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# ON CHARACTERIZING THE BIVARIATE EXPONENTIAL AND GEOMETRIC DISTRIBUTIONS

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**Abstract.** In this note, a characterization of the Gumbel's bivariate exponential distribution based on the properties of the conditional moments is discussed. The result forms a sort of bivariate analogue of the characterization of the univariate exponential distribution given by Sahobov and Geshev (1974) (cited in Lau and Rao ((1982), *Sankhyā Ser. A*, **44**, 87)). A discrete version of the property provides a similar conclusion relating to a bivariate geometric distribution.

*Key words and phrases*: Characterization, bivariate exponential and geometric distributions, conditional moments.

Although different forms of bivariate exponential distributions such as those of Gumbel (1960), Freund (1961), Marshall and Olkin (1967) and Block and Basu (1974) exist in literature, how far these distributions can be characterized by properties analogous to the results in the univariate case have not been fully explored. Among the characterizations of the univariate exponential distribution, Marshall and Olkin (1967) considered the extension of the lack of memory property to the bivariate case, defining the same as

$$\bar{F}(s_1 + t, s_2 + t) = \bar{F}(s_1, s_2)\, \bar{F}(t, t), \qquad s_1, s_2, t \geq 0 ,$$

where

$$\bar{F}(s, t) = P(X_1 > s, X_2 > t) , \tag{1}$$

and used it to characterize the distribution specified by

$$\bar{F}(x_1, x_2) = \exp[ - \lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max (x_1, x_2)] \qquad x_1, x_2 > 0; \quad \lambda_1, \lambda_2, \lambda_{12} \geq 0 .$$

Instead of utilizing the conditional probabilities, Shanbag (1970) employed the conditional expectations to prove that a random variable $Y$ has univariate exponential distribution if and only if

$$E(Y-y \mid Y>y)=E(Y)\,, \tag{2}$$

for all $y>0$. Nair and Nair (1986), in an attempt to generalize this result to the bivariate case, found that a straightforward extension of (2) by taking the vectors $\boldsymbol{X}=(x_1, x_2)$ and $\boldsymbol{t}=(t_1, t_2)$ this, would lead only to the trival result that the variables $X_1$ and $X_2$ are independent and exponentially distributed. Accordingly they took up the condition.

$$E(X_i-t_i \mid X>t)=E(X_i \mid X_{3-i}>t_{3-i}), \quad i=1,2\,, \tag{3}$$

and established that the Gumbel's bivariate exponential distribution.

$$\bar{F}(x_1, x_2)=\exp\left[-\alpha_1 x_1-\alpha_2 x_2-\theta x_1 x_2\right] \qquad x_1, x_2>0; \quad \alpha_1, \alpha_2>0; \quad \theta \geq 0\,, \tag{4}$$

can be characterized by it and proposed (3) as alternative definition of bivariate lack of memory.

The present note extends the characterizing property in terms of the conditional means in (3) to the conditional moments, thereby providing a bivariate analogue of the Sahobov and Geshev's (1974) result in the univariate case which states that for any non-negative random variable $Y$,

$$E[(Y-y)^k \mid Y \geq y]=E(Y^k)\,,$$

for every $y>0$ implies that $Y$ follows the exponential distribution.

THEOREM 1. *Let $\boldsymbol{X}=(X_1, X_2)$ be a vector of non-negative random variables admitting probability density function with respect to Lebesgue measure, given by $f(x_1, x_2)$. Then $\boldsymbol{X}$ follows Gumbel's bivariate exponential distribution specified by* (4) *or equivalently by*

$$f(x_1, x_2)=\left[(\alpha_2+\theta x_1)(\alpha_1+\theta x_2)-\theta\right] \exp\left[-\alpha_1 x_1-\alpha_2 x_2-\theta x_1 x_2\right] \qquad x_1, x_2>0; \quad \alpha_1, \alpha_2>0; \quad \theta \geq 0\,, \tag{5}$$

*if and only if for all positive integers $k$,*

$$E[(X_i-t_i)^k \mid X>t]=a_k^{(i)}(t_{3-i}), \quad i=1,2\,, \tag{6}$$

*where $X>t$ stands for $X_1>t_1$ and $X_2>t_2$,*

$$a_k^{(i)}(t_{3-i})=E(X_i^k \mid X_{3-i}>t_{3-i})\,, \tag{7}$$

*are non-increasing,* $a_k^{(1)}$ *is independent of* $t_1$, $a_k^{(2)}$ *is independent of* $t_2$ *for all* $t_1, t_2>0$ *with*

$$a_1^{(i)}(0) = \alpha_i^{-1} \tag{8}$$

PROOF. When the conditions of the theorem are true

$$\begin{aligned} a_k^{(i)}(t_{3-i})\, \bar{F}(t_1, t_2) &= \iint_{X>t} (X_i - t_i)^k\, dF \\ &= \int_{t_1}^{\infty} (X_1 - t_1) \frac{\partial}{\partial x_1}[F_1(x_1) - F(x_1, t_2)]\,. \end{aligned} \tag{9}$$

Where $F_i$ and $F$ are the distribution functions of $X_i$ and $X$ corresponding to $f$ and $\bar{F}$ is as defined in equation (1). Successive integration in (9) yields

$$a_k^{(i)}(t_{3-i}) \frac{\partial^k}{\partial t_1^k} \bar{F}(t_1, t_2) = (-1)^k k!\, \bar{F}(t_1, t_2)\,. \tag{10}$$

To solve the above system of differential equations, we consider the case when $k=1$. On integration,

$$\bar{F}(t_1, t_2) = C(t_{3-i}) \exp\left[\frac{-t_i}{a_k^{(i)}(t_{3-i})}\right], \tag{11}$$

where

$$C(t_{3-i}) = 1 - F_{3-i}(t_{3-i}), \quad i = 1, 2\,.$$

When $t_{3-i}$ tends to zero, by virtue of (8)

$$1 - F_i(t) = \exp[-\alpha_i t_i]\,. \tag{12}$$

Eliminating $\bar{F}$ between the two relations arising out of (11) when $i=1,2$ leaves the functional equation,

$$t_1 a_1^{(2)}(t_1) - t_2 a_1^{(1)}(t_2) = (\alpha_1 t_1 - \alpha_2 t_2)\, a_1^{(2)}(t_1)\, a_1^{(1)}(t_2)\,.$$

Under the conditions assumed in the theorem, the above equation can be solved using techniques discussed in Hille (1972) in connection with the dissolvent equations. The solution is

$$a_1^{(i)}(t_{3-i}) = (\alpha_i + \theta t_{3-i})^{-1}, \quad i = 1, 2; \quad \theta \geq 0\,. \tag{13}$$

A detailed discussion of the intermediate steps in arriving at (13) is available in

Nair and Nair (1986). Further, from (7) we reach the conclusion,

$$\bar{F}(t_1, t_2) = \exp[-\alpha_1 t_1 - \alpha_2 t_2 - \theta t_1 t_2] .$$

Since this expression for $\bar{F}$ is an exponential function, with the exponent a linear function of one of the variables, when the other is kept constant, the solution of the partial differential equation (10) will be the same except for the change in the multiplicative constant. Hence from (10)

$$a_k^{(i)}(t_{3-i}) = k!\,(\alpha_i + \theta t_{3-i})^{-1} . \tag{14}$$

That the density function of $X$ is as in (5) is evident from (10) and (13). The converse is easily verified from direct calculations.

The corresponding result for the bivariate geometric distribution is as follows.

THEOREM 2. *A discrete non-negative random vector* $\boldsymbol{X}=(X_1, X_2)$ *with support* $I_2^+=\{(x_1, x_2)/x_1, x_2=0, 1, 2,\dots\}$ *has bivariate geometric distribution.*

$$f(x_1, x_2) = p_1^{x_1} p_2^{x_2} \theta^{x_1 x_2 - 1}[(1 - p_1\theta^{x_2+1})(1 - p_2\theta^{x_1+1}) + \theta - 1] ,$$
$$0 < p_1, \quad p_2 < 1, \quad 0 \le \theta \le 1 ,$$

*if and only if for all positive integers k, the condition* (6) *with the statements that follow it in Theorem* 1 *holds for the discrete vector* $\boldsymbol{X}$, *where*

$$a_1^{(i)}(0) = p_i(1 - p_i)^{-1}, \quad i = 1, 2 .$$

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