langle d(e_\beta a) \cdot (e_\alpha c), x\rangle &= \lim_\beta \langle d(e_\beta a), (e_\alpha c) \cdot x\rangle \\
&= \langle \tilde{d}(a), (e_\alpha c) \cdot x\rangle = \langle \tilde{d}(a) \cdot (e_\alpha c), x\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\lim_\beta \langle (e_\beta a) \cdot d(e_\alpha c), x\rangle &= \lim_\beta \langle d(e_\alpha c), x \cdot (e_\beta a)\rangle \\
&= \lim_\beta \langle d(e_\alpha c), \tilde{\varphi}(e_\beta a)x\rangle \\
&= \lim_\beta \langle d(e_\alpha c), \tilde{\varphi}(e_\beta)\tilde{\varphi}(a)x\rangle \\
&= \langle d(e_\alpha c), \tilde{\varphi}(a)x\rangle \quad \text{since } \lim_\beta \tilde{\varphi}(e_\beta) = 1 \\
&= \langle d(e_\alpha c), x \cdot a\rangle = \langle a \cdot d(e_\alpha c), x\rangle.
\end{aligned}$$

The reason why equation (3.1.5) holds is similar. Therefore we have shown $\tilde{d} \in \mathcal{Z}^1(A, E^*)$ where $E \in \mathcal{M}_{\varphi}^A$.

Finally we show that d is the restriction of $\tilde{d}$ on I. For every $a \in I$, $\tilde{d}(a) = w^*$-$\lim_\alpha d(e_\alpha a) = d(a)$, since $\|a - e_\alpha a\| \to 0$ and d is continuous. Hence $d = d|_I$. By the left character amenability of A, there exists $f \in E^*$ such that

$$\tilde{d}(a) = \delta_f(a) = a \cdot f - f \cdot a.$$

Since $\tilde{d}|_I = d$ it follows that for every $a \in I$,

$$d(a) = \tilde{d}(a) = \delta_f(a) = a \cdot f - f \cdot a,$$

So I is left character amenable, which is what we wanted to show.

3.2. Bounded approximate identities and φ-amenability

The following characterization of left φ-amenability is due to Kaniuth, Lau and Pym in [**25**, Proposition 2.1, p. 90].

THEOREM 3.2.1. *Let A be a Banach algebra and $\varphi \in \sigma(A)$. If the ideal $I = \ker\varphi$ has a bounded left approximate identity, then A is left φ-amenable and has a bounded left approximate identity.*

PROOF. Let $(d_\beta)_\beta$ be a bounded left approximate identity for I. Choose $u \in A$ with $\varphi(u) = 1$. Trivially for each $a \in A$, $a = \varphi(a)u + (a - \varphi(a)u)$, so $A = I \oplus \mathbb{C}u$, and $\mathrm{codim}(I) = 1$. Therefore A also has a bounded left approximate identity [**9**, Proposition 7.1, p. 43], say $(e_\alpha)_\alpha$. By Cohen's factorization theorem, both $I \cdot A^*$ and $A \cdot A^*$ are closed linear subspaces of A^*. Since $A = I \oplus \mathbb{C}u$, it follows that $A \cdot A^* = I \cdot A^* + u \cdot A^*$. Since for every $a \in A$,

$$\langle u \cdot \varphi, a\rangle = \langle \varphi, au\rangle = \varphi(a)\varphi(u) = \varphi(a),$$

we have $u \cdot \varphi = \varphi(u)\varphi = \varphi$.

We claim that $u \cdot \varphi = \varphi \notin I \cdot A^*$. Assume towards a contradiction that $\varphi = u \cdot \varphi = b \cdot f$ for some $b \in I, f \in A^*$. Then

$$\langle f, b\rangle_{A^*,A} = \lim_\beta \langle f, d_\beta b\rangle = \lim_\beta \langle b \cdot f, d_\beta\rangle = \lim_\beta \langle \varphi, d_\beta\rangle = 0.$$

But on the other hand, since $u = \lim_\alpha(e_\alpha u)$, it follows that $1 = \lim_\alpha \varphi(e_\alpha u) = \lim_\alpha \varphi(e_\alpha)$, and hence

$$\langle f, b\rangle_{A^*,A} = \lim_\alpha \langle f, e_\alpha b\rangle = \lim_\alpha \langle b \cdot f, e_\alpha\rangle = \lim_\alpha \varphi(e_\alpha) = 1,$$

which is a contradiction. Thus $u\cdot\varphi \notin I\cdot A^*$. Since $I\cdot A^* \subset A\cdot A^*$ and $u\cdot\varphi \in A\cdot A^*$, by the Hahn-Banach theorem, there exists $n \in (A\cdot A^*)^*$ such that $n(I\cdot A^*) = \{0\}$ and $n(u\cdot\varphi) = 1$. Let us define Φ on A^* by $\Phi(g) = n(u\cdot g)$, $(g \in A^*)$. Then it is routinely checked that Φ is a bounded linear functional on A^* and $\Phi(\varphi) = n(u\cdot\varphi) = 1$.

Let $a = b + \lambda u$, with $b \in I, \lambda \in \mathbb{C}$. It follows that for every $f \in A^*$,

$$\begin{aligned}
\langle \Phi, a\cdot f\rangle &= \langle n, u\cdot(a\cdot f)\rangle = \langle n, u\cdot((b+\lambda u)\cdot f)\rangle \\
&= \langle n, u\cdot(b\cdot f)\rangle + \lambda\langle n, u^2\cdot f\rangle \\
&= \langle n, (ub)\cdot f\rangle + \lambda\langle n, (u^2-u)\cdot f\rangle + \lambda\langle n, u\cdot f\rangle \\
&= \lambda\langle n, u\cdot f\rangle \quad \text{since } ub, u^2-u \in \ker\varphi \\
&= \varphi(\lambda u)\langle\Phi, f\rangle = \varphi(b+\lambda u)\langle\Phi, f\rangle = \varphi(a)\langle\Phi, f\rangle.
\end{aligned}$$

Hence Φ is a φ-TLIE and $\Phi(\varphi) \neq 0$, which implies A is left φ-amenable. $\square$

Our next objective is to prove a converse to theorem 3.2.1. First however, we need some definitions. Richard Arens [**1**] defined two products on A^{**} under which A^{**} becomes a Banach algebra. More precisely, the first and second Arens products denoted by $\Box$ and $\Diamond$ respectively, are defined as follows:

$$A^{**}\times A^{**} \to A^{**}, \quad (\Phi,\Psi) \mapsto \Phi\Box\Psi$$

$$\langle \Phi\Box\Psi, f\rangle_{A^{**},A^*} := \langle \Phi, \Psi\cdot f\rangle_{A^{**},A^*}$$

and $\Psi\cdot f$ is defined by

$$\langle \Psi\cdot f, a\rangle_{A^*,A} = \langle \Psi, f\cdot a\rangle_{A^{**},A^*}, \quad (f \in A^*, a \in A). \tag{$*$}$$

Also, for the left multiplication,

$$A^{**}\times A^{**} \to A^{**}, \quad (\Phi,\Psi) \mapsto \Phi\Diamond\Psi$$

$$\langle \Phi\Diamond\Psi, f\rangle_{A^{**},A^*} := \langle \Psi, f\cdot\Phi\rangle_{A^{**},A^*}.$$

and $f \cdot \Phi$ is defined by

$$\langle f \cdot \Phi, a\rangle_{A^*,A} = \langle \Phi, a \cdot f\rangle_{A^{**},A^*}, \quad (f \in A^*, a \in A).$$

It is easy to show that A^* is a Banach left A^{**}-module with the module multiplication given by $(*)$. With either of there products, A can be viewed as a subalgebra of A^{**}. In general, the multiplication $(\Phi, \Psi) \mapsto \Phi\Box\Psi$ is not separately continuous with respect to the w^*-topology on A^{**}. But for fixed $\Psi \in A^{**}$, the maps $\Phi \to \Phi\Box\Psi$ and $\Phi \to \Psi\Diamond\Phi$ are both w^*-continuous.

We also need to use the following well-known result for the proof of which we refer to [**9**, Theorem 33.3, p. 224].

LEMMA 3.2.2. *Let A be a Banach algebra.*

(i) *$(A^{**}, \Box)$ has a right identity if and only if A has a bounded right approximate identity.*

(ii) *$(A^{**}, \Diamond)$ has a left identity if and only if A has a bounded left approximate identity.*

The following result which is a converse of theorem 3.2.1 is shown in [**25**, Proposition 2.2, p. 90].

THEOREM 3.2.3. *Let A be a Banach algebra and $\varphi \in \sigma(A)$. Suppose A is left φ-amenable, and A has a bounded left approximate identity, then $I = \ker\varphi$ has a bounded left approximate identity.*

PROOF. By lemma 3.2.2 it suffices to verify that $(\ker\varphi)^{**}$ has a left identity with respect to the second Arens product $\Diamond$. Let $\Phi \in A^{**}$ be such that $\langle \Phi, a \cdot f\rangle = \varphi(a)\langle \Phi, f\rangle$, for all $a \in A, f \in A^*$ and $\Phi(\varphi)$=1. Put

$$J(\varphi) = \{\Phi \in A^{**}, \langle \Phi, \varphi\rangle = 0\}.$$

We claim that $J(\varphi)$ is a w^*-closed ideal of $(A^{**}, \Diamond)$. Indeed, suppose $(\Phi_\alpha)_\alpha \subset J(\varphi)$ with $\Phi_\alpha \xrightarrow{w^*} \Phi,\ \Phi \in A^{**}$, then $\langle \Phi, \varphi\rangle = \lim_\alpha \langle \Phi_\alpha, \varphi\rangle = 0$, i.e., $\Phi \in J(\varphi)$. And for every $a \in A, \Psi \in A^{**}$ and $\Phi \in J(\varphi)$,

$$\langle \varphi \cdot \Phi, a\rangle_{A^*,A} = \langle \Phi, a \cdot \varphi\rangle_{A^{**},A^*} = \langle \Phi, \varphi(a)\varphi\rangle = \varphi(a)\langle \Phi, \varphi\rangle = 0.$$

Thus $\varphi \cdot \Phi = 0$, and hence

$$\langle \Phi \Diamond \Psi, \varphi \rangle_{A^{**},A^*} = \langle \Psi, \varphi \cdot \Phi \rangle = 0,$$

which implies $\Phi \Diamond \Psi \in J(\varphi)$. Similarly

$$\langle \varphi \cdot \Psi, a \rangle = \langle \Psi, a \cdot \varphi \rangle = \langle \Psi, \varphi(a)\varphi \rangle = \langle \Psi(\varphi)\varphi, a \rangle.$$

Thus $\varphi \cdot \Psi = \Psi(\varphi)\varphi$ and

$$\langle \Psi \Diamond \Phi, \varphi \rangle = \langle \Phi, \varphi \cdot \Psi \rangle = \langle \Phi, \Psi(\varphi)\varphi \rangle = \Psi(\varphi)\langle \Phi, \varphi \rangle = 0,$$

which implies $\Psi \Diamond \Phi \in J(\varphi)$. Moreover $J(\varphi)$ can be canonically identified with $I^{**} = (\ker \varphi)^{**}$. In fact,

$$(\ker \varphi)^{\perp} = \{f \in A^* : f|_{\ker \varphi} = 0\} = \mathbb{C}\varphi.$$

It follows that

$$((\ker \varphi)^*)^* \cong (A^*/(\ker \varphi)^{\perp})^* = (A^*/\mathbb{C}\varphi)^* \cong (\mathbb{C}\varphi)^{\perp} = J(\varphi).$$

Let $(e_\alpha)_\alpha$ be a bounded left approximate identity for A. Then there exist $\Phi_0 \in A^{**}$ such that Φ_0 is a w^*-cluster point of the canonical image of $(e_\alpha)_\alpha$ in A^{**} by Alaoglu's theorem [**10**, Theorem V.4.2, p. 424]. Without loss of generality, we may assume $\Phi_0 = w^*\text{-}\lim_\alpha e_\alpha$. Let $\Phi_1 = \Phi_0 - \Phi \in A^{**}$. It remains to show Φ_1 is the left identity for $J(\varphi)$. Firstly, $\Phi_1 \in J(\varphi)$ since

$$\langle \Phi_1, \varphi \rangle = \langle \Phi_0, \varphi \rangle - \langle \Phi, \varphi \rangle = \lim_\alpha \langle e_\alpha, \varphi \rangle - \Phi(\varphi) = \lim_\alpha \langle \varphi, e_\alpha \rangle - \Phi(\varphi) = 1 - 1 = 0.$$

Next, for $a \in I$ and $f \in A^*$,

$$\begin{aligned}
\langle \Phi_1 \cdot a, f \rangle &= \langle \Phi_1, a \cdot f \rangle = \langle \Phi_0 - \Phi, a \cdot f \rangle \\
&= \lim_\alpha \langle e_\alpha, a \cdot f \rangle - \langle \Phi, a \cdot f \rangle \\
&= \lim_\alpha \langle a \cdot f, e_\alpha \rangle - \varphi(a)\langle \Phi, f \rangle \quad \text{since } \Phi \text{ is } \varphi\text{-TLIE} \\
&= \lim_\alpha \langle f, e_\alpha a \rangle - \varphi(a)\langle \Phi, f \rangle \\
&= \langle f, a \rangle = \langle a, f \rangle \quad \text{since } a \in I = \ker \varphi.
\end{aligned}$$

Since $I \hookrightarrow I^{**}$ is w^*-dense in I^{**}, for each $\Psi \in J(\varphi) = I^{**}$, there exists a net $(a_\beta)_\beta \subset I$ such that $\Psi = w^*\text{-}\lim_\beta a_\beta$. It follows that for all $f \in A^*$,

$$\langle \Phi_1 \Diamond \Psi, f\rangle = \lim_\beta \langle \Phi_1 \cdot a_\beta, f\rangle = \lim_\beta \langle a_\beta, f\rangle = \langle \Psi, f\rangle.$$

That is $\Phi_1 \Diamond \Psi = \Psi$ for each $\Psi \in I^{**}$. Therefore Φ_1 is a left identity for $J(\varphi)$. $\square$

As an immediate consequence of theorem 3.2.1 and theorem 3.2.3 we obtain the following corollaries.

Corollary 3.2.4. *Let A be a Banach algebra and $\varphi \in \sigma(A)$. Then $I = \ker\varphi$ has a bounded left approximate identity if and only if A is left φ-amenable and A has a bounded left approximate identity.*

Corollary 3.2.5. *Let A be Banach algebra. A is left (right) character amenable if and only if $\ker\varphi$ has a bounded left (right) approximate identity for every $\varphi \in \sigma(A) \cup \{0\}$.*

Corollary 3.2.6. *Every C^*-algebra is character amenable.*

Proof. The result follows from corollary 3.2.5 and the fact that every closed two-sided ideal of a C^*-algebra has a bounded approximate identity [**32**, Theorem 7.4, p. 27]. $\square$

3.3. Character amenability of projective tensor products

We turn to projective tensor product $A\widehat{\otimes}B$ of two Banach algebras A and B. Recall that, if $m \in A\widehat{\otimes}B$, then

$$m = \sum_{i=1}^{\infty} a_i \otimes b_i, \quad \sum_{i=1}^{\infty} \|a_i\|\|b_i\| < \infty, \quad (a_i \in A,\ b_i \in B),$$

$$\text{and } \|m\|_\pi = \inf\left\{\sum_{i=1}^{\infty} \|a_i\|\,\|b_i\|, \quad m = \sum_{i=1}^{\infty} a_i \otimes b_i\right\}.$$

For $f \in A^*, g \in B^*$, we denote by $f \otimes g$ the element in $(A\widehat{\otimes}B)^*$ which is defined by $(f \otimes g)(a \otimes b) := f(a)g(b)$. In particular, if $\varphi \in \sigma(A), \phi \in \sigma(B)$, then we have $\varphi \otimes \phi \in \sigma(A\widehat{\otimes}B)$. In fact, for

$$m = \sum_i a_i \otimes b_i, \quad m' = \sum_j a'_j \otimes b'_j,$$

$$
\begin{aligned}
\langle \varphi \otimes \phi, mm' \rangle &= \langle \varphi \otimes \phi, \sum_i \sum_j a_i a'_j \otimes b_i b'_j \rangle \\
&= \sum_i \sum_j \langle \varphi \otimes \phi, a_i a'_j \otimes b_i b'_j \rangle \\
&= \sum_i \sum_j \varphi(a_i)\varphi(a'_j)\phi(b_i)\phi(b'_j) \\
&= (\sum_i \varphi(a_i)\phi(b_i))(\sum_j \varphi(a'_j)\phi(b'_j)) \\
&= \langle \varphi \otimes \phi, m \rangle \langle \varphi \otimes \phi, m' \rangle.
\end{aligned}
$$

To see whether $A\widehat{\otimes}B$ also preserves the character amenability if its factor algebras are character amenable, we need the following lemma in [**25**, lemma 3.1, p. 92].

Lemma 3.3.1. *Let A be a Banach algebra and I a closed ideal of A. Let $\varphi \in \sigma(A)$ be such that $\varphi|_I \neq 0$. If A is left φ-amenable, then I is left $\varphi|_I$-amenable.*

Proof. Let $\Phi \in A^{**}$ be a φ-TLIE such that $\Phi(\varphi) \neq 0$. Since $A \hookrightarrow A^{**}$ is w^*-dense in A^{**}, for such $\Phi \in A^{**}$, there exists a net $(e_\alpha)_\alpha$ in A such that $\Phi = w^*$-$\lim_\alpha e_\alpha$. Then for every $f \in I^\perp, a \in I$,

$$\langle \Phi, a \cdot f \rangle = \lim_\alpha \langle e_\alpha, a \cdot f \rangle = \lim \langle a \cdot f, e_\alpha \rangle = \lim_\alpha \langle f, e_\alpha a \rangle = 0.$$

Choose $a \in I$ with $\varphi(a) = 1$, we conclude that $\Phi(I^\perp) = \{0\}$. Since $I^{**} = (I^*)^* = (A^*/I^\perp)^*$, it follows that for $g \in A^*$, $\widetilde{\Phi}(g + I^\perp) := \Phi(g)$ will be a well defined element of I^{**}. So

$$\widetilde{\Phi}(\varphi|_I) = \widetilde{\Phi}(\varphi + I^\perp) = \Phi(\varphi) \neq 0.$$

Moreover, for every $a \in I, g \in A^*$,

$$\langle \widetilde{\Phi}, a \cdot (g + I^\perp) \rangle = \langle \widetilde{\Phi}, a \cdot g + I^\perp \rangle = \langle \Phi, a \cdot g \rangle = \varphi(a)\langle \Phi, g \rangle = \varphi|_I(a)\langle \widetilde{\Phi}, g + I^\perp \rangle,$$

which implies $\widetilde{\Phi}$ is a $\varphi|_I$-TLIE. Therefore I is left $\varphi|_I$-amenable. $\square$

Theorem 3.3.2. *Let A and B be Banach algebras and $\varphi \in \sigma(A)$, $\phi \in \sigma(B)$. Then $A\widehat{\otimes}B$ is left $\varphi \otimes \phi$-amenable if and only if A is left φ-amenable and B is left ϕ-amenable.*

PROOF. ($\Rightarrow$) If $A\widehat{\otimes}B$ is left $\varphi\otimes\phi$-amenable, then there exists $\Phi \in (A\widehat{\otimes}B)^{**}$ such that $\Phi(\varphi\otimes\phi)=1$ and for all $a\otimes b\in A\widehat{\otimes}B, T\in (A\widehat{\otimes}B)^*=\mathcal{B}(A,B^*)$,

$$\langle \Phi,(a\otimes b)\cdot T\rangle = (\varphi\otimes\phi)(a\otimes b)\langle\Phi,T\rangle = \varphi(a)\phi(b)\langle\Phi,T\rangle.$$

In particular, if $f\in A^*$, then $T=f\otimes\phi\in(A\widehat{\otimes}B)^*$ and we have

$$\langle\Phi,(a\times b)\cdot(f\otimes\phi)\rangle_{(A\widehat{\otimes}B)^{**},(A\widehat{\otimes}B)^*} = (\varphi\otimes\phi)(a\otimes b)\langle\Phi,f\otimes\phi\rangle = \varphi(a)\phi(b)\langle\Phi,f\otimes\phi\rangle.$$

Choose $a_0\in A, b_0\in B$ with $\varphi(a_0)=\phi(b_0)=1$ and define $\Phi_\phi\in A^{**}$ by $\Phi_\phi(f)=\Phi(f\otimes\phi),\ f\in A^*$. So $\Phi_\phi(\varphi)=\Phi(\varphi\otimes\phi)=1$. Furthermore, for all $a\in A, f\in A^*$,

$$\begin{aligned}
\langle\Phi_\phi,a\cdot f\rangle &= \Phi((a\cdot f)\otimes\phi)\\
&= \varphi(a_0)\ \phi(b_0)\langle\Phi,(a\cdot f)\otimes\phi\rangle\\
&= \langle\Phi,(a_0\otimes b_0)\cdot((a\cdot f)\otimes\phi)\rangle \qquad (3.3.1)\\
&= \langle\Phi,(a_0\cdot(a\cdot f))\otimes(b_0\cdot\phi)\rangle\\
&= \langle\Phi,((a_0a)\cdot f)\otimes(b_0\cdot\phi)\rangle\\
&= \langle\Phi,((a_0a)\otimes b_0)\cdot(f\otimes\phi)\rangle \qquad (3.3.2)\\
&= \varphi(a_0a)\ \phi(b_0)\langle\Phi,f\otimes\phi\rangle \qquad (3.3.3)\\
&= \varphi(a)\langle\Phi,f\otimes\phi\rangle\\
&= \varphi(a)\langle\Phi_\phi,f\rangle.
\end{aligned}$$

The equations (3.3.1) and (3.3.3) hold since Φ is a $\varphi\otimes\phi$-TLIE. The identity (3.3.2) holds since for $x\otimes y\in A\widehat{\otimes}B$,

$$\begin{aligned}
\langle((a_0a)\otimes b_0)\cdot(f\otimes\phi),x\otimes y\rangle &= \langle f\otimes\phi,(x\otimes y)((a_0a)\otimes b_0)\rangle\\
&= \langle f\otimes\phi,(xa_0a)\otimes(yb_0)\rangle\\
&= \langle f,x(a_0a)\rangle\langle\phi,yb_0\rangle\\
&= \langle(a_0a)\cdot f,x\rangle\langle b_0\cdot\phi,y\rangle\\
&= \langle((a_0a)\cdot f)\otimes(b_0\cdot\phi),x\otimes y\rangle
\end{aligned}$$

Thus A is left φ-amenable. Likewise, if we define $\Phi_\varphi \in B^*$ by $\Phi_\varphi(g) = \Phi(\varphi \otimes g)$, $g \in B^*$, then Φ_φ is a ϕ-TLIE with $\Phi_\varphi(\phi) = 1$, hence B is also left ϕ-amenable.

Conversely, let A and B be left φ-amenable and left ϕ-amenable, respectively. Suppose $A^\sharp$ and $B^\sharp$ are unitization of A and B, and e_A, e_B are their identities, then an arbitrary element of $A^\sharp \widehat{\otimes} B^\sharp$ is of the form

$$\alpha e_A \otimes e_B + e_A \otimes b + a \otimes e_B + \sum_{n=1}^{\infty} a_n \otimes b_n, \quad (\alpha \in \mathbb{C}, a, a_i \in A, b, b_i \in B).$$

It follows that $A \widehat{\otimes} B$ is a closed two-sided ideal of $A^\sharp \widehat{\otimes} B^\sharp$. In addition, if φ' and ϕ' are the extension of φ and ϕ to $A^\sharp$ and $B^\sharp$, then $\varphi' \otimes \phi'|_{A \widehat{\otimes} B} = \varphi \otimes \phi$. It follows from lemma 3.3.1 that if we can show $A^\sharp \widehat{\otimes} B^\sharp$ is left $\varphi' \otimes \phi'$-amenable, then $A \widehat{\otimes} B$ is left $\varphi \otimes \phi$-amenable. So without loss of generality, we can always assume A and B are unital with respective identities e_A and e_B.

It remains to show that if E is a Banach $A \widehat{\otimes} B$-bimodule such that

$$x \cdot (a \otimes b) = (\varphi \otimes \phi)(a \otimes b)x = \varphi(a)\phi(b)x, \quad (x \in E, a \otimes b \in A \widehat{\otimes} B),$$

then $\mathcal{H}^1(A \widehat{\otimes} B, E^*) = \{0\}$. Let $d \in \mathcal{Z}^1(A \widehat{\otimes} B, E^*)$. Since E can be identified with a Banach A-bimodule with the following operations:

$$a \cdot x = (a \otimes e_B) \cdot x, \quad x \cdot a = x \cdot (a \otimes e_B) = \varphi(a)\phi(e_B)x = \varphi(a)x, \quad (a \in A, x \in E),$$

it follows that $E \in \mathcal{M}_\varphi^A$. Define $d_A : A \to E^*$ by $d_A(a) = d(a \otimes e_B)$, $a \in A$. It is easy to check that d_A is a continuous derivation from A to E^*. By the left φ-amenability of A, there exists $f \in E^*$ such that for every $a \in A$, $d_A(a) = a \cdot f - f \cdot a$. Let δ_f be the inner derivation from $A \widehat{\otimes} B$ to E^*. Then

$$d(a \otimes e_B) = d_A(a) = a \cdot f - f \cdot a = (a \otimes e_B) \cdot f - f \cdot (a \otimes e_B) = \delta_f(a \otimes e_B).$$

So $\tilde{d} := d - \delta_f = 0$ on $A \otimes e_B$. Since $A \otimes e_B$ and $e_A \otimes B$ commute, i.e., for all $a \in A,\ b \in B$,

$$a \otimes b = (a \otimes e_B)(e_A \otimes b) = (e_A \otimes b)(a \otimes e_B),$$

it follows that

$$\tilde{d}(a \otimes b) = (a \otimes e_B) \cdot \tilde{d}(e_A \otimes b) = \tilde{d}(e_A \otimes b) \cdot (a \otimes e_B).$$

So

$$(a \otimes e_B) \cdot \tilde{d}(e_A \otimes b) - \tilde{d}(e_A \otimes b) \cdot (a \otimes e_B) = 0,$$

and therefore by taking closure in w^*-topology of E^*,

$$\delta_g(A \otimes e_B) = \{0\} \quad \text{for every} \quad g \in \overline{\tilde{d}(e_A \otimes B)}^{w^*}. \tag{$*$}$$

Let F be the annihilator of $\tilde{d}(e_A \otimes B)$ in E. Viewing E as a Banach B-bimodule in which the module structures are given by

$$b \cdot x = (e_A \otimes b) \cdot x, \quad x \cdot b = x \cdot (e_A \otimes b) = \varphi(e_A)\phi(b)x = \phi(b)x, \quad (x \in E, b \in B),$$

then F is a Banach B-submodule of E. Indeed, for every $y \in F$ and $b_1,\ b_2 \in B$,

$$\begin{aligned}
\langle \tilde{d}(e_A \otimes b_1), y \cdot b_2 \rangle &= \langle \tilde{d}(e_A \otimes b_1), y \cdot (e_A \otimes b_2) \rangle \\
&= \langle \tilde{d}(e_A \otimes b_1), \phi(b_2) y \rangle \\
&= \phi(b_2) \langle \tilde{d}(e_A \otimes b_1), y \rangle = 0,
\end{aligned}$$

which shows $y \cdot b_2 \in F$ and hence F is a right B-module. Moreover,

$$\begin{aligned}
\langle \tilde{d}(e_A \otimes b_1), b_2 \cdot y \rangle &= \langle \tilde{d}(e_A \otimes b_1), (e_A \otimes b_2) \cdot y \rangle \\
&= \langle \tilde{d}(e_A \otimes b_1) \cdot (e_A \otimes b_2), y \rangle \\
&= \langle \tilde{d}((e_A \otimes b_1)(e_A \otimes b_2)) - (e_A \otimes b_1) \cdot \tilde{d}(e_A \otimes b_2), y \rangle \\
&= \langle \tilde{d}(e_A \otimes b_1 b_2) - (e_A \otimes b_1) \cdot \tilde{d}(e_A \otimes b_2), y \rangle \\
&= \langle \tilde{d}(e_A \otimes b_1 b_2), y \rangle - \langle (e_A \otimes b_1) \cdot \tilde{d}(e_A \otimes b_2), y \rangle \\
&= \langle \tilde{d}(e_A \otimes b_1 b_2), y \rangle - \langle \tilde{d}(e_A \otimes b_2), y \cdot (e_A \otimes b_1) \rangle \\
&= -\langle \tilde{d}(e_A \otimes b_2), y \cdot b_1 \rangle = 0,
\end{aligned}$$

where the last identity holds since $y \in F$ and $y \cdot (e_A \otimes b_1) \in F$ by the previous step. Therefore E/F is a Banach B-bimodule satisfying

$$(x + F) \cdot b = x \cdot b + F = \phi(b)x + F = \phi(b)(x + F), \quad (x \in E,\ b \in B),$$

that is, $E/F \in \mathcal{M}_\phi^B$. Moreover, by the bipolar theorem [**6**, corollary 1.9, p. 127], we have

$$(E/F)^* = F^\perp = ({}^\perp\tilde{d}(e_A \otimes B))^\perp = \overline{\tilde{d}(e_A \otimes B)}^{w^*}.$$

Define $d_B(b) = \tilde{d}(e_A \otimes b)$. Then d_B is a continuous derivation of B into $(E/F)^*$. Since B is left ϕ-amenable, it follows that there exists $g \in (E/F)^* \subset E^*$ such that for all $b \in B$,

$$\tilde{d}(e_A \otimes b) = d_B(b) = b \cdot g - g \cdot b = (e_A \otimes b) \cdot g - g \cdot (e_A \otimes b).$$

For such $g \in (E/F)^*, \delta_g|_{A\otimes e_B} = 0$ by the identity $(*)$, so $\tilde{d} - \delta_g$ is a continuous derivation of $A\widehat{\otimes}B$ vanishing on $A \otimes e_B$ and $e_A \otimes B$. Since $(A \otimes e_B) \cup (e_A \otimes B)$ generates $A\widehat{\otimes}B$, it follows that $\tilde{d} - \delta_g$ vanishes on all of $A\widehat{\otimes}B$, thus

$$\delta_g = \tilde{d} = d - \delta_f,$$

and hence $d = \delta_{f+g}$, as required. □

In general, it is not known whether for two arbitrary Banach algebras A and B we must have

$$\sigma(A\widehat{\otimes}B) = \sigma(A) \times \sigma(B). \tag{$*$}$$

[**14**]. However, the identity in $(*)$ is known to be true for some special cases, for example it both A and B are commutative or when both A and B are unital. The result in the commutative case was shown independently by Tomiyama [**33**, Theorem 2, p. 150] and by Gelbaum [**15**, Proposition 2, p. 529].

If A and B are unital, then every $\Phi \in \sigma(A\widehat{\otimes}B)$ is of the form $\Phi = \varphi \otimes \phi$ with $\varphi \in \sigma(A)$ and $\phi \in \sigma(B)$, where

$$\varphi(a) := \Phi(a \otimes e_B), \quad \phi(b) := \Phi(e_A \otimes b), \quad (a \in A, b \in B).$$

Corollary 3.3.3. *Let A and B be Banach algebra such that $\sigma(A\widehat{\otimes}B) = \sigma(A) \times \sigma(B)$. Then $A\widehat{\otimes}B$ is left character amenable if and only if A and B are left character amenable.*

Proof. We only need to show the existence of bounded left approximate identity in $A\widehat{\otimes}B$ versus their existences in A and B. In fact, the equivalence was shown in [**9**, Theorem 8.2, p. 48] and [**9**, Theorem 8.3, p. 49] since the projective tensor product norm is an algebra admissible norm. □

CHAPTER 4

Banach Function Algebras and their Character Amenability

4.1. Banach function algebras

In this chapter, we will discuss character amenability of Banach function algebras and uniform algebras. We will show in theorem 4.2.4 that in the case of natural unital uniform algebras, character amenability is completely determined by its Choquet boundary. We first introduce some elementary definitions.

Definition 4.1.1. Let S be a nonempty set and $\mathbb{C}^S$ be the commutative algebra of all functions on S. Let E be a subset of $\mathbb{C}^S$.

(i) E separates the points of S if for each $s, t \in S$ with $s \neq t$, there exists $f \in E$ such that $f(s) \neq f(t)$. E separates strongly the points of S if E separates the points of S and if for each $s \in S$ there exists $f \in E$ such that $f(s) \neq 0$.

(ii) The weakest topology τ on S such that each $f \in E$ is continuous with respect to τ is called E-topology on S.

(iii) If $f \in \mathbb{C}^S$, and $F \subset S$, we write $\|f\|_F = \sup\limits_{x \in F} |f(x)|$.

Definition 4.1.2. Let X be a topological space.

(i) A is a function algebra on X if A is a subalgebra of $\mathbb{C}^X$ which separates strongly the points of X and the A-topology on X is the given topology.

(ii) A Banach function algebra on X is a function algebra on X which is also a Banach algebra with respect to some norm.

(iii) A Banach function algebra A is called natural if $\sigma(A) = \{\tau_x : x \in X\}$, where τ_x is the evaluation functional at x.

(iv) A uniform algebra on X is a Banach function algebra on X with the norm $\|\cdot\|_X$.

If A is a Banach function algebra on X, then for each $f \in A$, and $x \in X$,

$$|f(x)| = |\tau_x(f)| \leq \|\tau_x\|\|f\| \leq \|f\|.$$

It follows that $\|f\|_X \leq \|f\|$ for every $f \in A$ since τ_x is a character hence continuous and $\|\tau_x\| \leq 1$ [**8**, Theoerm 2.1.29 (ii), p. 167].

Definition 4.1.3. Let X be a compact space and A is a unital Banach function algebra on X. The Choquet boundary of A, denoted by $\Gamma_0(A)$, is the set of all $x \in X$ such that the point mass δ_x is the unique probability measure μ on X with $f(x) = \int_X f d\mu$ for every $f \in A$.

Lemma 4.1.4. *Let A be a unital Banach function algebra on a compact space X and $x \in X$. If $M_x = \{f \in A : f(x) = 0\}$ has a bounded approximate identity, then there exists α, β with $0 < \alpha < \beta < 1$, such that for each open neighborhood N of x there exists $f \in A$, with $\|f\| \leq 1, f(x) > \beta$ and $f(y) < \alpha$ for all $y \notin N$.*

Proof. Let $(f_\alpha)_\alpha$ be a bounded approximate identity for M_x with $\|f_\alpha\| \leq M$ for some $M \geq 0$. Let $r = \frac{1}{1+M}$. Since A separates strongly the points of X, it follows that for each $y \in X \setminus N$, there exists $f \in A$, such that $f(y) \neq 0$ and $f(x) = 0$. Multiplying f by some constant if necessary, we may assume $f(y) > 1$. So there exists a neighborhood V of y such that $\|f\|_V > 1$. Compactness of $X \setminus N$ implies that we can find $f_1, f_2, \cdots, f_n \in M$, such that for every $y \in X \setminus N$, there exists $1 \leq k \leq n$ with $|f_k(y)| > 1$. Let $0 < \epsilon < \frac{r}{2}$ and $\alpha_0 \in I$ be such that $\|f_k f_{\alpha_0} - f_k\| < \epsilon$, for each $k = 1, 2, \cdots n$. Define $f = \frac{1-f_{\alpha_0}}{1+M}$, then $f \in A$, $f(x) = \frac{1}{1+M} = r$ and $\|f\| < \frac{1+M}{1+M} = 1$. Also, for every $y \in X \setminus N$ and for every $1 \leq k \leq n$,

$$|(f_k f)(y)| = \frac{1}{1+M}|f_k(y) - f_k(y)f_{\alpha_0}(y)| \leq \frac{\|f_k - f_k f_\alpha\|}{1+M} < \epsilon.$$

But since for at least one k, $|f_k(y)| > 1$, we have $|f(y)| < \epsilon$ for every $y \in X \setminus N$. The assertion of the lemma follows if we put $\alpha = \epsilon$ and $\beta = r - \epsilon$. □

4.2. Character amenability of Banach function algebras

The following lemma which we shall need later is shown in [**5**, Theorem 2.2.1, p. 88].

LEMMA 4.2.1. *Let* X *be a compact space and* A *be a unital Banach function algebra on* X. *Suppose there exists constants* α, β *with* $0 < \alpha < \beta < 1$, *such that for every open neighborhood* N *of* x, *there exists* $f \in A$, *with* $\|f\| \leq 1$, $f(x) > \beta$, *and* $|f(y)| < \beta$ *for every* $y \notin N$. *Then* $x \in \Gamma_0(A)$.

If we combine the previous two lemmas, we will get the following result due to Hu, Sangani Monfared and Traynor in [**21**, Theorem 5.1, p. 69].

THEOREM 4.2.2. *If* A *is a character amenable unital Banach function algebra on a compact space* X, *then* $\Gamma_0(A) = X$.

PROOF. Clearly $\Gamma_0(A) \subset X$, to complete the proof, it remains to show the converse containment. Since A is character amenable, it follows from corollary 3.2.5 that $\ker\varphi$ has a bounded approximate identity for every $\varphi \in \sigma(A)$. In particular, for each $x \in X$, $\ker\tau_x = M_x$ has a bounded approximate identity. So by lemma 4.1.4 and lemma 4.2.1, $x \in \Gamma_0(A)$ for every $x \in X$. □

For unital uniform algebras, we have the following characterization of the Choquet boundary, for the proof of which we refer to [**8**, Theorem 4.3.5, p. 448].

LEMMA 4.2.3. *Let* A *be a unital uniform algebra on a compact space* X *and* $x \in X$. *Then the following are equivalent:*

(i) $x \in \Gamma_0(A)$.

(ii) x *is a strong boundary point for* A, *that is, for every open neighborhood of* x, *there exists* $f \in A$ *such that* $f(x) = \|f\|_X = 1$ *and* $|f(y)| < 1$ *for all* $y \notin N$.

(iii) $M_x = \{f \in A : f(x) = 0\}$ *has a bounded approximate identity.*

Choquet boundary can completely characterize character amenability of a natural unital uniform algebra, in fact:

THEOREM 4.2.4. *A natural unital uniform algebra* A *on a compact space* X *is character amenable if and only if* $\Gamma_0(A) = X$.

PROOF. By theorem 4.2.2, character amenability of A implies that $\Gamma_0(A) = X$. For the converse, if $\Gamma_0(A) = X$ and $\sigma(A) = \{\tau_x,\ x \in X\}$, then for every $x \in X =$

$\Gamma_0(A)$, $\ker \tau_x = \{f \in A, f(x) = 0\}$ has a bounded approximate identity by lemma 4.2.3. Then by corollary 3.2.5, A must be character amenable. $\square$

Let X be a nonempty compact space. It is known that uniform algebra $\mathcal{C}(X)$ is amenable. M.V. Sheinberg showed that if A is a unital amenable uniform algebra on X, then $A = \mathcal{C}(X)$ [**8**, Theorem 5.6.2, p.709]. A natural question is to ask whether classical amenability can be replaced by character amenability in Sheinberg's result. As we will show below, the analogue of Sheinberg's result does not hold for character amenable uniform algebras. Before that, we first define various standard uniform algebras on a compact subset of $\mathbb{C}^n$.

Definition 4.2.5. Let K be a nonempty compact subset of $\mathbb{C}^n$.

(i) P(K) is the subalgebra of $\mathcal{C}(K)$ consisting of uniform limits of polynomials.
(ii) R(K) is the subalgebra of $\mathcal{C}(K)$ consisting of uniform limits of rational functions which have the form p/q, where p and q are polynomials and $0 \notin q(K)$.
(iii) A(K) is the subalgebra of $\mathcal{C}(K)$ consisting of functions analytic on the interior of K.

Definition 4.2.6. Let A be an algebra of functions on a topological space X.

(i) A subset S of X is a peak set for A if there exists $f \in A$ such that $f(x) = 1$ for every $x \in S$ and $|f(y)| < 1$ for $y \in X \setminus S$.
(ii) A point $x \in X$ is called a peak point for A if $\{x\}$ is a peak set.

Definition 4.2.7. Let A be a unital uniform algebra on a compact space X and $\varphi \in \sigma(A)$. A probability measure μ on X is a Jensen measure for φ if for every $f \in A$,

$$\log|\varphi(f)| \leq \int_X \log|f| d\mu.$$

Example 4.2.8. In this example, we show that there exists a character amenable uniform algebra other than the C^*-algebra $\mathcal{C}(X)$. Let M be the closed unit ball in $\mathbb{C}^2$, that is, $M = \{(z, w) : |z|^2 + |w|^2 \leq 1\}$ and K be an arbitrary compact subset of the open unit disk in $\mathbb{C}$. We associate with K a subset of the 3-sphere

$\{|z|^2 + |w|^2 = 1\}$ by defining

$$\Omega_K = \{(z,w) \in \partial M,\ z \in K\} = \{(z,w) \in \mathbb{C}^2,\ z \in K,\ |z|^2 + |w|^2 = 1\}.$$

Suppose that K is chosen so that the only Jensen measures for $R(K)$ are point mass measures and $R(K) \neq C(K)$. For this Ω_k, Basener [**4**, Lemma 10, p. 372] showed that $R(\Omega_k) \neq C(\Omega_k)$. However, for any $(z_0, w_0) \in \Omega_k$, there exists a polynomial, such as, $p(z,w) = (\bar{z}_0 z + \bar{w}_0 w + 1)/2$ peaking at (z_0, w_0). Since Ω_k is metrizable, it follows that the Choquet boundary and the set of all peak points for $R(\Omega_k)$ coincide [**8**, p. 447]. In other words, $\Gamma_0(R(\Omega_k)) = \Omega_k$. Note that for compact subset K in $\mathbb{C}$, $R(K)$ is natural [**8**, Proposition 4.3.12 (iii), p. 453], by theorem 4.2.4, we have $R(\Omega_k)$ is character amenable.

In the remaining of this section we show that for a compact subset K of $\mathbb{C}$, character amenable version of Sheinberg's result holds for $P(K)$.

Definition 4.2.9. Let K be a compact subset of $\mathbb{C}^n$. Then polynomially convex hull of K, denoted by $\widehat{K}$, is defined by

$$\widehat{K} = \{z \in \mathbb{C}^n : \quad |p(z)| \leq \|p\|_K \quad \text{for all polynomials } p\}.$$

K is called polynomially convex if $\widehat{K} = K$.

Lemma 4.2.10. *Let K be a compact subset of $\mathbb{C}^n$. If $P(K)$ is character amenable, then K has empty interior and is polynomially convex.*

Proof. Suppose $P(K)$ is character amenable, let $\varphi \in \sigma(P(K))$ and $E = \mathbb{C}_{\varphi,\varphi}$. It follows from character amenability of $P(K)$ that every continuous derivation $d : P(K) \to \mathbb{C}_{\varphi,\varphi}$ is inner. But for each $z \in \mathbb{C}_{\varphi,\varphi}$, the inner derivation $\delta_z(f) = z \cdot f - f \cdot z = 0$ for all $f \in P(K)$. Thus $P(K)$ doesn't have any non-zero inner derivations. So K must have an empty interior. Indeed, assume towards a contradiction that $w = (z_1, z_2, \cdots, z_n) \in \text{int}K$. For any $1 \leq k \leq n$, the map

$$D : P(K) \to \mathbb{C}_{\varphi,\varphi}, \quad f \mapsto \frac{\partial f}{\partial z_k}(w)$$

is a non-zero continuous derivation since

$$D(fg) = \frac{\partial(fg)}{\partial z_k}(w) = \frac{\partial f}{\partial z_k}(w)g(w) + f(w)\frac{\partial g}{dz_k}(w) = D(f)g(w) + f(w)D(g),$$

which is a contraction since character amenability of $P(K)$ implies that D must be inner and hence 0.

Let $\widehat{K}$ be the polynomially convex hull of K. Since $\sigma(P(K)) = \widehat{K}$ [**8**, Proposition 4.3.12, p. 453], $P(K) = P(\widehat{K})$ [**13**, Theorem 1.4, p. 27], and $P(K)$ is character amenable, it follows that $\widehat{K} = \Gamma_0(P(\widehat{K}))$ and $\Gamma_0(P(K)) = K$ by theorem 4.2.2. It remains to show $\Gamma_0(P(\widehat{K})) \subset \Gamma_0(P(K))$. In fact, since K is metrizable, Choquet boundary is just the set of peak points. If $x_0 \in \widehat{K}$ is a peak point for $P(\widehat{K})$, then there exists $f \in P(\widehat{K})$ such that $f(x_0) = 1$ and $|f(y)| < 1$ for every $y \neq x_0$. Assume by contradiction that $x_0 \notin K$, then by the definition of $\widehat{K}$, for each polynomial p, $p(x_0) \leq \|p\|_K$. Let $p_n \xrightarrow{\|\cdot\|_{\widehat{K}}} f$, for some sequence polynomials (p_n). Let $y_0 \in K$ be arbitrary but fixed. Then for some $\epsilon > 0$,

$$f(x_0) > 1 - \epsilon > |f(y_0)|.$$

Since $p_n \xrightarrow{\|\cdot\|_{\widehat{K}}} f$, there exists n_0 such that

$$|p_{n_0}(x_0) - 1| < \frac{\epsilon}{2}, \quad \text{and} \quad |p_{n_0}(y_0) - f(y_0)| < \frac{\epsilon}{2}.$$

But then

$$|p_{n_0}(x_0)| > 1 - \frac{\epsilon}{2}, \quad \text{and} \quad |p_{n_0}(y_0)| < |f(y_0)| + \frac{\epsilon}{2} < 1 - \frac{\epsilon}{2},$$

which is a contradiction since $|p(x_0)| \leq \|p\|_K$. Thus $x_0 \in K$ and $x_0 \in \Gamma_0(P(K))$. Hence K is polynomially convex. $\square$

The following characterization of character amenability of $P(K)$ is due to Hu, Sangani Monfared and Traynor in [**21**, Theorem 5.4, p. 71].

THEOREM 4.2.11. *For a compact subset K of $\mathbb{C}$, the uniform algebra $P(K)$ is character amenable if and only if $P(K) = C(K)$.*

PROOF. Since $C(K)$ is a C^*-algebra, it follows that $C(K)$ is character amenable by corollary 3.2.6.

Conversely if $P(K)$ is character amenable then K is polynomially convex and has empty interior by lemma 4.2.10. Applying the Lavrentieff's theorem [**13**, II Theorem 8.7, p. 48] we have $P(K) = C(K)$. $\square$

CHAPTER 5

Reduction of Order of Cohomology Groups and Splitting Properties of Modules

5.1. Reduction of order formula

In previous chapters, we introduced the first cohomology groups of Banach algebras with coefficients in Banach bimodules. In this chapter, we study the higher order cohomology groups of A with coefficients in a Banach A-bimodule E. As it turns out, it is always possible to express $\mathcal{H}^n(A, E)$ as the first cohomology group of A with coefficients in another Banach A-bimodule, using an identity which is called the reduction of order (dimension) formula. One application of reduction of order formula is the following theorem of Johnson shown for example in Runde [**29**, Theorem 2.4.7, p. 58].

THEOREM 5.1.1. *Let A be a Banach algebra. Then the following are equivalent:*

(i) *A is amenable.*

(ii) *$\mathcal{H}^n(A, E^*) = \{0\}$ for every Banach A-bimodule E and for all $n \in \mathbb{N}$.*

Our main objective in this chapter is to show that the character amenable version of theorem 5.1.1 also holds. Before that we need some preparations.

DEFINITION 5.1.2. Let A be a Banach algebra and A^{op} be the Banach algebra with multiplication $\circ$ defined by $a \circ b = ba$. Then A^{op} is called the opposite algebra of A.

REMARK 5.1.3. It is clear that $\sigma(A) = \sigma(A^{op})$. Moreover, every Banach A-bimodule E has a canonical A^{op}-bimodule structure given by

$$a \circ x = x \cdot a, \quad x \circ a = a \cdot x, \quad (a \in A^{op}, x \in E). \tag{$*$}$$

In fact,

$$a \circ (b \circ x) = a \circ (x \cdot b) = (x \cdot b) \cdot a = x \cdot (ba) = (ba) \circ x = (a \circ b) \circ x,$$

$$(x \circ a) \circ b = b \cdot (x \circ a) = b \cdot (a \cdot x) = (ba) \cdot x = x \circ (ba) = x \circ (a \circ b),$$

$$(a \circ x) \circ b = b \cdot (a \circ x) = b \cdot (x \cdot a) = (b \cdot x) \cdot a = (x \circ b) \cdot a = a \circ (x \circ b).$$

In particular, let $\varphi \in \sigma(A) \cup \{0\}$. If $E \in {}_{\varphi}\mathcal{M}^A$, then $E \in \mathcal{M}_{\varphi}^{A^{op}}$ since

$$x \circ a = a \cdot x = \varphi(a)x, \quad (a \in A^{op},\ x \in E).$$

Denote $\mathcal{B}^n(A, E)$ to be the space of bounded n-linear maps from $A \times \cdots \times A$ (n-times) to E and we put $\mathcal{B}^0(A, E)$ to be E.

Definition 5.1.4. Let A be a Banach algebra and E be a Banach A-bimodule. For $x \in E$, define $\delta^0(x) = \delta_x \in \mathcal{B}^1(A, E)$ (where δ_x is the inner derivation by x) and for $n \in \mathbb{N}$, define a continuous linear map

$$\delta^n : \mathcal{B}^n(A, E) \to \mathcal{B}^{n+1}(A, E)$$

by

$$(\delta^n T)(a_1, a_2, \cdots a_{n+1}) = a_1 \cdot T(a_2, a_3, \cdots, a_{n+1}) + (-1)^{n+1} T(a_1, a_2, \cdots, a_n) \cdot a_{n+1}$$
$$+ \sum_{j=1}^{n} (-1)^j T(a_1, a_2, \cdots, a_{j-1}, a_j a_{j+1}, a_{j+2}, a_{j+3}, \cdots, a_{n+1}).$$

The maps δ^n, $n = 0, 1, \cdots$ are called the connecting maps. Moreover, the elements of $\ker \delta^n$ and $\operatorname{im} \delta^{n-1}$ are called the n-cocycles and the n-coboundaries, respectively. We denote these linear spaces by

$$\mathcal{Z}^n(A, E) = \ker \delta^n, \quad \mathcal{N}^n(A, E) = \operatorname{im} \delta^{n-1}.$$

Remark 5.1.5. The definition of $\mathcal{Z}^1(A, E)$ and $\mathcal{N}^1(A, E)$ coincide with our previous notation in definition 1.2.6. A direct but tedious calculation shows that $\delta^n \circ \delta^{n-1} = 0$ and so $\operatorname{im} \delta^{n-1} \subset \ker \delta^n$, that is, $\mathcal{N}^n(A, E) \subset \mathcal{Z}^n(A, E)$.

Definition 5.1.6. Let A be a Banach algebra and E be a Banach A-bimodule. For $n \in \mathbb{N}$, the nth cohomology group of A with coefficients in E is defined as the quotient vector space

$$\mathcal{H}^n(A, E) = \mathcal{Z}^n(A, E)/\mathcal{N}^n(A, E).$$

Also

$$\mathcal{H}^0(A,E) = \mathcal{Z}^0(A,E) = \ker \delta^0 = \{x \in E : \ a \cdot x = x \cdot a, \quad a \in A\}.$$

REMARK 5.1.7. The quotient space $\mathcal{H}^n(A,E)$ is in general only a seminormed space since $\mathcal{N}^n(A,E)$ may not be a closed subspace of $\mathcal{Z}^n(A,E)$ and therefore the norm on $\mathcal{Z}^n(A,E)$ (which is the same as that of $\mathcal{B}^n(A,E)$) can only induce a seminorm on the quotient space $\mathcal{H}^n(A,E)$. It should be also noted that the group structure on $\mathcal{H}^n(A,E)$ is the vector space addition which it has as a quotient space.

THEOREM 5.1.8. *Let A be a Banach algebra and E be a Banach A-bimodule. Suppose E is turned into a Banach A^{op}-bimodule in $(*)$. Then for every $n \geq 1$,*

$$\mathcal{H}^n(A,E) \cong \mathcal{H}^n(A^{op},E).$$

PROOF. Define the map

$$\pi : \mathcal{B}^n(A,E) \to \mathcal{B}^n(A^{op},E), \quad T \mapsto T^0$$

where $T^0(a_1,a_2,a_3,\cdots,a_n) := T(a_n,a_{n-1},\cdots,a_1), \quad (a_1,a_2,\cdots,a_n \in A).$

It is easy to check that π is an isometric linear isomorphism. We will show that this map sends $\mathcal{Z}^n(A,E)$ onto $\mathcal{Z}^n(A^{op},E)$ and $\mathcal{N}^n(A,E)$ onto $\mathcal{N}^n(A^{op},E)$. Then the theorem will follow by passing to the quotient. In fact, let $T \in \mathcal{Z}^n(A,E)$. Then $\delta^n T = 0$ and hence

$$\delta^n T(a_1,a_2,a_3,\cdots,a_{n+1}) = 0, \qquad (a_1,a_2,a_3,\cdots,a_{n+1} \in A).$$

This means that

$$a_1 \cdot T(a_2,a_3,\cdots,a_{n+1}) + (-1)^{n+1} T(a_1,a_2,\cdots,a_n) \cdot a_{n+1}$$
$$+ \sum_{j=1}^{n} (-1)^j T(a_1,a_2,\cdots,a_{j-1},a_j a_{j+1},a_{j+2},a_{j+3},\cdots,a_{n+1}) = 0.$$

Then $(-1)^{n+1} a_1 \cdot T(a_2,a_3,\cdots,a_{n+1}) + T(a_1,a_2,\cdots,a_n) \cdot a_{n+1}$

$$+ \sum_{j=1}^{n} (-1)^{n+j+1} T(a_1,a_2,\cdots,a_{j-1},a_j a_{j+1},a_{j+2},a_{j+3},\cdots,a_{n+1}) = 0,$$

if we multiply both sides of the equation by $(-1)^{n+1}$. Put $A_0 = (-1)^{n+1}a_1 \cdot T(a_2, a_3, \cdots, a_{n+1})$ and $B_0 = T(a_1, a_2, \cdots, a_n) \cdot a_{n+1}$. Then

$$A_0+B_0+(-1)^n \sum_{j=1}^{n} (-1)^{j+1}T(a_1, a_2, \cdots, a_{j-1}, a_j a_{j+1}, a_{j+2}, a_{j+3}, \cdots, a_{n+1}) = 0. \quad (**)$$

We use the above identity to verify that $\delta^n T^0 = 0$. In fact, for $a_1, a_2, \cdots, a_{n+1} \in A^{op}$, we have

$$\begin{aligned}
&(\delta^n\, T^0)(a_{n+1}, \cdots, a_1) = a_{n+1} \circ T^0(a_n, \cdots, a_1) + (-1)^{n+1}T^0(a_{n+1}, \cdots, a_2) \circ a_1 \\
&\quad + \sum_{j=0}^{n-1}(-1)^{j+1}T^0(a_{n+1}, \cdots, a_{n+1-j} \circ a_{n-j}, \cdots, a_1). \\
&= T(a_1, \cdots, a_n) \cdot a_{n+1} + (-1)^{n+1}a_1 \cdot T(a_2, \cdots, a_{n+1}) \\
&\quad + \sum_{j=0}^{n-1}(-1)^{j+1}T(a_1, \cdots, a_{n-j}a_{n+1-j}, \cdots, a_{n+1}) \\
&= B_0 + A_0 - T(a_1, a_2, \cdots, a_{n-1}, a_n a_{n+1}) + T(a_1, a_2, \cdots, a_{n-1}a_n, a_{n+1}) \\
&\quad - T(a_1, a_2, \cdots, a_{n-2}a_{n-1}, a_n, a_{n+1}) + \cdots + (-1)^n T(a_1 a_2, a_3, \cdots, a_n, a_{n+1}) \\
&=A_0 + B_0 + (-1)^n T(a_1 a_2, a_3, \cdots, a_n, a_{n+1}) + (-1)^{n-1}T(a_1, a_2 a_3, \cdots, a_{n+1}) \\
&\quad + \cdots + (-1)^{n-(n-3)}T(a_1, a_2, \cdots, a_{n-2}a_{n-1}, a_n, a_{n+1}) \\
&\quad + (-1)^{n-(n-2)}T(a_1, a_2, \cdots, a_{n-1}a_n, a_{n+1}) + (-1)^{n-(n-1)}T(a_1, a_2, \cdots, a_{n-1}, a_n a_{n+1}) \\
&=A_0 + B_0 + (-1)^n[T(a_1 a_2, a_3, \cdots, a_{n+1}) - T(a_1, a_2 a_3, \cdots, a_{n+1}) + \cdots \\
&\quad + (-1)^{n-3}T(a_1, \cdots, a_{n-2}a_{n-1}, a_n, a_{n+1}) + (-1)^{n-2}T(a_1, \cdots, a_{n-1}a_n, a_{n+1}) \\
&\quad + (-1)^{n-1}T(a_1, \cdots, a_n a_{n+1})] \\
&=A_0 + B_0 + (-1)^n \sum_{j=1}^{n} (-1)^{j+1}T(a_1, a_2, \cdots, a_{j-1}, a_j a_{j+1}, a_{j+2}, a_{j+3}, \cdots, a_{n+1}) \\
&=0,
\end{aligned}$$

by $(**)$. Essentially the same argument also shows that given $T \in \mathcal{Z}^n(A^{op}, E)$, then $T^0 \in \mathcal{Z}^n((A^{op})^{op}, E) = \mathcal{Z}^n(A, E)$ and of course $(T^0)^0 = T$. Thus the map $T \mapsto T^0$ sends $\mathcal{Z}^n(A, E)$ onto $\mathcal{Z}^n(A^{op}, E)$.

Next we show if $T \in \mathcal{N}^n(A, E)$, then $T^0 \in \mathcal{N}^n(A^{op}, E)$. Since $T \in \mathcal{N}^n(A, E)$, it follows that there exists $S \in \mathcal{B}^{n-1}(A, E)$ such that $T = \delta^{n-1} S$.

Thus for all $a_1, a_2, a_3, \cdots, a_n \in A$,

$$\begin{aligned} T(a_1, a_2, a_3, \cdots, a_n) = \delta^{n-1} S(a_1, \cdots, a_n) = a_1 \cdot S(a_2, a_3, \cdots, a_n) \\ + (-1)^n S(a_1, \cdots, a_{n-1}) \cdot a_n + \sum_{j=1}^{n-1} (-1)^j S(a_1, \cdots, a_j a_{j+1}, \cdots, a_n). \end{aligned}$$

If we put $A' = S^0(a_n, \cdots, a_2) \circ a_1$ and $B' = (-1)^n a_n \circ S^0(a_{n-1}, \cdots, a_1)$, then

$$\begin{aligned} T^0(a_n, \cdots, a_1) =& T(a_1, a_2, \cdots, a_n) = S^0(a_n, \cdots, a_2) \circ a_1 + (-1)^n a_n \circ S^0(a_{n-1}, \cdots, a_1) \\ &+ \sum_{j=1}^{n-1} (-1)^j S^0(a_n, \cdots, a_{j+1} \circ a_j, \cdots, a_1) \\ =& A' + B' + \sum_{j=1}^{n-1} (-1)^j S^0(a_n, \cdots, a_{j+1} \circ a_j, \cdots, a_1). \end{aligned}$$

So that

$$\begin{aligned} & T^0(a_n, \cdots, a_1) \\ =& A' + B' + (-1) S^0(a_n, \cdots, a_3, a_2 \circ a_1) + (-1)^2 S^0(a_n, \cdots, a_3 \circ a_2, a_1) \\ &+ (-1)^3 S^0(a_n, \cdots, a_4 \circ a_3, a_2, a_1) + \cdots + (-1)^{n-3} S^0(a_n, a_{n-1}, a_{n-2} \circ a_{n-3}, \cdots, a_1) \\ &+ (-1)^{n-2} S^0(a_n, a_{n-1} \circ a_{n-2}, \cdots, a_1) + (-1)^{n-1} S^0(a_n \circ a_{n-1}, a_{n-2}, \cdots, a_1). \end{aligned}$$

Rewriting the above expression in reverse order and using the identity $(-1)^j = (-1)^{2n-j}$, we get

$$\begin{aligned} & T^0(a_n, \cdots, a_1) \\ =& A' + B' + (-1)^{n+1} S^0(a_n \circ a_{n-1}, a_{n-2}, \cdots, a_1) + (-1)^{n+2} S^0(a_n, a_{n-1} \circ a_{n-2}, \cdots, a_1) \\ &+ (-1)^{n+3} S^0(a_n, a_{n-1}, a_{n-2} \circ a_{n-3}, \cdots, a_1) + \cdots + (-1)^{2n-3} S^0(a_n, \cdots, a_4 \circ a_3, a_2, a_1) \\ &+ (-1)^{2n-2} S^0(a_n, \cdots, a_3 \circ a_2, a_1) + (-1)^{2n-1} S^0(a_n, \cdots, a_3, a_2 \circ a_1) \\ =& A' + B' + \sum_{j=1}^{n-1} (-1)^{n+j} S^0(a_n, \cdots, a_{n-j+1} \circ a_{n-j}, \cdots, a_1). \end{aligned}$$

By substituting the values of A' and B', we get

$$\begin{aligned}
&T^0(a_n, \cdots, a_1)\\
=&a_n \circ [(-1)^n S^0](a_{n-1}, \cdots, a_1) + (-1)^n [(-1)^n S^0](a_n, a_{n-1}, \cdots, a_2) \circ a_1\\
&+ \sum_{j=1}^{n-1} (-1)^j [(-1)^n S^0](a_n, \cdots, a_{n-j+1} \circ a_{n-j}, \cdots, a_1)\\
=&\delta^{n-1}[(-1)^n S^0](a_n, \cdots, a_1).
\end{aligned}$$

Thus $T^0 = \delta^{n-1}[(-1)^n S^0] \in \mathcal{N}^n(A^{op}, E)$. □

COROLLARY 5.1.9. *Let A be a Banach algebra. Then A is left (right) character amenable if and only if A^{op} is right (left) character amenable.*

PROOF. Let A be left character amenable, and let $\varphi \in \sigma(A^{op}) \cup \{0\}$, $E \in {}_\varphi\mathcal{M}^{A^{op}}$. By remark 5.1.3, we have $\varphi \in \sigma(A) \cup \{0\}$ and

$$E \in \mathcal{M}_\varphi^{(A^{op})^{op}} = \mathcal{M}_\varphi^A.$$

Since A is left character amenable, it follows that $\mathcal{H}^1(A, E^*) = \{0\}$. Then

$$\mathcal{H}^1(A^{op}, E^*) \cong \mathcal{H}^1(A, E^*) = \{0\}.$$

Hence A^{op} is right character amenable. The rest of the assertions follows similarly. □

For the convenience of reference, we mention the following results from Dales [**8**, p. 132].

LEMMA 5.1.10. *Let A be a Banach algebra and E be a Banach A-bimodule and $n \in \mathbb{N}$. Then $\mathcal{B}^n(A, E)$ can be viewed as a Banach A-bimodule, using the following module actions:*

$$\begin{aligned}
(T * a)(a_1, a_2, \cdots, a_n) = T(aa_1, a_2, \cdots, a_n) + (-1)^n T(a, a_1, a_2, \cdots, a_{n-1}) \cdot a_n\\
+ \sum_{j=1}^{n-1} T(a, a_1, \cdots, a_j a_{j+1}, \cdots, a_n),
\end{aligned}$$

$$(a * T)(a_1, a_2, \cdots, a_n) = a \cdot T(a_1, a_2, \cdots, a_n), \quad (T \in \mathcal{B}^n(A, E)).$$

LEMMA 5.1.11. *Let A be a Banach algebra and E be a Banach A-bimodule and $k, p \in \mathbb{N}$. Then $\mathcal{H}^{k+p}(A, E)$ and $\mathcal{H}^k(A, (\mathcal{B}^p(A, E), *))$ are linearly isomorphic as seminormed spaces. Moreover, this identification is induced by the linear map*

$$\Lambda_{k,p} : \mathcal{B}^{k+p}(A, E) \to \mathcal{B}^k(A, \mathcal{B}^p(A, E)),$$

$$[(\Lambda_{k,p}T)(a_1, a_2, \cdots, a_k)](a_{k+1}, a_{k+2}, \cdots, a_{k+p}) = T(a_1, a_2, \cdots, a_k, a_{k+1}, \cdots, a_{k+p}),$$

for $a_1, a_2, \cdots, a_{k+p} \in A$ and $T \in \mathcal{B}^{k+p}(A, E)$.

In particular, the above lemma asserts that every Hochschild cohomology group $\mathcal{H}^n(A, E)$ of order n can be viewed as a first Hochschild cohomology group. For $\mathcal{H}^n(A, E^*)$, we have the following result shown by Johnson [**24**].

LEMMA 5.1.12. *Let A be a Banach algebra and E be a Banach A-bimodule. Denote $A \widehat{\otimes} A \widehat{\otimes} \cdots \widehat{\otimes} A \widehat{\otimes} E$ by $\mathcal{B}_n(A, E)$.*

(i) *For $n \geq 1$, $(\mathcal{B}_n(A, E))^*$ and $\mathcal{B}^n(A, E^*)$ are isometrically isomorphic as Banach A-bimodules, where the module actions on $\mathcal{B}_n(A, E)$ are defined by*

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes x) * a = a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes (x \cdot a),$$

and

$$\begin{aligned} a * (a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes x) &= aa_1 \otimes \cdots \otimes a_n \otimes x + (-1)^n a \otimes a_1 \otimes a_2 \otimes \cdots \otimes (a_n \cdot x) \\ &+ \sum_{j=1}^{n-1} (-1)^j a \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n \otimes x. \end{aligned}$$

(ii) *For $k, p \in \mathbb{N}$, $\mathcal{H}^{k+p}(A, E^*)$ and $\mathcal{H}^k(A, (\mathcal{B}_p(A, E))^*)$ are linearly isomorphic as seminormed spaces. More specifically, the identification is induced by the linear map*

$$\Lambda_{k,p} : \mathcal{B}^{k+p}(A, E^*) \to \mathcal{B}^k(A, (\mathcal{B}_p(A, E)^*),$$

$$\langle \Lambda_{k,p}T(a_1, a_2, \cdots, a_k), (a_{k+1} \otimes a_{k+2} \otimes \cdots a_{k+p} \otimes x) \rangle = \langle T(a_1, a_2, \cdots, a_{k+p}), x \rangle$$

for $a_1, a_2, \cdots, a_{k+p} \in A$, $x \in E$ and $T \in \mathcal{B}^{k+p}(A, E^*)$.

Lemma 5.1.11 and lemma 5.1.12 are often referred to as the reduction of order or dimension formula. An application of reduction of order formula is the following:

THEOREM 5.1.13. *If A is left (right) character amenable, then* $\mathcal{H}^n(A, E^*) = \{0\}$ *for all* $E \in \mathcal{M}_\varphi^A$ ($E \in {}_\varphi\mathcal{M}^A$), *where* $\varphi \in \sigma(A) \cup \{0\}$.

PROOF. Let A be left character amenable and $E \in \mathcal{M}_\varphi^A$. By the reduction of order formula mentioned in lemma 5.1.12 (ii), we have

$$\mathcal{H}^{n+1}(A, E^*) \cong \mathcal{H}^1(A, (\mathcal{B}_n(A, E))^*),$$

where the module actions on $\mathcal{B}_n(A, E)$ are given by

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes x) * a = a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes (x \cdot a) = \varphi(a)(a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes x),$$

and

$$\begin{aligned} a * (a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes x) &= aa_1 \otimes \cdots \otimes a_n \otimes x + (-1)^n a \otimes a_1 \otimes a_2 \otimes \cdots \otimes (a_n \cdot x) \\ &+ \sum_{j=1}^{n-1} (-1)^j a \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n \otimes x. \end{aligned}$$

for every $a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes x \in \mathcal{B}_n(A, E), a \in A$. Therefore $\mathcal{B}_n(A, E) \in \mathcal{M}_\varphi^A$. So

$$\mathcal{H}^1(A, (\mathcal{B}_n(A, E))^*) = \{0\}.$$

Thus

$$\mathcal{H}^{n+1}(A, E^*) = \{0\} \quad \text{for } n \geq 0.$$

When A is right character amenable, we use the opposite algebra A^{op}. By corollary 5.1.9 we have A^{op} is left character amenable. If we equip E with natural module action on A^{op}, then $E \in \mathcal{M}_\varphi^{A^{op}}$. Applying the above argument to A^{op}, we have

$$\mathcal{H}^{n+1}(A^{op}, E^*) \cong \mathcal{H}^1(A, (\mathcal{B}_n(A^{op}, E))^*) = \{0\}.$$

Thus by theorem 5.1.8,

$$\mathcal{H}^{n+1}(A, E^*) = \{0\} \quad \text{for every } n \geq 0. \quad \square$$

COROLLARY 5.1.14. *For a Banach algebras A, the following are equivalent:*

(i) *A is left (right) character amenable.*

(ii) $\mathcal{H}^n(A, E^*) = \{0\}$ *for every* $E \in \mathcal{M}_\varphi^A$ ($E \in {}_\varphi\mathcal{M}^A$), $\varphi \in \sigma(A) \cup \{0\}$, *and* $n \in \mathbb{N}$.

If A is a commutative amenable Banach algebra, then it is known that $\mathcal{H}^1(A, E) = \mathcal{H}^2(A, E) = \{0\}$ for all Banach A-bimodules E [**8**, Theorm 2.8.74, p. 303].

A natural question is to ask whether $\mathcal{H}^n(A, E) = \{0\}$ if A is commutative character amenable Banach algebras and E is an arbitrary Banach A-bimodule.

We will show this result is valid provided that E is of finite dimension. The general case remains an open question. Before that, we first mention the following result in [**12**, Lemma 2.10, p. 3651].

Lemma 5.1.15. *Let A be a commutative character amenable Banach algebra and let E be a Banach A-bimodule. Then there exists $\varphi_i, \psi_i \in \sigma(A) \cup \{0\}$, $i = 1, \cdots, n$, such that $E \cong \bigoplus_{i=1}^{n} \mathbb{C}_{\varphi_i, \psi_i}$ as Banach A-bimodules.*

Proof. For each $a \in A$, let $\pi(a)$ and $\pi'(a) \in \mathcal{B}(E)$ be defined by

$$\pi(a)x := a \cdot x, \quad \pi'(a)x := x \cdot a.$$

Since A is commutative, the families of operators $\mathfrak{F} = \{\pi(a), \pi'(a) : a \in A\}$ is commutative and hence we can find a suitable basis of E such that every element of $\mathfrak{F}$ can be represented as an upper-triangular matrix [**27**, Theorem 1.1.5]. Thus we may write

$$\pi(a) = \begin{pmatrix} \alpha_{11}(a) & \alpha_{12}(a) & \cdots & \alpha_{1n}(a) \\ 0 & \alpha_{22}(a) & \cdots & \alpha_{2n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{nn}(a) \end{pmatrix}; \pi'(a) = \begin{pmatrix} \alpha'_{11}(a) & \alpha'_{12}(a) & \cdots & \alpha'_{1n}(a) \\ 0 & \alpha'_{22}(a) & \cdots & \alpha'_{2n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha'_{nn}(a) \end{pmatrix}$$

where $a \in A$, $\alpha_{ij}, \alpha'_{ij} \in A^*$ and $\alpha_{ii}, \alpha'_{ii} \in \sigma(A) \cup \{0\}$ for all $1 \leq i \leq j \leq n$. By induction we show that for $i < j$, $\alpha_{ij} : A \to \mathbb{C}$ is a continuous derivation of the form

$$d(ab) = \varphi(a)d(b) + d(a)\psi(b), \quad (a, b \in A, \ \varphi, \psi \in \sigma(A) \cup \{0\}),$$

and hence it must be zero by the definition of character amenability. Let $1 \leq i \leq n$ be fixed. Using the above representation for $\pi(a)$ and the identity $\pi(ab) = \pi(a)\pi(b)$ we have

$$\alpha_{i\,i+1}(ab) = \alpha_{i\,i}(a)\alpha_{i\,i+1}(b) + \alpha_{i\,i+1}(a)\alpha_{i+1\,i+1}(b),$$

for which it follows that $\alpha_{i\,i+1} = 0$. We assume $\alpha_{i\,i+k} = 0$, for $k = 1, \cdots l-1 < n-i$. Then once again we may write

$$\alpha_{i\;i+l}(ab) = \sum_{k=0}^{l} \alpha_{i\,i+k}(a)\alpha_{i+k\;i+l}(b) = \alpha_{i\,i}(a)\alpha_{i\,i+l}(b) + \alpha_{i\,i+l}(a)\alpha_{i+l\,i+l}(b),$$

and hence $\alpha_{i\,i+l} = 0$, as required. By a similar argument, we can show that $\alpha'_{ij} = 0$ if $i \neq j$. Now the theorem follows if we put $\varphi_i := \alpha_{i\,i}$ and $\psi_i := \alpha'_{i\,i}$, $i = 1, \cdots, n$. □

The following result is well-known (see, [**8**, p. 127]).

LEMMA 5.1.16. *Let A be a Banach algebra and E be a Banach A-bimodule. Suppose that $E = F \oplus G$ where F and G are two Banach A-submodules of E. Then*

$$\mathcal{H}^n(A, E) = \mathcal{H}^n(A, F) \bigoplus \mathcal{H}^n(A, G).$$

The following result is due to Sangani Monfared in [**31**, Theorem 3.1, p. 702].

THEOREM 5.1.17. *If A is a commutative character amenable Banach algebra and E is a finite-dimensional Banach A-bimodule, then $\mathcal{H}^n(A, E) = \{0\}$ for every $n \in \mathbb{N}$.*

PROOF. Since A is left character amenable, by lemma 5.1.15 it follows that there is a decomposition for E such that

$$E = \bigoplus_{i=1}^{n} \mathbb{C}_{\varphi_i,\psi_i}, \quad \varphi_i, \psi_i \in \sigma(A) \cup \{0\}.$$

Moreover, applying the lemma 5.1.16, we have

$$\mathcal{H}^n(A, E) = \bigoplus_{i=1}^{n} \mathcal{H}^n(A, \mathbb{C}_{\varphi_i,\psi_i}).$$

It suffices to verify for every $\varphi_i, \psi_i \in \sigma(A) \cup \{0\}$,

$$\mathcal{H}^n(A, \mathbb{C}_{\varphi_i,\psi_i}) = \{0\}.$$

But this follows from corollary 5.1.14 since A is character amenable. □

5.2. Splitting properties of modules

Definition 5.2.1. Let A and B be two Banach algebras and I be a closed two-sided ideal of B. If the Banach algebra B/I is isomorphic to A, then we call B an extension of A by I. We may denote an extension by a short exact sequence $\Sigma = \Sigma(B, I)$,

$$\Sigma : 0 \to I \xrightarrow{\iota} B \xrightarrow{\pi} A \to 0,$$

where ι is the natural inclusion map and $\pi : B \to A$ is a continuous surjective algebra homomorphism such that $\ker \pi = I$. The extension $\Sigma(B, I)$ is called

(i) finite dimensional if I is finite-dimensional as a vector space, we say Σ is of dimension m if $\dim(I) = m$,
(ii) singular if $ab = 0$ for all $a, b \in I$.

Definition 5.2.2. An extension

$$\Sigma : 0 \to I \xrightarrow{\tau} B \xrightarrow{\pi} A \to 0$$

is called admissible if there exists a continuous linear map $\theta : A \to B$ such that $\pi \circ \theta = id_A$. The extension splits strongly if there exists a continuous algebra homomorphism $\theta : A \to B$ such that $\pi \circ \theta = id_A$.

Obviously every short exact sequence of Banach algebras that splits strongly is admissible. Admissibility is equivalent to the decomposition $B = I \oplus \theta(A)$ as a Banach space direct sum, while strong splitting is equivalent to $B = I \oplus A$ as a Banach space direct sum.

Let $\Sigma = \Sigma(B, I)$ be a singular extension of a Banach algera A. Clearly I can be viewed as a Banach B-bimodule using the product actions on B. Moreover, I is also a Banach A-bimodule with respect to the actions:

$$a \cdot x = bx, \quad x \cdot a = xb \quad (x \in I,\ a \in A,\ b \in B \text{ with } \pi(b) = a).$$

We show that these actions are well defined. Suppose there exists $b_1, b_2 \in B$ such that $\pi(b_1) = \pi(b_2) = a$, then $(b_1 - b_2) \in \ker \pi = I$, so $(b_1 - b_2)x = 0$, since I is singular. Thus $b_1 x = b_2 x$ and similarly we have $xb_1 = xb_2$.

Definition 5.2.3. Let $\Sigma(B, I)$ be a singular extension of a Banach algebra A. Let E be a Banach A-bimodule. The extension Σ is called a singular extension of A by E if I is isomorphic to E as a Banach A-bimodule.

Let A be a Banach algebra and E be a Banach A-bimodule. Johnson [**22**, Corollary 2.2, p. 868] showed that $\mathcal{H}^2(A, E) = \{0\}$ if and only if every singular admissible extension of A by E splits strongly. For finite dimensional extension, Bade, Dales, and Lykova [**3**, Theorem 1.8 (ii), p. 13] have shown the following result.

Theorem 5.2.4. *Let A be a Banach algebra. Suppose that every singular extension of dimension at most m splits strongly. Then every extension of dimension at most m splits strongly.*

In view of Johnson's result and the above theorem, the following result follows immediately [**3**, Theorem 2.6, p. 28].

Theorem 5.2.5. *Let A be a Banach algebra. Then the following are equivalent:*

(i) $\mathcal{H}^2(A, E) = \{0\}$ *for every finite-dimensional Banach A-bimodule E.*
(ii) *Every singular, finite-dimensional extension of A splits strongly.*
(iii) *Every finite-dimensional extension of A splits strongly.*

If we compare the statements in theorem 5.1.17 and theorem 5.2.5, we have the following result shown in [**31**, Corollary 3.2, p. 704].

Corollary 5.2.6. *Let A be a commutative character amenable Banach algebra. Then every finite-dimensional extension of A splits strongly.*

Dales [**8**, Propostion 2.8.24, p. 283] showed that if A is a commutative unital Banach algebra, and $\varphi, \phi \in \sigma(A)$ with $\varphi \neq \phi$, then $\mathcal{H}^1(A, \mathbb{C}_{\varphi,\phi}) = \mathcal{H}^2(A, \mathbb{C}_{\varphi,\phi}) = \{0\}$. If we inspect the proof of theorem 5.1.17, we see that we did not use the commutativity of A to show that $\mathcal{H}^n(A, \mathbb{C}_{\varphi,\phi}) = \{0\}$. So we obtain the following variant of Dale's result, in which the two characters φ and ϕ of A may be the same and A is not necessarily commutative, but character amenability of A is required.

COROLLARY 5.2.7. *Let A be a left character amenable Banach algebra. Then for all $\varphi, \phi \in \sigma(A) \cup \{0\}$ and $n \in \mathbb{N}$, we have $\mathcal{H}^n(A, \mathbb{C}_{\varphi,\phi}) = \{0\}$.*

We now take a look at splitting properties of exact short sequences of Banach modules.

DEFINITION 5.2.8. Let A be a Banach algebra and X, Y, Z be Banach left A-modules. Let Σ be a short exact sequence

$$\Sigma : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0,$$

in which f and g are continuous left module homomorphisms. The short exact sequence Σ is called admissible if there exists a continuous linear map $G : Z \to Y$ such that $g \circ G = id_Z$. Furthermore, Σ splits strongly if there exists a continuous left module homomorphism $G : Z \to Y$ such that $g \circ G = id_Z$.

Clearly, every short exact sequence of Banach left A-modules that splits strongly is admissible. For modules over character amenable Banach algebras we have a partial converse. The following is an analogue of splitting property [**7**, Theorem 2.3, p. 94] for amenable Banach algebras.

THEOREM 5.2.9. *Let A be a left character amenable Banach algebra and X be a right Banach A-module such that $X \in \mathcal{M}_{\varphi}^{A}$. Let Y and Z be left Banach A-modules. Then every admissible short exact sequence of Banach left A-modules*

$$\Sigma : 0 \to X^* \xrightarrow{f} Y \xrightarrow{g} Z \to 0,$$

splits strongly.

PROOF. If Σ is admissible then there exists $\widetilde{G} \in \mathcal{B}(Z, Y)$ such that $g \circ \widetilde{G} = id_Z$. Since $\mathcal{B}(Z, Y)$ can be viewed as a Banach A-bimodules with respect to the following module actions:

$$(a \cdot T)(z) := a \cdot T(z), \quad (T \cdot a)(z) := T(a \cdot z) \quad (a \in A,\ T \in \mathcal{B}(Z, Y),\ z \in Z),$$

we can define the map

$$d : A \to \mathcal{B}(Z, Y), \quad a \mapsto a \cdot \widetilde{G} - \widetilde{G} \cdot a.$$

The map d is a continuous inner derivation. Moreover, for $z \in Z, a \in A$,

$$\begin{aligned}(g \circ d(a))(z) &= g(d(a)(z)) \\ &= g((a \cdot \widetilde{G})(z) - (\widetilde{G} \cdot a)(z)) \\ &= g(a \cdot \widetilde{G}(z) - \widetilde{G}(a \cdot z)) \\ &= a \cdot (g \circ \widetilde{G})(z) - (g \circ \widetilde{G})(a \cdot z) \text{ since } g \text{ is a left } A \text{ module homomorphism} \\ &= a \cdot z - a \cdot z = 0.\end{aligned}$$

So $d(A) \subset \mathcal{B}(Z, \ker g) = \mathcal{B}(Z, \mathrm{im}\ f)$. Since $f(X^*) \cong X^*$, we can view d as a continuous derivation

$$d : A \to \mathcal{B}(Z, X^*) \cong (Z \widehat{\otimes} X)^*.$$

Note that the canonical A-module action on $Z \widehat{\otimes} X$ are given by

$$a \cdot (z \otimes x) = (a \cdot z) \otimes x \quad (z \otimes x) \cdot a = z \otimes (x \cdot a) = \varphi(a)(z \otimes x) \quad (a \in A,\ z \in Z,\ x \in X).$$

That is, $Z \widehat{\otimes} X \in \mathcal{M}_\varphi^A$. By the assumption of left character amenability of A, d must be inner, and there exists $Q \in (Z \widehat{\otimes} X)^* = \mathcal{B}(Z, X^*)$ such that $d(a) = a \cdot Q - Q \cdot a$, for every $a \in A$. Viewing Q as an element of $\mathcal{B}(Z, Y)$, we have

$$d(a) = a \cdot \widetilde{G} - \widetilde{G} \cdot a = a \cdot Q - Q \cdot a.$$

Put $G = \widetilde{G} - Q \in \mathcal{B}(Z, Y)$. Then $a \cdot G = G \cdot a$ for every $a \in A$. Thus for every $z \in Z, a \in A$,

$$G(a \cdot z) = (G \cdot a)(z) = (a \cdot G)(z) = a \cdot G(z).$$

Hence G is a left A-module homomorphism. The problem is reduced to showing G is also a bounded right inverse for g. Indeed, for every $z \in Z$,

$$g \circ G(z) = g \circ \widetilde{G}(z) - g \circ Q(z) = z - g(Q(z)) = z - 0 = z,$$

since $g \circ \widetilde{G} = id_Z$ and $\mathrm{im}\ Q \subset \ker g$. $\square$

CHAPTER 6

Conclusion and Future Work

Character amenability is weaker than the classical amenability introduced by B.E. Johnson. The definition requires continuous derivations from A into dual Banach A-bimodules to be inner, but only those modules are considered where either of the left or right module action is defined by a character of A. In chapter 2, we characterized character amenability in terms of bounded approximate identities and certain topological invariant elements of the second dual. We also saw the existence of certain topological invariant elements in the second dual is equivalent to the existence of a bounded left φ-approximate diagonal, which in turn is equivalent to the existence of a left φ-virtual diagonal. In theorem 2.2.17, we showed that the character amenability for each of the Banach algebras $L^1(G)$ and $A_p(G)$ is equivalent to the amenability of G, which is the main advantage of character amenability compared with classical amenability.

In chapter 3, we discussed the main hereditary properties of character amenability. If we inspect the original proof of 3.1.1 (ii), we used the equivalent characterization of character amenability in terms of certain topological invariant element of the second dual. The author's main contribution in this thesis, was to give a direct proof using only the original definition of character amenability involving derivations. We showed that if A is left character amenable and I has a bounded two-sided approximate identity, then I is left character amenable.

In chapter 4, we studied character amenability of Banach function algebras. In theorem 4.2.2, we showed that if a unital Banach function algebra A on a compact space X is character amenable, then the Choquet boundary of A must coincide with X. In the case of uniform algebras we obtained complete characterization of character amenability in term of the Choquet boundary of the underlying space (Corollary 4.2.4).

In chapter 5, we introduced character amenable version of the reduction of order formula. We also discussed splitting properties of modules over character amenable Banach algebras. In theorem 5.1.17, we showed triviality of cohomological groups with coefficients in finite-dimensional Banach modules over character amenable commutative Banach algebras. As a consequence we concluded that all finite-dimensional extensions of commutative character amenable Banach algebra split strongly. The section ends with another splitting property of short exact sequences over character amenable Banach algebras.

There are also some open questions about character amenability which require further investigation.

(1) Even though character amenability shows greater flexibility for particular types of Banach algebras, are there some other properties of character amenability not shared by classical amenability?
(2) It is not known whether for two arbitrary Banach algebras A and B we must have
$$\sigma(A\widehat{\otimes}B) = \sigma(A) \times \sigma(B).$$
(3) It is not known whether $\mathcal{H}^n(A, E) = \{0\}$ if A is commutative character amenable Banach algebras and E is an arbitrary Banach A-bimodule.
(4) Curits and Loy [**7**] showed that a Banach algebra A is amenable if and only if A has a bounded approximate identity and for each essential Banach A-bimodule E, every admissible short exact sequence
$$\Sigma : 0 \to X^* \xrightarrow{f} Y \xrightarrow{g} Z \to 0,$$
splits strongly. It is not known whether the analogue characterization of character amenability also holds.

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