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**ON CHARACTERISTIC HYPERSURFACES OF SUBMANIFOLDS
IN EUCLIDEAN SPACE**

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The main purpose of this paper is to prove that $M^n \subset E^N$, where $N = n(n+1)/2$, the characteristic $(n-1)$ -dimensional submanifolds of M^n are the asymptotic hypersurfaces.

1. Introduction. The concept of a characteristic submanifold of a given solution for a differential system, was introduced by E. Cartan in his theory of partial differential equations ([2], p. 79). Its importance appears in the treatment of the Cauchy problem.

Given an n -dimensional submanifold M^n of the Euclidean space E^N , we can define geometrically the notion of asymptotic submanifolds of M^n . The asymptotic lines have been used extensively for the study of the geometry of a surface in E^3 . For higher dimension and codimension some results have been obtained, using the generalized concept [3], [4], [9], [10]. It is well known, that the characteristic curves of a surface in E^3 are the asymptotic lines ([2], p. 143).

In §2 we start with a brief introduction to the Cartan-Kähler theory of differential equations. Then given a Riemannian manifold M^n , we consider the differential ideal, whose integral submanifolds determine local isometries of M^n into E^N , $N = n(n+1)/2$. Next assuming $M^n \subset E^N$, we characterize the $(n-1)$ -dimensional characteristic submanifolds of M^n .

In §3, we define the concept of asymptotic submanifolds of $M^n \subset E^N$, prove the main result and obtain a first order partial differential equation whose solutions are the characteristic hypersurfaces of M .

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2. Characteristic submanifold. Let M be an n -dimensional differentiable manifold. We denote by $A_k(M)$ the vector space of differential k -forms on M and $A(M) = \sum_{k=0}^n A_k(M)$. A *differential ideal* is an ideal U in $A(M)$ which is finitely generated, homogeneous (i.e., $U = \sum_{k=0}^n U_k$ where $U_k = U \cap A_k(M)$) are closed under exterior differentiation. We assume that U is a differential ideal which does not contain functions i.e., $U_0 = 0$. A p -dimensional submanifold S of M is said to be an (p -dimensional) *integral submanifold* for U , if $i^*(U) = 0$ i.e., $i^*(U_p) = 0$ where $i: S \rightarrow M$ is the inclusion map.

We denote by $T_x M$ the tangent space to M at $x \in M$; $G_x^p(M)$ denotes the Grassman manifold of p -dimensional subspaces of $T_x M$

and $G^p(M) = \bigcup_{x \in M} G_x^p(M)$ is given the usual manifold structure. An element $E_x^p \in G_x^p(M)$ is said to be an *integral element* for U , if all the differential forms of U vanish when restricted to the elements of E_x^p .

Let $I_x^p(U)$ denote the set of p -dimensional integral elements for U at x , and let $I^p(U) = \bigcup_{x \in M} I_x^p(M)$ be given the topology as a subspace of $G^p(M)$. If E_x^p is an integral element for U generated by $\{v_1, \dots, v_p\}$, we define the *polar space* $H(E_x^p)$ by

$$H(E_x^p) = \{v \in T_x M; \phi(v, v_1, v_2, \dots, v_p) = 0, \forall \phi \in U_{p+1}\}.$$

An integral element E_x^p , $p \geq 1$ is said to be *ordinary* if there exist integral elements $E_x^0, E_x^1, \dots, E_x^{p-1}$ with $E_x^0 \subset E_x^1 \subset \dots \subset E_x^{p-1} \subset E_x^p$ such that $\dim H(E_x^i)$ is constant on a neighborhood of E_x^i in $I^i(U)$ for $i = 0, 1, \dots, p-1$. A zero-dimensional integral element E_x^0 is said to be *regular* if $\dim H(E_x^0)$ is constant on a neighborhood of E_x^0 in $I^0(U)$. A p -dimensional integral element E_x^p , $p \geq 1$ is said to be *regular* if it is ordinary and $\dim H(E_x^p)$ is constant on a neighborhood of E_x^p in $I^p(U)$. We remark that when M is connected, this definition of regularity is equivalent to Cartan's ([2], pp. 61-67) according to which, an integral element E_x^p is regular if it is ordinary and $\dim H_x^p$ is equal to the dimension of a generic p -dimensional ordinary integral element.

It follows from Cartan-Kähler theorem ([2, pp. 68-74], [7, p. 26]) under the assumption that the manifold M and the differential forms are analytic, that given a q -dimensional ordinary integral element E_x^q , then there exists a q -dimensional integral submanifold S , which contains x and satisfies the requirement $T_x S = E_x^q$.

An integral submanifold S for U is said to be *singular* if $\forall x \in S$, the integral element $T_x S$ is not ordinary. We remark, that an integral submanifold S may be singular because none of its points is regular, or none of its tangential subspaces of dimension one, or two, \dots , etc., or $p-1$ is regular, where p is the dimension of S . Hence one may have different classes of singular integral submanifolds, whose degree of singularity decreases in a certain sense when one goes from one class to the next one.

Let S be a p -dimensional nonsingular integral submanifold for U , a submanifold $\bar{S} \subset S$ of dimension $q < p$ is called *characteristic* if $\forall x \in \bar{S}$, the integral element $T_x \bar{S}$ is not regular.

The concepts introduced above, can be found with more details in [2] and [7]. The Cartan-Janet theorem [1], [6] asserts that any real analytic, n -dimensional, Riemannian manifold can be locally mapped by a real analytic isometric embedding, into a Euclidean space E^N of dimension $N = n(n+1)/2$. In what follows we consider the differential ideal, whose integral submanifolds give local isome-

tries of M into E^N . Next assuming $M \subset E^N$, we characterize the $(n - 1)$ -dimensional characteristic submanifolds of M . We adopt the following indices convention

$$\begin{aligned} 1 \leq i, j, k, l \leq n; & \quad n + 1 \leq \lambda, \mu, \alpha \leq N; \\ 1 \leq I, J, K \leq N; & \quad N = n(n + 1)/2 \end{aligned}$$

and the summation convention with regard to repeated indices.

Let M be an n -dimensional Riemannian manifold with metric g . Let $F(M)$ denote the bundle of orthonormal frames over M , with the usual manifold structure. Under the action of the orthogonal group $O(n)$, $F(M)$ is a principal fiber bundle over M , with structural group $O(n)$. Let $\pi: F(M) \rightarrow M$ be the usual projection. We define the canonical forms $\omega^1, \dots, \omega^n$ on $F(M)$ by

$\pi_{*z}(v) = \omega^i(v)e_i$ where $z = (x, e_1, \dots, e_n) \in F(M)$ and $v \in T_z(F(M))$, hence $\pi^*g = \sum_i \omega^i \otimes \omega^i$. The connection forms ω_i^j on $F(M)$ are uniquely defined by

$$d\omega^i = \omega^j \wedge \omega_j^i, \quad \omega_i^i + \omega_j^j = 0.$$

Finally, if we consider

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j$$

then there exist functions R_{ijkl} , the components of the Riemann curvature tensor, defined on $F(M)$ such that

$$\Omega_i^j = -\frac{1}{2}R_{ijkl}\omega^k \wedge \omega^l, \quad R_{ijkl} = -R_{jilk}.$$

Similarly for E^N , we denote by $F(E^N)$ the bundle of orthonormal frames over E^N , $\bar{\pi}: F(E^N) \rightarrow E^N$ the projection, $\bar{\omega}^I$ the canonical forms on $F(E^N)$, $\bar{\omega}_I^J$ the connection forms on $F(E^N)$.

We consider the product manifold $B = F(M) \times F(E^N)$, and define the differential ideal on B . Let $\rho: B \rightarrow F(M)$ and $\bar{\rho}: B \rightarrow F(E^N)$ be the usual projections. Using ρ and $\bar{\rho}$ we can pull the differential forms $\omega^i, \omega_i^j, \bar{\omega}^I, \bar{\omega}_I^J$ back to B , we will denote the pulled-back forms by the same symbols. Let U be the differential ideal on B generated by

$$\begin{aligned} & \bar{\omega}^i - \omega^i \\ & \bar{\omega}^I \\ (*) & \bar{\omega}_i^j - \omega_i^j \\ & \omega^i \wedge \bar{\omega}_i^j \\ & \bar{\omega}_i^j \wedge \bar{\omega}_j^i + \frac{1}{2}R_{ijk} \omega^i \wedge \omega^k. \end{aligned}$$

We remark that there is a left action of $O(n)$ on B which preserves the differential ideal U . Namely if $A = (a_{ij}) \in O(n)$ we consider $L_A: B \rightarrow B$, which associates to

$$z = ((x, e_1, \dots, e_n), (\bar{x}, \bar{e}_1, \dots, \bar{e}_N)) \in B$$

the point

$$L_A(z) = \left(\left(x, \sum_i a_{1i} e_i, \dots, \sum_i a_{ni} e_i \right), \left(\bar{x}, \sum_i a_{1i} \bar{e}_i, \dots, \sum_i a_{ni} \bar{e}_i, \bar{e}_{n+1}, \dots, \bar{e}_N \right) \right).$$

It is not difficult to verify that $L_A^*(U \cap A_1(B)) \subset U \cap A_1(B)$ and hence $L_A^*(U) = U$.

Since we want to determine the $(n -)$ -dimensional characteristic submanifolds of $M^n \subset E^N$, we start characterizing the nonregular $(n - 1)$ -dimensional integral elements E_z^{n-1} for U in B , whose projections $\pi_* \circ \rho_*(E_z^{n-1})$ are $(n - 1)$ -dimensional. This characterization is obtained in Lemma 1(c).

Let p be an integer $0 \leq p < n$, we adopt the additional index conventions

$$1 \leq a, b, c \leq p; \quad p + 1 \leq r, s, t \leq n.$$

Suppose that E_z^p is a p -dimensional integral element for U , generated by vectors e_1, \dots, e_p such that

$$\omega^a(e_b) = \delta_b^a, \quad \omega^r(e_b) = 0.$$

If we denote, $h_{ia}^\lambda = \bar{\omega}_i^\lambda(e_a)$ then it follows, from the fact that the generators of U vanish when restricted to E_z^p , that

$$(1) \quad h_{ab}^\lambda = h_{ba}^\lambda$$

$$(2) \quad \sum_\lambda (h_{ia}^\lambda h_{jb}^\lambda - h_{ib}^\lambda h_{ja}^\lambda) - R_{ijab} = 0.$$

Denote by

$$H_{ia} = (h_{ia}^{n+1}, \dots, h_{ia}^N)$$

the vector in the $(N - n)$ -dimensional Euclidean space.

Let J^p denote the set of p -dimensional integral elements E_z^p , which satisfy the following conditions:

1. $\omega^1 \wedge \dots \wedge \omega^p \neq 0$ and $\omega^{p+1} = \dots = \omega^n = 0$ when restricted to E_z^p .

2. the vectors $\{H_{ma}: 1 \leq a \leq p, a \leq m \leq n - 1\}$ are linearly independent. Let $V^p = \{E_z^p \in I^p(U): L_A^*(E_z^p) \in J^p \text{ for some } A \in O(n)\}$.

Then V^p is an open subset of $I^p(U)$. Part of the next lemma is proved following ([5], with the obvious modifications).

LEMMA 1.

- (a) If $0 \leq p < n$, then $\dim H(E_z^p)$ is constant on V^p ;
- (b) For $0 \leq p < n$, if $E_z^p \in V^p$, then it is a regular element;
- (c) If $p = n - 1$, and E_z^{n-1} is an integral element such that $\pi_* \circ \rho_*(E_z^{n-1})$ is $(n - 1)$ -dimensional, then E_z^{n-1} is regular if and only if $E_z^{n-1} \in V^{n-1}$.

Proof. (a) Since $L_\lambda^*(U) = U$ it suffices to show that $\dim H(E_z^p)$ is constant on J^p . Assume that E_z^p is generated by e_1, \dots, e_p such that $\omega^a(e_b) = \delta_b^a$ and $\omega^r(e_b) = 0$. We consider the polar space

$$\begin{aligned} H(E_z^p) &= \{v \in T_z B; \phi(v, e_1, \dots, e_p) = 0 \forall \phi \in U_n\} \\ &= \{v \in T_z B; \phi_1(v) = 0 \text{ and } \phi_2(v, e_a) = 0 \forall \phi_1 \in U_1, \phi_2 \in U_2\} \end{aligned}$$

where last equality follows from the fact that U is generated by (*). Hence $H(E_z^p)$ consists of vectors $v \in T_z B$ which satisfy the following system of equations:

- (3) $\bar{\omega}^i(v) - \omega^i(v) = 0$
- (4) $\bar{\omega}^\lambda(v) = 0$
- (5) $\bar{\omega}_i^j(v) - \omega_i^j(v) = 0$
- (6) $h_{ia}^\lambda \omega^i(v) - \bar{\omega}_a^\lambda(v) = 0$
- (7) $\sum_\lambda h_{ja}^\lambda \bar{\omega}_i^\lambda(v) + \sum_\lambda h_{ia}^\lambda \bar{\omega}_j^\lambda(v) - R_{ijta} \omega^t(v) = 0, \quad i < j.$

If we specify $\omega^i(v)$, $\omega_i^j(v)$ then equations (3)-(6) will uniquely determine $\bar{\omega}^i(v)$, $\bar{\omega}_i^j(v)$ and $\bar{\omega}_a^\lambda(v)$. Moreover we remark that for $1 \leq i, j \leq p$, equation (7) is an immediate consequence of (1), (2) and (6). So we need only to consider (7) where $1 \leq i \leq p$, $p + 1 \leq j \leq n$ and $p + 1 \leq i < j \leq n$, i.e.,

$$\begin{aligned} (8) \quad &\sum_\lambda h_{sa}^\lambda \bar{\omega}_b^\lambda(v) + \sum_\lambda h_{ba}^\lambda \bar{\omega}_s^\lambda(v) - R_{bsia} \omega^i(v) = 0 \\ &\sum_\lambda h_{sa}^\lambda \bar{\omega}_i^\lambda(v) + \sum_\lambda h_{ia}^\lambda \bar{\omega}_s^\lambda(v) - R_{isia} \omega^i(v) = 0. \end{aligned}$$

Since in (8), for $a \neq b$, interchanging a and b does not modify the equation, we need only to consider

$$(9) \quad \sum_\lambda h_{ba}^\lambda \bar{\omega}_s^\lambda(v) = \left(\sum_\lambda h_{sa}^\lambda h_{ib}^\lambda - R_{bsia} \right) \omega^i(v), \quad a \leq b$$

$$(10) \quad \sum_\lambda h_{sa}^\lambda \bar{\omega}_i^\lambda(v) - \sum_\lambda h_{ia}^\lambda \bar{\omega}_s^\lambda(v) = R_{isia} \omega^i(v), \quad s < t.$$

Denote the vectors

$$H_i(v) = (\bar{\omega}_i^{p+1}(v), \dots, \bar{\omega}_i^N(v)).$$

We determine the vectors $H_{p+1}(v), \dots, H_n(v)$ so that they satisfy (9) and (10). The system (9) determines the dot product of $H_{p+1}(v)$ with the $p(p+1)/2$ linearly independent vectors $H_{b,a}$, $a \leq b$. Once we have chosen a particular $H_{p+1}(v)$ which satisfies this linear system of rank $p(p+1)/2$, the dot product of $H_{p+2}(v)$ with each of the $p(p+1)/2 + p$ linearly independent vectors $\{H_{m,a}: 1 \leq a \leq p, a \leq m \leq p+1\}$ is completely determined by (9) and (10). We continue in this fashion. Finally we find that the dot product of $H_n(v)$ with each of the $p(p+1)/2 + p(n-p-1)$ linearly independent vectors $\{H_{m,a}: 1 \leq a \leq p, a \leq m \leq n-1\}$ is completely determined. Hence we find that $\bar{\omega}_i^j(v)$ must satisfy a consistent system of linear equations which has rank $np(n-p)/2$. The polar system of E_z^p consists of these equations together with (3)-(6). Hence $\dim H(E_z^p)$ depends only on n and p whenever $E_z^p \in J^p$.

(b) Suppose that $E_z^p \in J^p$ is generated by e_1, \dots, e_p , such that $\omega^a(e_b) = \delta_b^a$ and $\omega^r(e_b) = 0$. If $0 \leq q \leq p$, we let E_z^q be the q -dimensional integral element generated by e_1, \dots, e_q . Then $E_z^q \in J^q$ and hence $\dim H(E_z^q)$ is constant in a neighborhood of E_z^q in $I^q(U)$. It follows that E_z^q is regular. Consequently if $E_z^p \in V^p$, then it is a regular integral element.

(c) From (b) we only need to prove that if E_z^{n-1} is a regular integral element then $E_z^{n-1} \in V^{n-1}$. Since $\pi_* \circ \rho_*(E_z^{n-1})$ is $(n-1)$ -dimensional, we can find an element $A \in O(n)$ such that $\omega^n = 0$ on $L_{A^*}(E_z^{n-1})$. Hence, we can assume that E_z^{n-1} is generated by e_1, \dots, e_{n-1} , such that $\omega^a(e_b) = \delta_b^a$ and $\omega^n(e_b) = 0$, where $1 \leq a, b \leq n-1$. Since E_z^{n-1} is regular, it follows that $\dim H(E_z^{n-1})$ is constant in a neighborhood of E_z^{n-1} in $I^{n-1}(U)$. The polar system of E_z^{n-1} is given by (3)-(6) and (7) reduces to

$$(11) \quad \sum_{\lambda} h_{\lambda a}^i \bar{\omega}_n^\lambda(v) = \left(\sum_{\lambda} h_{n\alpha}^\lambda h_{i\beta}^\lambda - R_{bna} \right) \omega^i(v), \quad a \leq b.$$

As in (a) if we specify $\omega^i(v)$, $\omega^j(v)$ then $\bar{\omega}^I(v)$, $\bar{\omega}^j(v)$ and $\bar{\omega}_n^i(v)$ will be uniquely determined by (3)-(6). Moreover the $n(n-1)/2$ components $\bar{\omega}_n^i(v)$ must satisfy the linear system (11) which has exactly $n(n-1)/2$ equations. Hence, if $\dim H(E_z^{n-1})$ is constant in a neighborhood of E_z^{n-1} , then the determinant of the coefficient matrix in (11) is nonzero, i.e., the vectors $\{H_{b,a}: 1 \leq a \leq b \leq n-1\}$ are linearly independent, which implies $E_z^{n-1} \in J^{n-1}$.

Let M be an n -dimensional Riemannian manifold and $f: M \rightarrow E^N$ an isometric imbedding. If $x_0 \in M$, there exists a neighborhood V of

x_0 in M and a section $\bar{\sigma}: V \rightarrow F(E^N)$ such that if $\bar{\sigma}(x) = (f(x), \bar{e}_1(x), \dots, \bar{e}_n(x))$, then $\bar{e}_1(x), \dots, \bar{e}_n(x)$ are tangent to $f(M)$. We consider the section $\sigma: V \rightarrow F(M)$, defined by $\sigma(x) = (x, e_1(x), \dots, e_n(x))$ where $f_*(e_i(x)) = \bar{e}_i(x)$. For simplicity, we denote by ω^i, ω_i^j the differential forms $\sigma^*\omega^i, \sigma^*\omega_i^j$ induced on V and similarly $\bar{\omega}^i, \bar{\omega}_i^j$ will denote the pulled-back forms $\bar{\sigma}^*\bar{\omega}^i, \bar{\sigma}^*\bar{\omega}_i^j$ on V . Consider the map $\Gamma: V \rightarrow B$ defined by $\Gamma(x) = (\sigma(x), \bar{\sigma}(x))$. Since f is an isometry, $\Gamma(V)$ is an integral submanifold for U in B . We say that a q -dimensional vector space $L^q \subset T_{x_0}M$, $0 \leq q < n$ is regular if $\Gamma_*(L)$ is a regular integral element for U . Similarly, a q -dimensional submanifold S of V is said to be characteristic, if $\Gamma(S)$ is a characteristic submanifold of $\Gamma(V)$. The characteristic hypersurfaces of M have at each point a nonregular tangent space. Our next lemma characterizes the nonregular $(n-1)$ -dimensional spaces tangent to M .

We denote the matrix $H^\lambda = (h_{ij}^\lambda)$ where $h_{ij}^\lambda = \bar{\omega}_i^\lambda(e_j)$. Moreover, given a matrix A , we denote by A_b the b th row of A and A_b^t denotes the transpose of A_b . Assume $\Gamma(V)$ is not a singular integral submanifold for U , then as an immediate consequence of Lemma 1(c), we obtain

LEMMA 2. Let $u_i\omega^i = 0$ be an $(n-1)$ -dimensional subspace of $T_{x_0}M$. We may assume that $\sum_{i=1}^n u_i^2 = 1$. Choose $A = (a_{ij}) \in O(n)$ such that $a_{ni} = u_i$. Then $u_i\omega^i = 0$ is nonregular if and only if the vectors

$$(A_a H^{n+1} A_b^t, \dots, A_a H^N A_b^t), \quad 1 \leq a \leq b \leq n-1$$

are linearly dependent, as vectors in E^{N-n} .

We remark that this condition determines a first order partial differential equation, and the characteristic hypersurfaces of M are the solutions of this equation. In the next section as a consequence of Lemma 3, the partial differential equation will be given in another form, which will not involve the choice of matrix A .

3. Asymptotic submanifolds; proof of main result. Let M be an n -dimensional C^∞ submanifold of E^N , $N = n(n+1)/2$ with the induced metric and such that the inclusion $i: M \rightarrow E^N$ is nondegenerate. Let $x \in M$ and denote by s the second fundamental form. A q -dimensional $0 < q < n$ linear subspace L of the tangent space $T_x M$ is called *asymptotic* if there exists a vector ξ normal to $T_x M$ such that $\langle s(X, Y), \xi \rangle = 0, \forall X, Y \in L$ where \langle, \rangle denotes the Euclidean metric. If L is of codimension one, we have an *asymptotic hyperplane* at x . A q -dimensional submanifold V of M , $q < n$ is called *asymptotic at $x \in V$* if $T_x V$ is asymptotic and asymptotic if this is

true for each $x \in V$. It is not difficult to see that V is an asymptotic hypersurface of M if and only if there exists a normal to the osculating space of V , which is also normal M . The notation of asymptotic submanifold in a more general context can be found in [4].

Let e_1, \dots, e_N be an orthonormal frame defined on a neighborhood of $x \in M$, such that e_1, \dots, e_n are tangent to M and e_{n+1}, \dots, e_N are normal to M . Let $\omega^1, \dots, \omega^N$ be the dual frame. With the same indices convention as in §2, we denote by $h^i_j = \omega^i(e_j)$ where ω^i are the connection forms. It follows from the definition that a hyperplane $u_i \omega^i = 0$ is asymptotic if and only if the second fundamental forms $h^i_j \omega^i \otimes \omega^j$ are linearly dependent when restricted to $u_i \omega^i = 0$.

The following algebraic lemma shows that the condition obtained in Lemma 2 is equivalent to saying that $u_i \omega^i = 0$ is asymptotic. As in §2 given a matrix A we denote by A_b the b th row of A and A_b^t denotes the transpose of A_b .

LEMMA 3. *Let φ^λ be $n \times n$ symmetric matrices $\lambda = n + 1, \dots, N$. $N = n(n + 1)/2$ and let $A = (a_{ij}) \in O(n)$. Then the vectors*

$$(A_b \varphi^{n+1} A_c^t, \dots, A_b \varphi^N A_c^t), \quad 1 \leq b \leq c \leq n - 1$$

are linearly dependent, as vectors in E^{N-n} , if and only if the quadratic forms $\varphi_{ij} \omega^i \otimes \omega^j$ are linearly dependent when restricted to $u_i \omega^i = 0$, where ω^i are n independent 1-forms.

Proof. The vectors $(A_b \varphi^{n+1} A_c^t, \dots, A_b \varphi^N A_c^t)$ are linearly dependent iff $\exists \alpha_\lambda \in \mathcal{R}$ not all zero, such that

$$A_b \left(\sum_{\lambda=n+1}^N \alpha_\lambda \varphi^\lambda \right) A_c^t = 0, \quad \forall 1 \leq b \leq c \leq n - 1.$$

We denote by D the matrix $D = \sum_{\lambda} \alpha_\lambda \varphi^\lambda$ and $W = (\omega^1, \dots, \omega^n)$. We will prove that $A_b D A_c^t = 0 \quad \forall 1 \leq b \leq c \leq n - 1$ if and only if $W D W^t = 0$ whenever $A_n W^t = 0$.

Consider

$$(12) \quad W D W^t = W A^t (A D A^t) A W^t.$$

Suppose $A_b D A_c^t = 0, \quad \forall 1 \leq b \leq c \leq n - 1$, then since D is symmetric

$$W D W^t = [W A_1, \dots, W A_{n-1}^t, W A_n^t] \begin{bmatrix} 0 & A_1 D A_n^t & \\ & \vdots & \\ & A_{n-1} D A_n^t & \\ A_n D A_1^t & \dots & A_n D A_n^t \end{bmatrix} \begin{bmatrix} A_1 W^t \\ \vdots \\ A_{n-1} W^t \\ A_n W^t \end{bmatrix}.$$

Hence if $A_n W^t = 0$ then $WDW^t = 0$, i.e., the quadratic forms $W\varphi^t W^t$ are linearly dependent whenever $A_n W^t = 0$.

Conversely, suppose $WDW^t = 0$ when $A_n W^t = 0$, then it follows from (12) that

$$(13) \quad 0 = \sum_{b=1}^{n-1} A_b DA_b^t \left(\sum_{k=1}^n a_{bk} \omega^k \right)^2 + 2 \sum_{\substack{b,c=1 \\ b < c}}^{n-1} A_b DA_c^t \left(\sum_{k,l=1}^n a_{bk} a_{cl} \omega^k \otimes \omega^l \right).$$

Let e_i be the dual basis of ω^i , i.e., $\omega^i(e_j) = \delta_j^i$. If we evaluate (13) at the pair (e_k, e_k) we get

$$\sum_{b=1}^{n-1} A_b DA_b^t a_{bk}^2 + 2 \sum_{\substack{b,c=1 \\ b < c}}^{n-1} A_b DA_c^t a_{bk} a_{ck} = 0, \quad \forall k = 1, \dots, n.$$

Adding over k , since $A \in O(n)$ we get

$$(14) \quad \sum_{b=1}^{n-1} A_b DA_b^t = 0.$$

If we apply (13) to the pairs (e_k, e_l) , (e_l, e_k) , $l \neq k$ and subtract we get

$$(15) \quad \sum_{\substack{b,c=1 \\ b < c}}^{n-1} A_b DA_c^t (a_{bk} a_{cl} - a_{bl} a_{ck}) = 0, \quad \forall 1 \leq k \leq l \leq n.$$

This is an homogeneous linear system of $n(n-1)/2$ equations with $(n-1)(n-2)/2$ unknowns $A_b DA_c^t$, $1 \leq b < c \leq n-1$. We claim that the rank of this system is $(n-1)(n-2)/2$. In fact, otherwise it follows from Sylvester-Franke theorem on determinants ([8], p. 94, take $m=2$), that the cofactor of a_{ni} in A is zero, $\forall i = 1, \dots, n$, which contradicts the fact that $\det A \neq 0$. Hence from (15) we have that

$$(16) \quad A_b DA_c^t = 0, \quad 1 \leq b < c \leq n-1.$$

Now (13) reduces to

$$(17) \quad \sum_{b=1}^{n-1} A_b DA_b^t \left(\sum_{k=1}^n a_{bk} \omega^k \right)^2 = 0$$

and from (14) we have

$$(18) \quad A_{n-1} DA_{n-1}^t = - \sum_{b=1}^{n-2} A_b DA_b^t.$$

If we substitute (18) in (17) we get

$$\sum_{b=1}^{n-2} A_b DA_b^t \left(\sum_{k=1}^n (a_{bk} - a_{n-1k}) \omega^k \right) \left(\sum_{k=1}^n (a_{bk} + a_{n-1k}) \omega^k \right) = 0.$$

Applying this equation to the pairs of vectors (e_k, e_l) , (e_l, e_k) , $l \neq k$ and subtracting we get

$$\sum_{b=1}^{n-2} A_b DA_b^i (a_{bk} a_{n-1l} - a_{n-1k} a_{bl}) = 0, \quad 1 \leq k < l \leq n.$$

This is a linear system of $n(n - 1)/2$ equations with $n - 2$ unknowns $A_b DA_b^i$, $1 \leq b \leq n - 2$. The rank of this system is $n - 2$. Otherwise, using Laplace's development of a determinant in the general version (i.e., the determinant is a linear function of the minors comprised in any number of lines) we get that the system (15) has rank lower than $(n - 1)(n - 2)/2$, which is a contradiction. Therefore $A_b DA_b^i = 0$ for $b = 1, \dots, n - 2$ and finally from (16) and (18) we conclude that $A_b DA_b^i = 0 \forall 1 \leq b \leq c \leq n - 1$.

Let $f: M \rightarrow E^N$ be an isometric embedding, with the same notation as in 2, we say that f is *singular* if $\forall x \in M, \Gamma_*(T_x M)$ is not an ordinary integral element for U in B . Then our main result follows immediately from Lemmas 2 and 3:

THEOREM. *Let $f: M \rightarrow E^N$ be a nonsingular isometric imbedding. An $(n - 1)$ -dimensional submanifold of M is characteristic if and only if it is asymptotic.*

We remark that f being nonsingular implies that f is nondegenerate, but for $n > 2$ it may exist a nondegenerate isometric imbedding which is singular; in this case all hypersurfaces would be asymptotic.

We observe that it is not difficult to prove that $u_i \omega^i = 0$ is asymptotic if and only if there exist real numbers a_i, b_i not all zero, such that

$$a_i h_{ij}^i \omega^i \otimes \omega^j \equiv u_i \omega^i \otimes b_j \omega^j.$$

This reduces to a homogeneous equation in u_i of degree n , $P(u_1, u_2, \dots, u_n) = 0$. In order to describe the polynomial P we consider the matrices

$$U_0 = \begin{bmatrix} u_1 & & & 0 \\ & u_2 & & \\ & & \ddots & \\ 0 & & & u_n \end{bmatrix} \quad U_p = \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ u_{p+1} & u_{p+2} & \dots & u_n \\ u_p & & & 0 \\ & u_p & & \\ & & \ddots & \\ 0 & & & u_p \end{bmatrix}$$

where U_p has the first $(p - 1)$ rows equal to zero, $1 \leq p \leq n - 1$

$$A_0 = \begin{bmatrix} h_{11}^{n+1} & h_{22}^{n+1} & \cdots & h_{nn}^{n+1} \\ \vdots & \vdots & & \vdots \\ h_{11}^N & h_{22}^N & \cdots & h_{nn}^N \end{bmatrix} \quad A_p = 2 \begin{bmatrix} h_{pp+1}^{n+1} & h_{pp+2}^{n+1} & \cdots & h_{pn}^{n+1} \\ \vdots & \vdots & & \vdots \\ h_{pp+1}^N & h_{pp+2}^N & \cdots & h_{pn}^N \end{bmatrix},$$

$$1 \leq p \leq n-1.$$

Then

$$P(u_1, u_2, \dots, u_n) = \det \begin{bmatrix} U_0 & U_1 & \cdots & U_{n-1} \\ A_0 & A_1 & \cdots & A_{n-1} \end{bmatrix} = 0.$$

Hence the characteristic hypersurfaces of M are the solutions of the first order partial differential equation defined by $P(u_1, \dots, u_n) = 0$. For $n = 3$ this equation was obtained by Cartan ([2], p. 208).

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