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ON CHARACTERS OF IRREDUCIBLE UNITARY REPRESENTATIONS OF GENERAL LINEAR GROUPS

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Introduction

Founding harmonic analysis on classical simple complex groups, I.M. Gelfand and M.A. Naimark in their classical book [GN] posed three basic questions: unitary duals, characters of irreducible unitary representations and Plancherel measures.

In the case of reductive p-adic groups, the only series of reductive groups where unitary duals are known are general linear groups. In this paper we reduce characters of irreducible unitary representations of $\mathrm{GL}(n)$ over a non-archimedean local field F, to characters of irreducible square-integrable representations of $\mathrm{GL}(m)$, with $m \leq n$ (we get an explicit expression for characters of irreducible unitary representations in terms of characters of irreducible square-integrable representations). In other words, we express characters of irreducible unitary representations in terms of the standard characters. We get also a formula expressing the characters of irreducible unitary representations in terms of characters of segment representations of Zelevinsky (the formula for the Steinberg character of $\mathrm{GL}(n)$ is a very special case of this formula).

The classification of irreducible square-integrable representations of $\mathrm{GL}(m, F)$'s has recently been completed ([Z], [BuK], [Co]). The characters of these representations are not yet known in the full generality, although there exists a lot of information about them ([Ca2], [CoMoSl], [K], [Sl]).

Zelevinsky's segment representations supported by minimal parabolic subgroups are one dimensional, so their characters are obvious. Therefore, we get the complete formulas for characters of irreducible unitary representations supported by minimal parabolic subgroups.

By the classification theorem for general linear groups over any locally compact non-discrete field, any irreducible unitary representation is parabolically induced by a tensor product of representations $u(\delta, n)$ where δ is an irreducible essentially square integrable representation of some general linear group and n a positive integer (see the second section for precise statements). Since there exists a simple formula for characters of parabolically induced representations in terms of the characters of inducing representations ([D]), it is enough to know the characters of $u(\delta, n)$'s.

Our idea in getting the formula for characters of irreducible unitary representations was to use the fact that unitary duals in archimedean and non-archimedean case can be expressed in the same way. Using the fact that there also exists a strong similarity of behavior of ends of complementary series, we relate in these two cases the formulas that express characters of $u(\delta, n)$'s in terms of standard characters. G. Zuckerman obtained the formula for the trivial character (of any reductive group) in terms of standard characters ([V1], Proposition 9.4.16). Along this lines we obtain the formula for characters of $u(\delta, n)$ in the non-archimedean case. Our method completely avoids the question of multiplicities.

It is interesting to note that the formulas for the trivial character in terms of standard characters of $\mathrm{GL}(n)$, are very different in the complex and the p-adic case.

Denote by ν the character $|\det|_F$ of $\mathrm{GL}(n)$. To each segment in irreducible cuspidal representations of general linear groups $[\rho, \nu^k\rho] = \{\rho, \nu\rho, \nu^2\rho, \dots, \nu^k\rho\}$, one can attach a unique essentially square integrable subquotient of $\rho \times \nu\rho \times \nu^2\rho \times \cdots \times \nu^k\rho$, which is denoted by $\delta([\rho, \nu^k\rho])$ ($\times$ denotes the J. Bernstein's and A.V. Zelevinsky's notation for the parabolic induction in the setting of general linear groups, see the first section). The representations $u(\delta, n)$'s are convenient to write in the Langlands classification. In this classification, they can be written as $L(([\nu\rho, \nu^d\rho], [\nu^2\rho, \nu^{d+1}\rho], \dots, [\nu^n\rho, \nu^{d+n-1}\rho]))$, where ρ is an irreducible cuspidal representation of some general linear group (see the third section for all details regarding notation). Denote by R^F the algebra of Grothendieck groups of categories of representations of finite length of all general linear groups over F (see the first section). Now Theorem 5.4 says

Theorem. *Let $n, d \in \mathbb{Z}, n, d \geq 1$. Let W_n be the group of permutations of the set $\{1, 2, \dots, n\}$. Denote $W_n^{(d)} = \{w \in W_n; w(i) + d \geq i$ for all $1 \leq i \leq n\}$. Then we have the following identity in R^F*

$$L(([\nu\rho, \nu^d\rho], \dots, [\nu^n\rho, \nu^{d+n-1}\rho])) = \sum_{w \in W_n^{(d)}} (-1)^{\mathrm{sgn}(w)} \prod_{i=1}^{n} \delta([\nu^i\rho, \nu^{w(i)+(d-1)}\rho]).$$

From this formula one can get simply, by multiplication of polynomials, the expression for (character of) arbitrary irreducible unitary representation of $\mathrm{GL}(m, F)$.

A similar approach to the characters of irreducible unitary representations of $\mathrm{GL}(n, \mathbb{R})$ can be undertaken, since D. Vogan has obtained the formulas for ends of complementary series in the real case which are similar to those ones in the complex case.

The ends of complementary series of $\mathrm{GL}(2n, \mathbb{C})$ seem to be known to several people (up to a twist by a character, there is only one such end). Unfortunately, there is no written reference for it. D. Vogan outlined to us a proof based on his paper [V2]. S. Sahi's paper [Sh1] seems to be the closest to the complete reference. He proved that there are no more than two subquotients at the end ([Sh1], Theorem 3C). One subquotient is more or less evident (reducibility is also evident). One needs only to see what is the other subquotient. Sahi described to us an explicit intertwining giving the other subquotient (subrepresentation in this case). In the appendix to our paper, we outline what is necessary to get the complete argument from [Sh1]. Instead of writing down Vogan's or Sahi's intertwining, we give a relatively simple argument based on the expression of the trivial character in terms of the standard characters. This argument seems to us more in the spirit of this paper and also the shortest since the necessary notations for such proof were already introduced. We are thankful to both of them for communicating us their proofs.

We are thankful to D. Vogan for informing us of Proposition 9.4.16 of [V1] (he explained it to us in the setting of $\mathrm{GL}(n, \mathbb{C})$). A number of conversations with D. Miličić was very helpful in the process of preparing of this paper. I. Mirković remarks helped a lot to make this paper more readable.

Now we describe the content of this paper, according to sections. In the first section, we introduce general notation for general linear groups over any local field. The second section recalls the classification of the non-unitary and unitary dual. The third section gives a more precise picture of these classifications in the non-archimedean case. In the fourth section we reformulate the parameterization for $\mathrm{GL}(n,\mathbb{C})$ for the later use. We recall here the formula for the character of the trivial representation in terms of standard characters. In the fifth section we construct an isomorphism, using which we get the character formula in the non-archimedean case. In the sixth section we consider the question of multiplicities related to characters of irreducible unitary representations. In the appendix, we sketch the argument which gives a complete written reference for the description of the ends of the complementary series of $\mathrm{GL}(2n,\mathbb{C})$.

Questions of S.J. Patterson motivated our work on characters of $\mathrm{GL}(n)$. The other motivation, besides the general concept of the harmonic analysis on general linear groups, are possible applications of such character formulas in trace formulas where only unitary representations show up.

We did this work during our stay in Göttingen as a guest of SFB 170. We want to thank SFB 170 for their kind hospitality and support.

1. General notation for $\mathrm{GL}(n)$

We denote in this paper the set of positive integers by $\mathbb{N} = \{1, 2, 3, \dots\}$.

A locally compact non-discrete field is denoted by F. The modulus character of F is denoted by $|\ \ |_F$. It is determined by the condition that $|x|_F \int_F f(xa)da = \int_F f(a)da$ for any continuous compactly supported complex-valued function f on F. In the above formula, da denotes a Haar measure of the additive group $(F,+)$ of the field. Note that $|\ \ |_{\mathbb{C}}$ is the square of the usual absolute value on $\mathbb{C}$.

Set $G_n = \mathrm{GL}(n,F)$ $(n \in \mathbb{Z}, n \geq 0)$. We identify characters of $F^\times$ with characters of G_n using the determinant homomorphism. The character of G_n corresponding to $|\ \ |_F$ will be denoted by ν.

Suppose that F is archimedean. Then we shall denote by $\mathfrak{g}_n$, or simply by $\mathfrak{g}$, the Lie algebra of G_n, where we consider G_n as a real Lie group. We denote by K_n, or simply by K, a maximal compact subgroup in G_n. If $F \cong \mathbb{R}$, then we can fix K_n to be the group $\mathrm{O}(n)$ of all real orthogonal matrices in G_n. If $F \cong \mathbb{C}$, then we can take K_n to be the group $\mathrm{U}(n)$ of all unitary matrices in G_n. The choice of maximal compact subgroups is not important for our purposes.

By an admissible module of G_n we shall mean an admissible representation of G_n when F is non-archimedean. If F is archimedean, then by an admissible G_n-module we shall mean an admissible $(\mathfrak{g}_n, K_n)$-module. Admissible G_n-modules form a category in a standard way.

The Grothendieck group of the category of all admissible G_n-modules of finite length, will be denoted by R_n^F. The set $\widetilde{G}_n$ of all equivalence classes of irreducible G_n-modules form a $\mathbb{Z}$-basis of R_n^F. For an admissible G_n-module π of finite length, we denote the semi simplification of π by s.s.$(\pi) \in R_n^F$. There is a natural partial order on R_n^F ($\widetilde{G}_n$ generates the cone of positive elements in R_n^F).

Let π_i be an admissible G_{n_i}-modules of finite length, for $i = 1, 2$. We can consider $\pi_1 \otimes \pi_2$ as a $G_{n_1} \times G_{n_2}$-module. We can identify $G_{n_1} \times G_{n_2}$ in an obvious way with a Levi factor of the following parabolic subgroup

$$\left\{ \begin{bmatrix} g_1 & * \\ 0 & g_2 \end{bmatrix} ; g_i \in G_{n_i} \right\}.$$

We denote by $\pi_1 \times \pi_2$ the $G_{n_1+n_2}$-module parabolically induced by $\pi_1 \otimes \pi_2$ from the above parabolic subgroup. Then $(\pi_1 \times \pi_2) \times \pi_3 \cong \pi_1 \times (\pi_2 \times \pi_3)$. Also $\chi(\pi_1 \times \pi_2) \cong (\chi\pi_1) \times (\chi\pi_2)$, for any character χ of $F^\times$.

Since the semi simplification of $\pi_1 \times \pi_2$ depends only on semi simplifications of π_1 and π_2, we can lift $\times$ to a $\mathbb{Z}$-bilinear mapping $\times : R_{n_1}^F \times R_{n_2}^F \to R_{n_1+n_2}^F$. Set $R^F = \oplus_{n \geq 0} R_n^F$. One lifts $\times$ to an operation $\times : R^F \times R^F \to R^F$ in an obvious way. In this way, R^F becomes an associative graded ring. This ring is commutative. For a fixed character χ of $F^\times$, lift $\pi \mapsto \chi\pi$ to a $\mathbb{Z}$-linear map $\chi : R^F \to R^F$. Then this map on R^F is an endomorphism of the graded ring.

Natural orders on R_n^F's determine a natural order $\leq$ on R^F in an obvious way. An additive homomorphism $\phi : R^F \to R^F$ is called positive if $x \in R^F$ and $x \geq 0$ implies $\phi(x) \geq 0$. In the sequel we shall denote R_n^F and R^F often simply by R_n and R. We shall use notation R_n^F and R^F only when we shall work with more than one field F in the same time.

2. Non-unitary and unitary dual

We say that a G_n-module is essentially square integrable if there exists a character of $F^\times$ such that the module, after twisting with that character, becomes square integrable modulo center. Denote by D^u (resp. by D) the set of all square integrable classes (resp. essentially square integrable classes) in $\cup_{n \geq 1} \widetilde{G}_n$. For each $\delta \in D$ there exists a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. We shall denote $\nu^{-e(\delta)}\delta$ by δ^u. Thus $\delta = \nu^{e(\delta)}\delta^u$, where $e(\delta) \in \mathbb{R}$ and δ^u is unitarizable.

Denote by $M(D)$ the set of all finite multisets in D. For $d = (\delta_1, \delta_2, \ldots, \delta_k) \in M(D)$, chose a permutation p of $\{1, 2, \ldots, k\}$ such that $e(\delta_{p(1)}) \geq e(\delta_{p(2)}) \geq \cdots \geq e(\delta_{p(k)})$. Then the module

$$\lambda(d) = \delta_{p(1)} \times \delta_{p(2)} \times \cdots \times \delta_{p(k)}$$

has a unique irreducible quotient, whose class depends only on d (not on p as above). We denote this irreducible quotient by $L(d)$. This is the Langlands classification for the groups G_n ([BoWh], [J]). It is easy to see that for a character χ of $F^\times$,

$$\chi L(\delta_1, \delta_2, \ldots, \delta_k) \cong L(\chi\delta_1, \chi\delta_2, \ldots, \chi\delta_k).$$

Define on $M(D)$ the structure of the additive semigroup by

$$(\delta_1, \delta_2, \ldots, \delta_k) + (\delta_1', \delta_2', \ldots, \delta_{k'}') = (\delta_1, \delta_2, \ldots, \delta_k, \delta_1', \delta_2', \ldots, \delta_{k'}').$$

Then $L(d + d')$ is always a subquotient of $L(d) \times L(d')$, for $d, d' \in M(D)$.

To shorten notation, we shall often denote s.s.$(\lambda(d)) \in R$ simply by $\lambda(d) \in R$ (this will cause no confusion). Since $\lambda(d) \in R$, $d \in M(D)$, form a basis of R, we have:

2.1. Proposition. *Ring R is a polynomial ring over D.* □

This fact was first observed by A.V. Zelevinsky in the non-archimedean case ([Z], Corollary 7.5).

The set of unitarizable classes in $\widetilde{G}_m$ will be denoted by $\widehat{G}_m$. For $\delta \in D^u$ and a positive integer n set

$$u(\delta, n) = L(\nu^{\frac{n-1}{2}}\delta, \nu^{\frac{n-3}{2}}\delta, \dots, \nu^{-\frac{n-1}{2}}\delta).$$

2.2. Theorem. *Set*

$$B = \{u(\delta, n),\ \ \nu^{\alpha}u(\delta, n) \times \nu^{-\alpha}u(\delta, n),\ \ \delta \in D^u,\ \ n \in \mathbb{N},\ \ 0 < \alpha < 1/2\}.$$

Then

(i) *If $\sigma_1, \dots, \sigma_k \in B$, then $\sigma_1 \times \dots \times \sigma_k \in \widehat{G}_p$ for some p.*

(ii) *If $\pi \in \widehat{G}_p$, then there exist $\sigma_1, \dots, \sigma_m \in B$, unique up to a permutation, such that*

$$\pi \cong \sigma_1 \times \dots \times \sigma_m.$$

Proof. In the non-archimedean case, this is Theorem 7.1 of [T4]. The proof there uses the positivity of the Zelevinsky involution (see the next section). The positivity was announced by J. Bernstein. Now there exist written proofs of this fact in preprints of A.-M. Aubert ([A]) and K.Procter ([P]). Let us note that a proof of the positivity is going to be contained also in a forthcoming paper of P. Schneider and U. Stuhler (see 14.19 of [LRaSu]).

In [T3] we gave a proof in the archimedean case. This proof uses a result announced by A.A. Kirillov for which there is no published proof yet. The complete proof is in [V1]. Our notation in the archimedean case follows [T3]. For a unified point of view on the above theorem, one can consult [T7]. □

3. Non-archimedean GL(n)

In this section we assume that F is non-archimedean.

Denote by $\mathcal{C}$ the set of all equivalence classes of irreducible cuspidal representation of all general linear groups $\mathrm{GL}(n, F)$, $n \geq 1$. Take $\rho \in \mathcal{C}$. Let $k \in \mathbb{Z}, k \geq 0$. The set

$$[\rho, \nu^k\rho] = \{\rho, \nu\rho, \nu^2\rho, \dots, \nu^k\rho\}$$

is called a segment of irreducible cuspidal representations. We shall sometimes write a segment

$$[\nu^{k_1}\rho, \nu^{k_2}\rho] = [k_1, k_2]^{(\rho)}$$

(here $k_1, k_2 \in \mathbb{R}$ such that $k_2 - k_1 \in \mathbb{Z}$ and $k_2 - k_1 \geq 0$). Denote the set of all such segments by $\mathcal{S}$. We denote by $M(\mathcal{S})$ the set of all finite multisets in $\mathcal{S}$. We consider the partial order on $M(\mathcal{S})$ which is introduced in 7.1 of [Z].

For $[\rho, \nu^k\rho] = \{\rho, \nu\rho, \nu^2\rho, \dots, \nu^k\rho\} \in M(\mathcal{S})$, the representation $\rho \times \nu\rho \times \nu^2\rho \times \dots \times \nu^k\rho$ has a unique irreducible subrepresentation, which we denote $\mathfrak{s}([\rho, \nu^k\rho])$, and a unique irreducible quotient, which we denote $\delta([\rho, \nu^k\rho])$. Representations $\mathfrak{s}([\rho, \nu^k\rho])$ will be called

Zelevinsky's segment representations. Representations $\delta([\rho,\nu^k\rho])$ are essentially square integrable.

The mapping $\mathfrak{s}(\Delta)\mapsto\delta(\Delta)$, $\Delta\in\mathcal{S}$, can be in a unique way extended to a ring morphism of R. A.V. Zelevinsky proved that this morphism is involutive ([Z], Proposition 9.12). This morphism, denoted $x\mapsto x^t$, is called the Zelevinsky involution. It carries irreducible representations into irreducible ones ([A], [P]). This fact is equivalent to the positivity of the involution.

Let $a=(\Delta_1,\Delta_2,\dots,\Delta_k)\in M(\mathcal{S})$. Choose a permutation p of $\{1,2,\dots,k\}$ such that $e(\delta(\Delta_{p(1)}))\geq e(\delta(\Delta_{p(2)}))\geq\cdots\geq e(\delta(\Delta_{p(k)}))$. Set

$$\lambda(a)=\delta(\Delta_{p(1)})\times\delta(\Delta_{p(2)})\times\cdots\times\delta(\Delta_{p(k)}),$$
$$\zeta(a)=\mathfrak{s}(\Delta_{p(1)})\times\mathfrak{s}(\Delta_{p(2)})\times\cdots\times\mathfrak{s}(\Delta_{p(k)}).$$

The representations $\lambda(a)$ and $\zeta(a)$ are determined up to an isomorphism by p as above. The representation $\zeta(d)$ has a unique irreducible subrepresentation which we denote by $Z(a)$. The representation $\lambda(d)$ has a unique irreducible quotient which we denote by $L(a)$.

If $a=(\Delta_1,\Delta_2,\dots,\Delta_k)\in M(\mathcal{S})$, then the number k will be denoted by $|a|$. For $\Delta=\{\rho,\nu\rho,\dots,\nu^\ell\rho\}\in\mathcal{S}$ we have in R

$$Z(\Delta)=\sum_{a\in M(\mathcal{S}),a\leq(\rho,\nu\rho,\dots,\nu^\ell\rho)}(-1)^{\ell+1-|a|}\ \text{s.s.}(\lambda(a)),\tag{3.1}$$

$$L(\Delta)=\sum_{a\in M(\mathcal{S}),a\leq(\rho,\nu\rho,\dots,\nu^\ell\rho)}(-1)^{\ell+1-|a|}\ \text{s.s.}(\zeta(a)).\tag{3.2}$$

This is a special case of Proposition 9.13 of [Z].

Fix an irreducible cuspidal representation ρ of a general linear group. Let d and n be positive integers. Denote $a(1,d)^{(\rho)}=[\nu^{-(d-1)/2}\rho,\nu^{(d-1)/2}\rho]\in\mathcal{S}$ and

$$a(n,d)^{(\rho)}=(a(1,d)^{(\nu^{-(n-1)/2}\rho)},a(1,d)^{(\nu^{1-(n-1)/2}\rho)},\dots,a(1,d)^{(\nu^{(n-1)/2}\rho)}).$$

We take formally $a(0,d)^{(\rho)}$ to be the empty multiset. Thus, $L(a(0,d)^{(\rho)})=Z(a(0,d)^{(\rho)})$ is the one-dimensional representation of G_0. This is identity of R. We take also $a(n,0)^{(\rho)}$ to be the empty multiset. So, again $L(a(n,0)^{(\rho)})=Z(a(n,0)^{(\rho)})$ is identity in R.

Note that

$$[\nu^{k_1}\rho,\nu^{k_2}\rho]=[k_1,k_2]^{(\rho)}=a(1,k_2-k_1+1)^{(\nu^{(k_1+k_2)}\rho)/2},\tag{3.3}$$

$$a(1,d)^{(\nu^\alpha\rho)}=[-(d-1)/2+\alpha,(d-1)/2+\alpha]^{(\rho)}.\tag{3.4}$$

3.1. Remark. Proposition 2.1 says that R is a polynomial ring over $\{\delta(\Delta),\Delta\in\mathcal{S}\}=\{L(a(1,d)^{(\rho)});d\in\mathbb{N}\text{ and }\rho\in\mathcal{C}\}$. Analogously, R is a polynomial ring over $\{\mathfrak{s}(\Delta),\Delta\in\mathcal{S}\}=\{Z(a(1,d)^{(\rho)});d\in\mathbb{N}\text{ and }\rho\in\mathcal{C}\}$.

For $\alpha\in\mathbb{R}$, one has directly the following isomorphisms

$$\nu^\alpha L(a(n,d)^{(\rho)})\cong L(a(n,d)^{(\nu^\alpha\rho)}),\tag{3.5}$$

$$\nu^\alpha Z(a(n,d)^{(\rho)})\cong Z(a(n,d)^{(\nu^\alpha\rho)}).\tag{3.6}$$

3.2. Theorem. *For positive integers n and d, and $\rho \in \mathcal{C}$, we have*

$$\nu^{1/2}L(a(n,d)^{(\rho)}) \times \nu^{-1/2}L(a(n,d)^{(\rho)}) = L(a(n+1,d)^{(\rho)}) \times L(a(n-1,d)^{(\rho)}) + L(a(n,d+1)^{(\rho)}) \times L(a(n,d-1)^{(\rho)})$$

and

$$\nu^{1/2}Z(a(n,d)^{(\rho)}) \times \nu^{-1/2}Z(a(n,d)^{(\rho)}) = Z(a(n+1,d)^{(\rho)}) \times Z(a(n-1,d)^{(\rho)}) + Z(a(n,d+1)^{(\rho)}) \times Z(a(n,d-1)^{(\rho)})$$

in R. Also

$$L(a(n,d)^{(\rho)}) = Z(a(d,n)^{(\rho)}).$$

Proof. Suppose that ρ is unitarizable. The last relation is just (iii) of Theorem A.9 of [T4]. For the proof of the first two relations, see Theorem 6.1 of [T5]. The case when ρ is not necessarily unitarizable, follows from the fact that $\nu^{\alpha},\ \alpha \in \mathbb{R}$, is an automorphism of R and the formulas (3.5) and (3.6). □

3.3. Lemma. *For positive integers n and d, and $\rho \in \mathcal{C}$, we have*

$$\big(Z(a(n,d)^{(\rho)})\big)^t = L(a(n,d)^{(\rho)}) = Z(a(d,n)^{(\rho)})$$

and

$$\big(L(a(n,d)^{(\rho)})\big)^t = Z(a(n,d)^{(\rho)}) = L(a(d,n)^{(\rho)}).$$

Proof. The lemma follows from the last relation of the above theorem, and [Ro]. We shall show here how one can get also the lemma directly from the relations in the above theorem. It is enough to prove only one of the above two relations. We shall prove the first one, by induction with respect to n. For $n = 1$, the relation holds by definition. It is enough to consider only the case $d \geq 2$.

Let $n \geq 1$ and suppose that the formula holds for $n' \leq n$. We shall show that it holds for $n+1$. By Theorem 3.2 we have

$$Z(a(n,d)^{(\nu^{1/2}\rho)}) \times Z(a(n,d)^{(\nu^{-1/2}\rho)}) = Z(a(n+1,d)^{(\rho)}) \times Z(a(n-1,d)^{(\rho)}) + Z(a(n,d+1)^{(\rho)}) \times Z(a(n,d-1)^{(\rho)}).$$

After applying the Zelevinsky involution and the inductive assumption, we get

$$L(a(n,d)^{(\nu^{1/2}\rho)}) \times L(a(n,d)^{(\nu^{-1/2}\rho)}) = Z(a(n+1,d)^{(\rho)})^t \times L(a(n-1,d)^{(\rho)}) + L(a(n,d+1)^{(\rho)}) \times L(a(n,d-1)^{(\rho)})$$

(if $n = 1$, then we take $Z(a(n-1,d)^{(\rho)})$ and $L(a(n-1,d)^{(\rho)})$ to be the identity of R, as before). From Theorem 3.2 we know

$$L(a(n,d)^{(\nu^{1/2}\rho)}) \times L(a(n,d)^{(\nu^{-1/2}\rho)}) = L(a(n+1,d)^{(\rho)}) \times L(a(n-1,d)^{(\rho)}) + L(a(n,d+1)^{(\rho)}) \times L(a(n,d-1)^{(\rho)}).$$

Taking the difference of these two relations, we get $(Z(a(n+1,d)^{(\rho)})^t - L(a(n+1,d)^{(\rho)})) \times L(a(n-1,d)^{(\rho)}) = 0$. Since R has no non-trivial zero divisors, we get $Z(a(n+1,d)^{(\rho)})^t = L(a(n+1,d)^{(\rho)})$. This finishes the proof of the Lemma. □

The formulas for the ends of the complementary series in the Theorem 3.2, give an algorithm to express $L(a(n,d)^{(\rho)})$ in terms of the standard representations $\lambda(a), a \in M(\mathcal{S})$. The same application can be done for $Z(a(n,d)^{(\rho)})$ which are expressed in terms of $\zeta(a), a \in M(\mathcal{S})$. We shall illustrate this algorithm in the case of the Langlands classification. To get the case of the Zelevinsky classification, one need only replace L by Z in the formulas below.

We can write the formula for the ends of the complementary series as

$$L(a(n+1,d)^{(\rho)}) = L(a(n-1,d)^{(\rho)})^{-1}$$
$$\Big(L(a(n,d)^{(\nu^{1/2}\rho)}) \times L(a(n,d)^{(\nu^{-1/2}\rho)}) - L(a(n,d+1)^{(\rho)}) \times L(a(n,d-1)^{(\rho)})\Big)$$

in the field of fractions of R (i.e., in the field of rational functions corresponding to R). We shall use these recursive relations for expressing characters of $L(a(n,d)^{(\rho)})$ in terms of the standard characters, for $n=2$ and $n=3$. For $n=2$ we have immediately the final formula

$$L(a(2,d)^{(\rho)}) = L(a(1,d)^{(\nu^{1/2}\rho)}) \times L(a(1,d)^{(\nu^{-1/2}\rho)}) - L(a(1,d+1)^{(\rho)}) \times L(a(1,d-1)^{(\rho)}).$$

For $n=3$ after a short calculation we get

$$\begin{aligned} L(a(3,d)^{(\rho)}) =& L(a(1,d)^{(\nu\rho)}) \times L(a(1,d)^{(\rho)}) \times L(a(1,d)^{(\nu^{-1}\rho)}) \\ & - L(a(1,d)^{(\nu\rho)}) \times L(a(1,d+1)^{(\nu^{-1/2}\rho)}) \times L(a(1,d-1)^{(\nu^{-1/2}\rho)}) \\ & - L(a(1,d+1)^{(\nu^{1/2}\rho)}) \times L(a(1,d-1)^{(\nu^{1/2}\rho)}) \times L(a(1,d)^{(\nu^{-1}\rho)}) \\ & + L(a(1,d+1)^{(\nu^{1/2}\rho)}) \times L(a(1,d+1)^{(\nu^{-1/2}\rho)}) \times L(a(1,d-2)^{(\rho)}) \\ & + L(a(1,d+2)^{(\rho)}) \times L(a(1,d-1)^{(\nu^{1/2}\rho)}) \times L(a(1,d-1)^{(\nu^{-1/2}\rho)}) \\ & - L(a(1,d+2)^{(\rho)}) \times L(a(1,d)^{(\rho)}) \times L(a(1,d-2)^{(\rho)}). \end{aligned}$$

4. GL($n,\mathbb{C}$)

In this section we assume $F = \mathbb{C}$. Denote $\mathbb{T} = \{z \in \mathbb{C}; |z|_{\mathbb{C}} = 1\}$. Then $\mathbb{C} \cong \mathbb{T} \times \mathbb{R}_+^\times$. By $|\ \ |$ we shall denote the usual absolute value on $\mathbb{C}$, i.e. $|\ \ | = |\ \ |_{\mathbb{C}}^{1/2}$.

Each character χ of $\mathbb{C}^\times$ can be written in a unique way as a product $\chi(z) = (z/|z|)^d\phi(z)$ where $d \in \mathbb{Z}$ and ϕ is trivial on $\mathbb{T}$. We shall say that $\phi \in (\mathbb{C}/\mathbb{T})\tilde{}$. This character will be denoted by $\chi(d)^{(\phi)}$. The character $\chi(d)^{(\phi)}$ determines via the determinant homomorphism the character of G_n which we denote by $\chi(n,d)^{(\phi)}$. Note that $\chi(1,d)^{(\phi)} = \chi(d)^{(\phi)}$. For $\phi' \in (\mathbb{C}/\mathbb{T})\tilde{}$ and $d' \in \mathbb{Z}$ we have $\chi(n,d')^{(\phi')}\chi(n,d)^{(\phi)} = \chi(n,d+d')^{(\phi'\phi)}$. In particular, for $\beta \in \mathbb{R}$ we have $\nu^\beta\chi(n,d)^{(\phi)} = \chi(n,d)^{(\nu^\beta\phi)}$. We take formally $\chi(0,d)^{(\phi)}$ to be the one-dimensional representation of G_0. This is identity in R. Recall that R is a polynomial ring over all $\chi(1,d)^{(\phi)}$, where $d \in \mathbb{Z}$ and ϕ is a character of $\mathbb{C}^\times$ trivial on $\mathbb{T}$.

The following proposition seems to be well known, but we could not find the complete reference for it. Very close to a complete reference is [Sh1]. See the appendix for an explanation of the proof.

4.1. Proposition. *Let $n, d \in \mathbb{Z}, n \geq 1$, and let ϕ be a character of $\mathbb{C}^\times$ trivial on $\mathbb{T}$. Then we have in R*

$$\nu^{1/2}\chi(n,d)^{(\phi)} \times \nu^{-1/2}\chi(n,d)^{(\phi)} = \chi(n+1,d)^{(\phi)} \times \chi(n-1,d)^{(\phi)} + \chi(n,d+1)^{(\phi)} \times \chi(n,d-1)^{(\phi)}$$

(in the above formula for $n=1$ we take $\chi(0,d)^{(\phi)}$ to be identity of R). □

4.2. Remark. It is enough to prove the above equality only for one d and one ϕ. Other equalities are obtained by twisting with characters, which are automorphisms of R.

Let W_n be the group of permutations of $\{1, 2, \ldots, n\}$. The parity of a permutation $w \in W_n$ will be denoted by $\mathrm{sgn}(w)$. The group W_n acts on $\mathbb{C}^n$ by permuting coordinates: $w(x_1, \ldots, x_n) = (x_{w^{-1}(1)}, \ldots, x_{w^{-1}(n)})$. Denote

$$\tau = (\tau_1, \tau_2, \ldots, \tau_n) = ((n-1)/2, (n-3)/2, \ldots, -(n-1)/2).$$

Here $\tau_i = (n-2i+1)/2$. Now we shall write a special case of the formula in Proposition 9.4.16 of [V1] for the character of the trivial representation. We are thankful to D. Vogan for telling us of that formula and for writing for us the specialization of the formula for $\mathrm{GL}(n, \mathbb{C})$. This proposition deals with unpublished results of G. Zuckerman.

4.3. Proposition. *For $n \in \mathbb{N}$ we have in R the following identity*

$$\chi(n,0)^{(1_{\mathbb{C}^\times})} = \sum_{w \in W_n} (-1)^{\mathrm{sgn}(w)} \chi(\tau_1 - \tau_{w(1)})^{(\nu^{(\tau_1+\tau_{w(1)})/2})} \times \cdots \times \chi(\tau_n - \tau_{w(n)})^{(\nu^{(\tau_n+\tau_{w(n)})/2})}.$$

□

Now obviously for $d \in \mathbb{Z}$ and $\phi \in (\mathbb{C}^\times/\mathbb{T})\tilde{}$ we have

$$(4.1) \quad \chi(n,d)^{(\phi)} = \sum_{w \in W_n} (-1)^{\mathrm{sgn}(w)} \chi(\tau_1 - \tau_{w(1)} + d)^{(\nu^{(\tau_1+\tau_{w(1)})/2}\phi)} \times \cdots \times \chi(\tau_n - \tau_{w(n)} + d)^{(\nu^{(\tau_n+\tau_{w(n)})/2}\phi)}.$$

5. Characters of non-archimedean GL(n)

In this section F will be non-archimedean.

Fix an irreducible cuspidal representation ρ of some $\mathrm{GL}(m, F)$. Let $R^F_{(\rho)}$ be the subring of R^F generated by

$$\{Z(a(1,d)^{(\nu^\alpha \rho)}); d \in \mathbb{N} \text{ and } \alpha \in \mathbb{R}\}.$$

Then $R^F_{(\rho)}$ is a polynomial ring over the above set (see Remark 3.1).

Fix some $\phi \in (\mathbb{C}^\times/\mathbb{T})\tilde{}$. Consider the mapping

$$\psi_{(\rho,\phi)} : \{Z(a(1,d)^{(\nu^\alpha\rho)}); d \in \mathbb{N} \text{ and } \alpha \in \mathbb{R}\} \to R^{\mathbb{C}}$$

given by $\psi_{(\rho,\phi)}(Z(a(1,d)^{(\nu^\alpha\rho)})) = \chi(1,d)^{(\nu^\alpha\phi)}$, when $d \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Then $\psi_{(\rho,\phi)}$ extends uniquely to a ring homomorphism $\Psi_{(\rho,\phi)} : R^F_{(\rho)} \to R^{\mathbb{C}}$. Obviously, $\Psi_{(\rho,\phi)}$ is injective, since it maps different indetermine to different indetermine. For any finite sum $\sum_{d\in\mathbb{N},\alpha\in\mathbb{R}} z_{d,\alpha}\ Z(a(1,d)^{(\nu^\alpha\rho)})$, where $z_{d,\alpha} \in \mathbb{Z}$ and $z_{d,\alpha} \neq 0$ can happen only for finitely many pairs $\alpha \in \mathbb{R}$, $d \in \mathbb{N}$, we have

$$\Psi_{(\rho,\phi)}(\sum_{d\in\mathbb{N},\alpha\in\mathbb{R}} z_{d,\alpha}\ Z(a(1,d)^{(\nu^\alpha\rho)})) = \sum_{d\in\mathbb{N},\alpha\in\mathbb{R}} z_{d,\alpha}\ \chi(1,d)^{(\nu^\alpha\phi)}. \tag{5.1}$$

Note that the homomorphism $\Psi_{(\rho,\phi)}$ does not respect the natural gradations of $R^F_{(\rho)} \subseteq R^F$ and $R^{\mathbb{C}}$. Nevertheless, $\Psi_{(\rho,\phi)}$ has the following interesting property:

5.1. Lemma. *Let $n, d \in \mathbb{N}$ and $\rho \in \mathcal{C}$. Suppose $n \leq d$. Then for any $\alpha \in \mathbb{R}$, $Z(a(n,d)^{(\nu^\alpha\rho)}) \in R^F_{(\rho)}$ and we have $\Psi_{(\rho,\phi)}(Z(a(n,d)^{(\nu^\alpha\rho)})) = \chi(n,d)^{(\nu^\alpha\rho)}$.*

Proof. We prove the lemma by induction with respect to n.

For $n = 1$, $Z(a(1,d)^{(\nu^\alpha\rho)}) \in R^F_{(\rho)}$ is by the definition of $R^F_{(\rho)}$. By the definition of $\Psi_{(\rho,\phi)}$, $\Psi_{(\rho,\phi)}$ extends $\psi_{(\rho,\phi)}$, and further $\psi_{(\rho,\phi)}(Z(a(1,d)^{(\nu^\alpha\rho)})) = \chi(1,d)^{(\nu^\alpha\phi)}$, for $d \geq 1$ and $\alpha \in \mathbb{R}$. Thus, the statement of the lemma holds for $n = 1$.

Let $n = 2$. Assume $d \geq 2$. Then we have from Theorem 3.2

$$\begin{aligned} &Z(a(2,d)^{(\nu^\alpha\rho)}) \\ =&Z(a(1,d)^{(\nu^{1/2}\nu^\alpha\rho)}) \times Z(a(1,d)^{(\nu^{-1/2}\nu^\alpha\rho)}) - Z(a(1,d+1)^{(\nu^\alpha\rho)}) \times Z(a(1,d-1)^{(\nu^\alpha\rho)}). \end{aligned}$$

This shows $Z(a(2,d)^{(\nu^\alpha\rho)}) \in R^F_{(\rho)}$. From Proposition 4.1 we have

$$\begin{aligned} &\chi(2,d)^{(\nu^\alpha\phi)} \\ =&\chi(1,d)^{(\nu^{1/2}\nu^\alpha\phi)} \times \chi(1,d)^{(\nu^{-1/2}\nu^\alpha\phi)} - \chi(1,d+1)^{(\nu^\alpha\phi)} \times \chi(1,d-1)^{(\nu^\alpha\phi)}. \end{aligned}$$

Thus, since $d \geq 2$ implies $d-1, d, d+1 \geq 1$; we have from the above formulas

$$\begin{aligned} \Psi_{(\rho,\phi)}(Z(a(2,d)^{(\nu^\alpha\rho)}) &= \Psi_{(\rho,\phi)}(Z(a(1,d)^{(\nu^{1/2}\nu^\alpha\rho)})) \times \Psi_{(\rho,\phi)}(Z(a(1,d)^{(\nu^{-1/2}\nu^\alpha\rho)})) \\ &\quad - \Psi_{(\rho,\phi)}(Z(a(1,d+1)^{(\nu^\alpha\rho)})) \times \Psi_{(\rho,\phi)}(Z(a(1,d-1)^{(\nu^\alpha\rho)})) \\ &= \chi(1,d)^{(\nu^{1/2}\nu^\alpha\phi)} \times \chi(1,d)^{(\nu^{-1/2}\nu^\alpha\phi)} - \chi(1,d+1)^{(\nu^\alpha\phi)} \times \chi(1,d-1)^{(\nu^\alpha\phi)} = \chi(2,d)^{(\nu^\alpha\phi)}. \end{aligned}$$

Therefore, the statement holds for $n = 2$. Note that $\Psi_{(\rho,\phi)}(Z(a(2,1)^{(\nu^\alpha\rho)})) \neq \chi(2,1)^{(\nu^\alpha\phi)}$.

Let $n \in \mathbb{Z}$, $n \geq 2$ and suppose that the statement of the lemma holds for $n' \leq n$. We shall show now that it holds for $n+1$. Fix some $d \in \mathbb{Z}$, $d \geq n+1$ and $\alpha \in \mathbb{R}$. First, the description of the composition series of representations $\zeta(a), a \in M(\mathcal{S})$, in Theorem 7.1 of [Z], implies $Z(a(n+1,d)^{(\nu^\alpha\rho)}) \in R^F_{(\rho)}$. Theorem 3.2 implies that we have in R^F

$$\begin{aligned} &Z(a(n,d)^{(\nu^{1/2}\nu^\alpha\rho)}) \times Z(a(n,d)^{(\nu^{-1/2}\nu^\alpha\rho)}) \\ =&Z(a(n+1,d)^{(\nu^\alpha\rho)}) \times Z(a(n-1,d)^{(\nu^\alpha\rho)}) + Z(a(n,d+1)^{(\nu^\alpha\rho)}) \times Z(a(n,d-1)^{(\nu^\alpha\rho)}). \end{aligned}$$

Applying $\Psi_{(\rho,\phi)}$ to the above identity, and using the inductive assumption (since $d \geq n$, $d \geq n-1$, $d+1 \geq n$ and $d-1 \geq n$), we get

$$\begin{aligned} &\chi(n,d)^{(\nu^{1/2}\nu^\alpha\rho)} \times \chi(n,d)^{(\nu^{-1/2}\nu^\alpha\rho)} \\ =&\Psi_{(\rho,\phi)}(Z(a(n+1,d)^{(\nu^\alpha\rho)})) \times \chi(n-1,d)^{(\nu^\alpha\rho)} + \chi(n,d+1)^{(\nu^\alpha\rho)} \times \chi(n,d-1)^{(\nu^\alpha\rho)}. \end{aligned}$$

Proposition 4.1 implies

$$\begin{aligned} &\chi(n,d)^{(\nu^{1/2}\nu^\alpha\phi)} \times \chi(n,d)^{(\nu^{-1/2}\nu^\alpha\phi)} \\ =&\chi(n+1,d)^{(\nu^\alpha\phi)} \times \chi(n-1,d)^{(\nu^\alpha\phi)} + \chi(n,d+1)^{(\nu^\alpha\phi)} \times \chi(n,d-1)^{(\nu^\alpha\phi)}. \end{aligned}$$

Subtracting the above two formulas, we get

$$\left(\Psi_{(\rho,\phi)}(Z(a(n+1,d)^{(\nu^\alpha\rho)})) - \chi(n+1,d)^{(\nu^\alpha\phi)}\right) \times \chi(n-1,d)^{(\nu^\alpha\phi)} = 0.$$

Since $R^{\mathbb{C}}$ is a polynomial ring, it has no non-trivial zero divisors. Clearly, $\chi(n-1,d)^{(\nu^\alpha\phi)} \neq 0$. Thus $\Psi_{(\rho,\phi)}(Z(a(n+1,d)^{(\nu^\alpha\rho)})) = \chi(n+1,d)^{(\nu^\alpha\phi)}$. This finishes the proof of the lemma. $\square$

The last lemma and formulas (4.1) and (5.1) imply directly the following

5.2. Corollary. *For $1 \leq n \leq d$ we have*

$$Z(a(n,d)^{(\rho)}) = \sum_{w\in W_n} (-1)^{\mathrm{sgn}(w)} \prod_{i=1}^{n} Z(a(1,\tau_i - \tau_{w(i)} + d)^{(\nu^{(\tau_i+\tau_{w(i)})/2}\rho)}). \quad \square$$

We shall write the above formula in a slightly different way:

$$\begin{aligned} Z(a(n,d)^{(\rho)}) &= \sum_{w\in W_n} (-1)^{\mathrm{sgn}(w)} \\ &\prod_{i=1}^{n} Z(a(1,(\tau_i+(d-1)/2)-(\tau_{w(i)}-(d-1)/2)+1)^{(\nu^{((\tau_i+(d-1)/2)+(\tau_{w(i)}-(d-1)/2))/2}\rho)}) \\ &= \sum_{w\in W_n} (-1)^{\mathrm{sgn}(w)} \prod_{i=1}^{n} \mathfrak{s}([\nu^{\tau_{w(i)}-(d-1)/2}\rho, \nu^{\tau_i+(d-1)/2}\rho]) \end{aligned}$$

by (3.3). Writing the last formula for $\tilde{\rho}$ and then taking the contragredient, gives

$$\begin{aligned} Z(a(n,d)^{(\rho)}) &= \sum_{w\in W_n} (-1)^{\mathrm{sgn}(w)} \prod_{i=1}^{n} \mathfrak{s}([\nu^{-\tau_i-(d-1)/2}\rho, \nu^{-\tau_{w(i)}+(d-1)/2}\rho]) \\ &= \sum_{w\in W_n} (-1)^{\mathrm{sgn}(w)} \prod_{i=1}^{n} \mathfrak{s}([\nu^{-(n+d)/2+i}\rho, \nu^{(d-n)/2+w(i)-1}\rho]) \end{aligned}$$

(here we have used the formula for the contragredient representation in the Zelevinsky classification, Theorem 7.1 of [Z]). Thus

$$\begin{aligned} \nu^{(n+d)/2} Z(a(n,d)^{(\rho)}) &= Z(([\nu\rho,\nu^d\rho],[\nu^2\rho,\nu^{d+1}\rho],\dots,[\nu^n\rho,\nu^{d+n-1}\rho])) \\ &= \sum_{w\in W_n} (-1)^{\mathrm{sgn}(w)} \prod_{i=1}^{n} \mathfrak{s}([\nu^{i}\rho, \nu^{w(i)+(d-1)}\rho]). \end{aligned} \tag{5.2}$$

5.3. Theorem. *Let $n,d\in\mathbb{N}$. Denote $W_n^{(d)} = \{w\in W_n; w(i)+d\geq i$ for any $1\leq i\leq n\}$. Then we have*

$$\begin{aligned} \nu^{(n+d)/2} Z(a(n,d)^{(\rho)}) &= Z(([\nu\rho,\nu^d\rho],[\nu^2\rho,\nu^{d+1}\rho],\dots,[\nu^n\rho,\nu^{d+n-1}\rho])) \\ &= \sum_{w\in W_n^{(d)}} (-1)^{\mathrm{sgn}(w)} \prod_{i=1}^{n} \mathfrak{s}([\nu^{i}\rho, \nu^{w(i)+(d-1)}\rho]), \end{aligned}$$

where in above formula we take the terms of the form $\mathfrak{s}([\nu^i\rho,\nu^j\rho])$ with $i>j$ to be the identity of R^F (i.e., they do not show up in the above formula).

Proof. For a segment $\Delta=[\sigma,\nu^k\sigma]\in\mathcal{S}$ set $\Delta^-=[\sigma,\nu^{k-1}\sigma]$ (if $k=0$, then set $\Delta^-=\emptyset$). Denote by $\mathcal{D}$ the endomorphism of the ring R^F which satisfy $\mathcal{D}(\mathfrak{s}(\Delta))=\mathfrak{s}(\Delta)+\mathfrak{s}(\Delta^-)$ (we take $\mathfrak{s}(\emptyset)$ to be identity of R^F). Then $\mathcal{D}$ is positive. See [Z] for more details regarding $\mathcal{D}$. One can easily see that $\mathcal{D}$ is injective. Further, let $x\in R^F, x\neq 0$. Write $\mathcal{D}(x)=\sum y_n$ where $y_n\in R_n^F$. Chose m so that $y_m\neq 0$ and $y_n=0$ for $n<m$. Then we call y_m the highest derivative of x and denote by h.d.(x). Note that h.d.$(x_1\times x_2)=$ h.d.$(x_1)\times$ h.d.(x_2).

We shall prove now the theorem. Note that one needs to prove the formula only for $1\leq d\leq n-1$. Fix such d and assume that the formula holds for $d+1$ and n. So,

$$\begin{aligned} \nu^{(n+d+1)/2} Z(a(n,d+1)^{(\rho)}) &= Z(([\nu\rho,\nu^{d+1}\rho],[\nu^2\rho,\nu^{d+2}\rho],\dots,[\nu^n\rho,\nu^{d+n}\rho])) \\ &= \sum_{w\in W_n^{(d+1)}} (-1)^{\mathrm{sgn}(w)} \prod_{i=1}^{n} \mathfrak{s}([\nu^{i}\rho, \nu^{w(i)+d}\rho]). \end{aligned}$$

Suppose that ρ is a representation of $\mathrm{GL}(m,F)$.

The highest derivative of the left-hand side is

$$\nu^{(n+d)/2} Z(a(n,d)^{(\rho)}) = Z(([\nu\rho,\nu^d\rho],[\nu^2\rho,\nu^{d+1}\rho],\dots,[\nu^n\rho,\nu^{d+n-1}\rho]))$$

([Z], Theorem 8.1). The degree is ndm.

For $w \in W_n^{(d+1)}$ consider h.d.$(\prod_{i=1}^n \mathfrak{s}([\nu^i\rho, \nu^{w(i)+d}\rho])) = \prod_{i=1}^n \mathfrak{s}([\nu^i\rho, \nu^{w(i)+d-1}\rho])$. If $i \leq w(i)+d$ for all $1 \leq i \leq n$ (i.e., if $w \in W_n^{(d)}$), then the degree is $\sum_{i=1}^n (w(i)+d-i)m = nmd$. In general, the degree is $m\sum_{i=1}^n \max\{w(i)+d-i, 0\}$. So if $w \in W_n^{(d+1)} \backslash W_n^{(d)}$, then $w(i_0)+d-i_0 \leq -1$ for some $i_0 \in \{1, 2, \ldots, n\}$. Then the degree is

$$m\sum_{i=1}^n \max\{w(i)+d-i, 0\} > m\sum_{i=1}^n w(i)+d-i = nmd.$$

This finishes the proof of the formula. $\square$

5.4. Theorem. *Let $n, d \in \mathbb{N}$ and $\rho \in \mathcal{C}$. Then*

$$\begin{aligned}\nu^{(n+d)/2}L(a(n,d)^{(\rho)}) &= L(([\nu\rho, \nu^d\rho], [\nu^2\rho, \nu^{d+1}\rho], \ldots, [\nu^n\rho, \nu^{d+n-1}\rho])) \\ &= \sum_{w \in W_n^{(d)}} (-1)^{\mathrm{sgn}(w)} \prod_{i=1}^n \delta([\nu^i\rho, \nu^{w(i)+(d-1)}\rho]).\end{aligned}$$

Proof. Applying the Zelevinsky involution to the formula in Theorem 5.3, one gets the above formula, using Lemma 3.3. $\square$

At the first moment, our formulas for characters of irreducible unitarizable representations do not seem very similar to the formula for the Steinberg character. We shall show now that the formula for the Steinberg character of G_n is a special case of our formulas. Fix $n \in \mathbb{N}$ and $\rho \in \mathcal{C}$.

By definition, $w \in (W_n^{(1)})^{-1}$ if and only if $w(i) \leq i+1$ for all $1 \leq i \leq n$. Each permutation can be written as a product of disjoint cycles (we shall write also cycles of length 1). It is easy to write all such w. They are in bijection with all possible sequences $1 \leq a_1 < a_2 < \cdots < a_k < n$ ($k \in \{0, 1, 2, \ldots, n-1\}$). To a such sequence is attached the following element of $(W_n^{(1)})^{-1}$:

$$(1\ 2\ \ldots\ a_1)\ (a_1{+}1\ \ a_1{+}2\ \ldots\ a_2)\ \ldots\ (a_{k-1}{+}1\ \ a_{k-1}{+}2\ \ldots\ a_k)\ (a_k{+}1\ \ a_k{+}2\ \ldots\ n),$$

written as a product of cycles. Note that the parity of the above permutation is $(-1)^{n-k-1}$. Since $(\ell\ \ \ell{+}1\ \ldots\ m{-}1\ m)^{-1} = (m\ \ m{-}1\ \ldots\ \ell{+}1\ \ \ell)$ for $\ell, m \in \mathbb{N}$, $\ell \leq m$, the formula in Theorem 5.3 gives for $d = 1$

$$\begin{aligned}\nu^{(n+1)/2}Z(a(n,1)^{(\rho)}) &= Z((\nu\rho, \nu^2\rho, \ldots, \nu^n\rho)) \\ &= \sum_{1 \leq a_1 < a_2 < \cdots < a_k < n} (-1)^{n-k-1} \prod_{i=0}^k \mathfrak{s}([\nu^{a_i+1}\rho, \nu^{a_{i+1}}\rho]),\end{aligned}$$

where we take in the formula $a_0 = 0$ and $a_{k+1} = n$. Taking for ρ the unramified character $\nu^{-(n+1)/2}$, one gets the formula for the Steinberg character of G_n (see [Ca1]).

5.5. Remark. Obviously, for $d \geq n-1$, $W_n^{(d)} = W_n$ and card $W_n^{(d)} = n!$. An easy counting of permutations gives card $W_n^{(d)} = d!(d+1)^{n-d}$ for $1 \leq d \leq n$. For $1 \leq d \leq n-2$, $W_n^{(d)}$ is not a subgroup of W_n.

6. On multiplicities related to characters (non-archimedean case)

In this section we assume that the field F is non-archimedean.

6.1. Proposition. *Let $n, d \in \mathbb{N}$ and $\rho \in \mathcal{C}$. Then $\zeta(a(n,d)^{(\rho)})$ is a multiplicity one representation, if and only if $d = 1$ or $n \leq 3$. Further, $\lambda(a(n,d)^{(\rho)})$ is a multiplicity one representation, if and only if $d = 1$ or $n \leq 3$.*

Proof. In the proof we shall consider only the case of $\zeta(a(n,d)^{(\rho)})$. The other case follows from this case using the fact that $(\zeta(a(n,d)^{(\rho)}))^t = \lambda(a(n,d)^{(\rho)})$ in R, and the positivity of the Zelevinsky involution.

If $d = 1$, then we are in the regular situation where we always have multiplicity one. If $n = 1$, then there is nothing to prove. The case $n = 2$ follows from Proposition 4.6 of [Z] (see also Theorem 3.2). It remains to see the multiplicity one for $n = 3$ and $d \geq 2$.

Using the highest derivatives, it is enough to prove the multiplicity one for $\zeta(a(3,2)^{(\rho)})$ (one proves this in the same way as Lemma 6.3 of [T5]). If $a \in M(\mathcal{S})$ and $a \leq a(3,2)^{(\rho)}$, then a consists of two or three segments. There are precisely six possibilities for such a. Four of them consist of three segments each. Looking at the highest derivatives, the representations corresponding to these segments have multiplicity one since $\zeta(a(3,1)^{(\rho)})$ is a multiplicity one representation. The fifth representation is $\mathfrak{s}(a(1,4)^{(\rho)}) \times \mathfrak{s}(a(1,2)^{(\rho)})$. This representation corresponds to a minimal among the six possible. Now (ii) of Proposition 3.5 in [T6] implies that it has multiplicity one. It remains to consider $Z(a(2,3)^{(\rho)})$. From $Z(a(2,3)^{(\rho)}) = \mathfrak{s}([\nu^{-3/2}\rho, \nu^{1/2}\rho]) \times \mathfrak{s}([\nu^{-1/2}\rho, \nu^{3/2}\rho]) - \mathfrak{s}([\nu^{-3/2}\rho, \nu^{3/2}\rho]) \times \mathfrak{s}([\nu^{-1/2}\rho, \nu^{1/2}\rho])$ we can compute the derivative. First

$$\begin{aligned}\mathcal{D}(\mathfrak{s}([\nu^{-3/2}\rho, \nu^{3/2}\rho]) \times \mathfrak{s}([\nu^{-1/2}\rho, \nu^{1/2}\rho])) = \mathfrak{s}([\nu^{-3/2}\rho, \nu^{3/2}\rho]) \times \mathfrak{s}([\nu^{-1/2}\rho, \nu^{1/2}\rho]) + \\ \mathfrak{s}([\nu^{-3/2}\rho, \nu^{1/2}\rho]) \times \mathfrak{s}([\nu^{-1/2}\rho, \nu^{1/2}\rho]) + \\ \mathfrak{s}([\nu^{-3/2}\rho, \nu^{3/2}\rho]) \times \nu^{-1/2}\rho + \mathfrak{s}([\nu^{-3/2}\rho, \nu^{1/2}\rho]) \times \nu^{-1/2}\rho.\end{aligned}$$

Subtracting this from $\mathcal{D}(\mathfrak{s}([\nu^{-3/2}\rho, \nu^{1/2}\rho]) \times \mathfrak{s}([\nu^{-1/2}\rho, \nu^{3/2}\rho]))$, we get

$$\begin{aligned}\mathcal{D}(Z(a(2,3)^{(\rho)})) = Z(a(2,3)^{(\rho)}) + Z(([\nu^{-3/2}\rho, \nu^{-1/2}\rho], [\nu^{-1/2}\rho, \nu^{3/2}\rho])) + \\ Z(([\nu^{-3/2}\rho, \nu^{-1/2}\rho], [\nu^{-1/2}\rho, \nu^{1/2}\rho])).\end{aligned}$$

The multiplicity of $Z(([\nu^{-3/2}\rho, \nu^{-1/2}\rho], [\nu^{-1/2}\rho, \nu^{1/2}\rho]))$ in

$$\mathfrak{s}([\nu^{-3/2}\rho, \nu^{-1/2}\rho]) \times \nu^{-1/2}\rho \times \nu^{1/2}\rho$$

is one. This follows from the proof of Proposition 11.4 of [Z] (there, $\pi(a_\ell)$ is a multiplicity one representation of length 3). Therefore, the multiplicity of

$$Z(([\nu^{-3/2}\rho, \nu^{-1/2}\rho], [\nu^{-1/2}\rho, \nu^{1/2}\rho]))$$

in $\mathcal{D}(\zeta(a(3,2)^{(\rho)}))$ is one (i.e., $Z(([\nu^{-3/2}\rho, \nu^{-1/2}\rho], [\nu^{-1/2}\rho, \nu^{1/2}\rho])) \leq \mathcal{D}(\zeta(a(3,2)^{(\rho)})$, but $2Z(([\nu^{-3/2}\rho, \nu^{-1/2}\rho], [\nu^{-1/2}\rho, \nu^{1/2}\rho])) \not\leq \mathcal{D}(\zeta(a(3,2)^{(\rho)})$ in R). This implies the multiplicity one of $Z(a(2,3)^{(\rho)})$ in $\zeta(a(3,2)^{(\rho)})$.

To finish the proof, we need to prove that $\zeta(a(n,d)^{(\rho)})$ is not a multiplicity one representation if $d \geq 2$ and $n \geq 4$. For this it is enough to prove that $\zeta(a(4,d)^{(\rho)})$ is not a multiplicity one representation for $d \geq 2$. Using the highest derivatives, it is enough to see that $\zeta(a(4,2)^{(\rho)})$ is not a multiplicity one representation. We shall show that $Z(a(2,4)^{(\rho)})$ has at least multiplicity two in $\zeta(a(4,2)^{(\rho)})$. We have in R the following equalities

$$\begin{aligned}\zeta(a(4,2)^{(\rho)}) &= \mathfrak{s}([\nu^{-2}\rho,\nu^{-1}\rho]) \times \mathfrak{s}([\nu^{-1}\rho,\rho]) \times \mathfrak{s}([\rho,\nu\rho]) \times \mathfrak{s}([\nu\rho,\nu^2\rho]) \\ &= \mathfrak{s}([\nu^{-2}\rho,\nu^{-1}\rho]) \times \mathfrak{s}([\nu^{-1}\rho,\rho]) \times \left[Z([\rho,\nu\rho],[\nu\rho,\nu^2\rho]) + \mathfrak{s}([\rho,\nu^2\rho]) \times \nu\rho\right].\end{aligned}$$

From this one gets easily that in R we have

$$\begin{aligned}\zeta(a(4,2)^{(\rho)}) \geq\ & Z([\nu^{-2}\rho,\nu^{-1}\rho],[\nu^{-1}\rho,\rho]) \times Z([\rho,\nu\rho],[\nu\rho,\nu^2\rho]) + \\ & \mathfrak{s}([\nu^{-2}\rho,\nu^{-1}\rho]) \times \mathfrak{s}([\nu^{-1}\rho,\rho]) \times \mathfrak{s}([\rho,\nu^2\rho]) \times \nu\rho.\end{aligned}$$

We know

$$\begin{aligned}& Z([\nu^{-2}\rho,\nu^{-1}\rho],[\nu^{-1}\rho,\rho]) \times Z([\rho,\nu\rho],[\nu\rho,\nu^2\rho]) = \\ & \qquad L([\nu^{-2}\rho,\nu^{-1}\rho],[\nu^{-1}\rho,\rho]) \times L([\rho,\nu\rho],[\nu\rho,\nu^2\rho])\end{aligned}$$

(see Theorem 3.2). Thus $Z(a(2,4)^{(\rho)}) = L(a(4,2)^{(\rho)})$ is a subquotient of

$$Z([\nu^{-2}\rho,\nu^{-1}\rho],[\nu^{-1}\rho,\rho]) \times Z([\rho,\nu\rho],[\nu\rho,\nu^2\rho]).$$

One has directly $a(2,4)^{(\rho)} \leq ([\nu^{-2}\rho,\nu^{-1}\rho],[\nu^{-1}\rho,\rho],[\rho,\nu^2\rho],\nu\rho)$. Now we see that the multiplicity of $Z(a(2,4)^{(\rho)})$ in $\zeta(a(4,2)^{(\rho)}$ is at least two. This finishes the proof of the proposition. □

Now we have a direct consequence of the above proof:

6.2. Corollary. *Let $n,d \in \mathbb{N}, d \geq 2$, and $\rho \in \mathcal{C}$. Then the representations $\zeta(a(4n,d)^{(\rho)})$ and $\lambda(a(4n,d)^{(\rho)})$ have irreducible subquotients which have multiplicities at least 2^n in that representations.* □

6.3. Remark. One delicate point in the solution of the unitarizability problem in [T4] was the proof that $Z(a(n,d)^{(\rho)})$'s are prime elements in R (Proposition 3.5 of [T4]). We present here one very simple proof of this fact, based on the Gelfand-Kazhdan derivatives. The proof here proceeds by induction on d. We shall assume that ρ is a representation of some $\mathrm{GL}(p,F)$. For $n = 1$, $Z(a(n,1)^{(\rho)})$ is an essentially square integrable representation. Since R is a polynomial ring over all essentially square integrable representations, $Z(a(n,1)^{(\rho)})$ is prime. Suppose that we have proved that $Z(a(n,d)^{(\rho)})$ is prime for some $d \geq 1$. Assume that $Z(a(n,d+1)^{(\rho)}) = P_1 \times P_2$. for some $P_1, P_2 \in R$. For the proof, it is enough to show that P_1 or P_2 must be in $\{\pm 1\}$. Since R is integral and graded, P_1 and P_2 are homogeneous with respect to the grading. Now on the level of the highest derivatives we have $Z(a(n,d)^{(\nu^{-1/2}\rho)}) = \text{h.d.}(P_1) \times \text{h.d.}(P_2)$. Thus, $\text{h.d.}(P_1)$ or $\text{h.d.}(P_2)$ is ± 1 by the inductive assumption. Assume $\text{h.d.}(P_2) \in \{\pm 1\}$. Without lost of generality we can assume it is 1. Then h.d.$(P_1) = Z(a(n,d)^{(\nu^{-1/2}\rho)})$. The theorem 8.1 of [Z] implies that if σ is an irreducible representation of $\mathrm{GL}(m,F)$ with $m \leq n(d+1)p$, whose highest derivative is $Z(a(n,d)^{(\nu^{-1/2}\rho)})$, then $\sigma = Z(a(n,d+1)^{(\rho)})$. Thus $m = n(d+1)p$, and the degree of P_1 is $m = n(d+1)p$. Therefore, the degree of P_2 is 0. Since $Z(a(n,d+1)^{(\rho)})$ is a primitive element of the free abelian group R, P_2 is also primitive. This implies $P_2 = 1$. This finishes the proof.

Appendix: Composition series of the ends of the complementary series of $\mathrm{GL}(2n,\mathbb{C})$

For $d \in \mathbb{Z}$ and $\beta \in \mathbb{C}$, we have denoted by $\chi(d)^{(\nu^\beta)}$ the character $\chi(d)^{(\nu^\beta)}(z) = (z/|z|)^d \nu^\beta(z)$ of $\mathbb{C}^\times$. For $x, y \in \mathbb{C}$ such that $x - y \in \mathbb{Z}$ we define a character $\gamma(x,y)$ of $\mathbb{C}^\times$ by

$$\gamma(x,y)(z) = (z/|z|)^{x-y}|z|^{x+y} = \chi(x-y)^{(\nu^{(x+y)/2})}.$$

Then

$$\chi(d)^{(\nu^\beta)} = \gamma(\beta + d/2, \beta - d/2),$$

for $d \in \mathbb{Z}, \beta \in \mathbb{C}$. Note that $\gamma(x,y) = \gamma(x',y')$ implies $x = x'$ and $y = y'$. Further

$$\begin{aligned}
\gamma(x,y)\gamma(x',y') &= \gamma(x+x', y+y'), & \\
\gamma(d/2,-d/2)(z) &= (z/|z|)^d = \chi(d)^{(\nu^0)}(z), & d \in \mathbb{Z}, \\
\gamma(\beta,\beta) &= \nu^\beta, & \beta \in \mathbb{C}, \\
\gamma(p,q)(z) &= z^p \bar{z}^q, & p, q \in \mathbb{Z}.
\end{aligned}$$

For characters χ_1, χ_2 of $\mathbb{C}^\times$, the representation $\chi_1 \times \chi_2$ is reducible if and only if $\chi_1\chi_2^{-1} = \gamma(p,q)$ where $p, q \in \mathbb{Z}$ and $pq > 0$. Thus, $\gamma(x_1,y_1) \times \gamma(x_2,y_2)$ is reducible if and only if $x_1 - x_2 \in \mathbb{Z}$ and $(x_1 - x_2)(y_1 - y_2) > 0$. In the case of reducibility, we have in the Grothendieck group:

$$\gamma(x_1,y_1) \times \gamma(x_2,y_2) = L(\gamma(x_1,y_1), \gamma(x_2,y_2)) + \gamma(x_1,y_2) \times \gamma(x_2,y_1).$$

For $(\chi_1, \dots, \chi_n) \in M(D)$ suppose that there exist $1 \leq i < j \leq n$ such that $\chi_i \times \chi_j$ is reducible. Write $\chi_i = \gamma(x_i,y_i), \chi_j = \gamma(x_j,y_j)$. Denote $\chi_i' = \gamma(x_i,y_j), \chi_j' = \gamma(x_j,y_i)$. Then we shall write

$$\begin{aligned}
&(\chi_1, \dots, \chi_{i-1}, \chi_i', \chi_{i+1}, \dots, \chi_{j-1}, \chi_j', \chi_{j+1}, \dots, \chi_n) \\
&\qquad\qquad \prec (\chi_1, \dots, \chi_{i-1}, \chi_i, \chi_{i+1}, \dots, \chi_{j-1}, \chi_j, \chi_{j+1}, \dots, \chi_n).
\end{aligned}$$

Generate by $\prec$ a minimal partial order $\leq$ on $M(D)$. If $(\chi_1, \dots, \chi_n), (\chi_1', \dots, \chi_n') \in M(D)$, then $L(\chi_1', \dots, \chi_n')$ is a composition factor of $\chi_1 \times \dots \times \chi_n$ if and only if $(\chi_1', \dots, \chi_n') \leq (\chi_1, \dots, \chi_n)$.

Now we shall rewrite the formula in Proposition 4.3 for the trivial character $1_{\mathrm{GL}(n,\mathbb{C})}$ of $\mathrm{GL}(n,\mathbb{C})$. We have in R

$$1_{\mathrm{GL}(n,\mathbb{C})} = \chi(n,0)^{(1_{\mathbb{C}^\times})} = L\left(\gamma\left(\tfrac{n-1}{2},\tfrac{n-1}{2}\right),\gamma\left(\tfrac{n-3}{2},\tfrac{n-3}{2}\right),\dots,\gamma\left(-\tfrac{n-1}{2},-\tfrac{n-1}{2}\right)\right)$$

$$= L\left(\gamma(\tau_1,\tau_1),\gamma(\tau_2,\tau_2),\dots,\gamma(\tau_n,\tau_n)\right) = \sum_{w\in W_n}(-1)^{\mathrm{sgn}(w)}\gamma(\tau_1,\tau_{w(1)})\times\cdots\times\gamma(\tau_n,\tau_{w(n)})$$

$$= \sum_{w\in W_n}(-1)^{\mathrm{sgn}(w)}\prod_{i=1}^n\gamma(\tau_i,\tau_{w(i)}) = \sum_{w\in W_n}(-1)^{\mathrm{sgn}(w)}\prod_{i=1}^n\gamma((n-2i+1)/2,(n-2w(i)+1)/2)$$

$$= \sum_{w\in W_n}(-1)^{\mathrm{sgn}(w)}\prod_{i=1}^n\gamma(i-(n+1)/2,w(i)-(n+1)/2)$$

(at the end we passed to the contragredients). From this one obtains that for $x, y \in \mathbb{C}$ such that $x-y\in\mathbb{Z}$, we have

$$\text{(A.1)}\quad L((\gamma(x+1,y+1),\gamma(x+2,y+2),\dots,\gamma(x+n,y+n))) = \sum_{w\in W_n}(-1)^{\mathrm{sgn}(w)}\prod_{i=1}^n\gamma(x+i,y+w(i)).$$

Set

$$\begin{aligned}
d_{-1/2} &= (\gamma(-n,0),\gamma(-n+1,1),\dots,\gamma(-1,n-1)),\\
d_{1/2} &= (\gamma(-n+1,1),\gamma(-n+2,2),\dots,\gamma(0,n)),\\
d_{\mathrm{GL}(n+1)} &= (\gamma(-n,0),\gamma(-n+1,1),\dots,\gamma(-1,n-1),\gamma(0,n)),\\
d_{\mathrm{GL}(n-1)} &= (\gamma(-n+1,1),\dots,\gamma(-1,n-1)),\\
d_- &= (\gamma(-n,1),\gamma(-n+1,2),\dots,\gamma(-1,n)),\\
d_+ &= (\gamma(-n+1,0),\gamma(-n+2,1),\dots,\gamma(0,n-1)).
\end{aligned}$$

Then

$$\begin{aligned}
\nu^{-1/2}\chi(n,-n)^{(\nu^0)} &= L(d_{-1/2}),\\
\nu^{1/2}\chi(n,-n)^{(\nu^0)} &= L(d_{1/2}),\\
\chi(n+1,-n)^{(\nu^0)} &= L(d_{\mathrm{GL}(n+1)}),\\
\chi(n-1,-n)^{(\nu^0)} &= L(d_{\mathrm{GL}(n-1)}),\\
\chi(n,-n-1)^{(\nu^0)} &= L(d_-),\\
\chi(n,-n+1)^{(\nu^0)} &= L(d_+).
\end{aligned}$$

In the following lemma we study some elements of R as polynomials in variables D (the basis of R is $\lambda(d), d\in M(D)$).

A.1. Lemma. *Let $n\in\mathbb{N}$, $n\ge 2$. Then*

(i) *Suppose that n is even. Then the monomial $\lambda(d_- + d_+)$ has in the polynomial $\nu^{-1/2}\chi(n,-n)^{(\nu^0)}\times\nu^{1/2}\chi(n,-n)^{(\nu^0)} = L(d_{-1/2})\times L(d_{1/2})$ coefficient equal to 1,* *and in the polynomial* $\chi(n+1,-n)^{(\nu^0)}\times\chi(n-1,-n)^{(\nu^0)} = L(d_{\mathrm{GL}(n+1)})\times L(d_{\mathrm{GL}(n-1)})$ *coefficient is* 0.

(ii) *Suppose that n is odd. Then the monomial $\lambda(d_- + d_+)$ has coefficient 0 in the polynomial $\nu^{-1/2}\chi(n,-n)^{(\nu^0)} \times \nu^{1/2}\chi(n,-n)^{(\nu^0)} = L(d_{-1/2}) \times L(d_{1/2})$, and -1 in the polynomial $\chi(n+1,-n)^{(\nu^0)} \times \chi(n-1,-n)^{(\nu^0)} = L(d_{\mathrm{GL}(n+1)}) \times L(d_{\mathrm{GL}(n-1)})$.*

(iii) *If $n \neq 3$, then there does not exist $d \in M(D)$ such that $L(d)$ is unitarizable and that*

$$d_- + d_+ < d < d_{1/2} + d_{-1/2} \quad (= d_{\mathrm{GL}(n+1)} + d_{\mathrm{GL}(n-1)}).$$

For $n = 3$ there exists exactly one such d. That one is $d_{\mathrm{GL}(4)} + (\gamma(-1,1),\gamma(-2,2))$.

(iv) *Suppose that $n = 3$. Then the monomial $\lambda(d_{\mathrm{GL}(4)} + (\gamma(-1,1),\gamma(-2,2)))$ has coefficients equal to -2 in both polynomials $\nu^{-1/2}\chi(3,-3)^{(\nu^0)} \times \nu^{1/2}\chi(3,-3)^{(\nu^0)} = L(d_{-1/2}) \times L(d_{1/2})$ and $\chi(4,-3)^{(\nu^0)} \times \chi(2,-3)^{(\nu^0)} = L(d_{\mathrm{GL}(4)}) \times L(d_{\mathrm{GL}(2)})$.*

Proof. First we shall see when the monomial $\lambda(d_- + d_+)$ can show up in the polynomial $L(d_{-1/2})\times L(d_{1/2})$. We shall use formula (A.1). Suppose that $\lambda(d_-+d_+)$ is coming from a monomial $L(d_{-1/2})$ and $L(d_{1/2})$ which correspond to permutations w_1 and w_2 respectively. Write $d_- + d_+ = (\gamma(x_1,y_1),\dots,\gamma(x_{2n},y_{2n})$. Note that $y_i - x_i = n-1$ or $n+1$.

Note that there is only one i with $x_i = -n$. This, and the fact that $y_i - x_i = n-1$ or $n+1$ imply $w_1(1) = 2$. Further, the fact that $y_i - x_i = n-1$ or $n+1$ implies $w_1(2) = 1$ or 3. If $w_1(2) = 3$, then $w_1^{-1}(1) \geq 3$. This shows that $\gamma(-n-1+w_1^{-1}(1),0)$ appears in the product. This contradicts the fact that $y_i - x_i = n-1$ or $n+1$. Continuing along this line, one gets that w_1 must be the product of cycles (1 2) (3 4) (5 6) . . . . If n is odd, then we must have $\gamma(-1,n-1)$ in the product. This contradicts $y_i - x_i = n-1$ or $n+1$. Thus, $\lambda(d_- + d_+)$ does not show up in $L(d_{-1/2}) \times L(d_{1/2})$. If n is even, then the only possibility for w_1 and w_2 is $w_1 = w_2 = (1\ 2)\ (3\ 4)\ (5\ 6)\dots(n{-}1\ n)$. This implies that $\lambda(d_- + d_+)$ has coefficient $(-1)^n = 1$ in the polynomial $L(d_{-1/2}) \times L(d_{1/2})$.

In a completely analogous way one proves that $\lambda(d_-+d_+)$ has coefficient 0 in $L(d_{\mathrm{GL}(n+1)})$ $\times L(d_{\mathrm{GL}(n-1)})$ if n is even, and $(-1)^n = -1$ if n is odd. This finishes the proof of (i) and (ii).

Take $d \in M(D)$ so that $L(d)$ is unitarizable and that $d_-+d_+ < d < d_{\mathrm{GL}(n+1)} + d_{\mathrm{GL}(n-1)} = d_{-1/2} + d_{1/2}$. Write

$$d = (\gamma(x'_1,y'_1),\dots,\gamma(x'_{2n},y'_{2n})), \quad d_{\mathrm{GL}(n+1)} + d_{\mathrm{GL}(n-1)} = (\gamma(x''_1,y''_1),\dots,\gamma(x''_{2n},y''_{2n})).$$

The first observation is that always $n-1 \leq y'_i - x'_i \leq n+1$. Only one $\gamma(x''_i,y''_i)$ has $x''_i = -n$ and also only one $\gamma(x'_j,y'_j)$ has $x'_j = -n$. After renumeration, we can assume $i = j = 1$. Obviously $y'_1 \neq -1$. Thus $y'_1 = 0$ or 1.

Suppose that $y'_1 = 0$. Unitarizability of $L(d)$ implies that $d = d_{\mathrm{GL}(n+1)} + d'$ for some $d' \in M(S)$. Write $d' = (\gamma(u_1,v_1),\dots,\gamma(u_{n-1},v_{n-1})$. There is only one $\gamma(u_i,v_i)$ with $u_i = -n+1$. Now in the same way as before we conclude $v_i = 1$ or 2. For the possibility $v_i = 2$ it must be $n \geq 3$. If $v_i = 1$, then unitarizability of $L(d)$ implies $d' = d_{\mathrm{GL}(n-1)}$. Thus, $d = d_{\mathrm{GL}(n+1)} + d_{\mathrm{GL}(n-1)}$. Therefore in this situation $v_i = 2$. The unitarizability of $L(d)$ implies $d' = (\gamma(-n+1,2),\dots,\gamma(-2,n-1)) + (\gamma(-1,1))$. For $n > 3$, this can not hold because it must be $n-1 \leq v_i - u_i \leq n+1$. For $n=3$ one easily sees that $d_- + d_+ < d_{\mathrm{GL}(4)} + (\gamma(-1,1), \gamma(-2,2)) < d_{\mathrm{GL}(4)} + d_{\mathrm{GL}(2)}$.

Suppose now $y_1' = 1$. Then unitarizability of $L(d)$ implies that $d = d_- + d'$ for some $d' \in M(D)$. Write $d' = (\gamma(u_1,v_1), \ldots, \gamma(u_n,v_n))$. There is only one $\gamma(u_i,v_i)$ with $u_i = -n+1$. If $v_i \geq 1$, then for $\gamma(u_j,v_j)$ for which $v_j = 0$ one has $u_j \geq -n+2$. Thus, $v_j - u_j < n-1$. This can not happen. Therefore, $v_i = 0$. This implies that $d' = d_+$. Thus, $d = d_- + d_+$. This completes the proof of (iii).

One gets (iv) directly from the formula (A.1). □

A.2. Corollary. *$\chi(n,-n-1)^{(\nu^0)} \times \chi(n,-n+1)^{(\nu^0)}$ is a subquotient of $\nu^{-1/2}\chi(n,-n)^{(\nu^0)} \times \nu^{1/2}\chi(n,-n)^{(\nu^0)}$, of multiplicity one.*

Proof. There exist integers $p_{d'} \geq 0$ ($d' \in M(D)$), so that $L(d_{-1/2}) \times L(d_{1/2}) = L(d_{\mathrm{GL}(n+1)}) \times L(d_{\mathrm{GL}(n-1)}) + \sum p_{d'} L(d')$, where the sum runs over all $d' \in M(D), d' < d_{-1/2} + d_{1/2}$. We need to show that $p_{L(d_- + d_+)} = 1$. Recall that $L(d') = \sum_{d'' \in M(D), d'' \leq d'} q_{d',d''} \lambda(d'')$, for some $q_{d',d''} \in \mathbb{Z}$, where $q_{d',d'} = 1$. Thus

$$\text{(A.2)}\quad L(d_{-1/2}) \times L(d_{1/2}) = L(d_{\mathrm{GL}(n+1)}) \times L(d_{\mathrm{GL}(n-1)}) + \sum_{d' \in M(D), d' < d_{-1/2} + d_{1/2}} p_{d'} \sum_{d'' \in M(D), d'' \leq d'} q_{d',d''} \lambda(d'').$$

Suppose $n \neq 3$. Now (i) and (ii) of the above lemma imply $p_{d'} q_{d', d_- + d_+} \neq 0$, for some $d' \in M(D)$ such that $d_- + d_+ \leq d' < d_{-1/2} + d_{1/2}$. The above lemma ((iii)) implies that the only possibility for d' is $d_- + d_+$. Now (i) and (ii) of the above lemma imply $p_{d_- + d_+} q_{d_- + d_+, d_- + d_+} = 1$. Thus, $p_{d_- + d_+} = 1$.

Suppose now that $n = 3$. Then $p_{L(d_{\mathrm{GL}(4)}) \times \gamma(-1,1) \times \gamma(-2,2)} = 0$ follows from (A.2) and (iv) of the above lemma by a similar type of analysis as above. Now we can repeat the above argument and get $p_{d_- + d_+} = 1$. This finishes the proof. □

The multiplicity of the irreducible representation $\chi(n+1,-n)^{(\nu^0)} \times \chi(n-1,-n)^{(\nu^0)}$ in $\nu^{-1/2}\chi(n,-n)^{(\nu^0)} \times \nu^{1/2}\chi(n,-n)^{(\nu^0)}$ is one. This follows directly from the fact that $L(d+d')$ is always a subquotient of $L(d) \times L(d')$, for $d, d' \in M(D)$ (this is Proposition 3.5 of [T3], one can see also the proof of Proposition 2.3 of [T6]; after minor changes this proof applies also to the archimedean case). As noted by S. Sahi, one can also argue in this case using the spherical representations (one needs to twist the situation by $\chi(n)^{(\nu^0)}$). Finally, Theorem 3C of [Sh1] implies that in R we have

$$\nu^{-1/2}\chi(n,-n)^{(\nu^0)} \times \nu^{1/2}\chi(n,-n)^{(\nu^0)} = \chi(n+1,-n)^{(\nu^0)} \times \chi(n-1,-n)^{(\nu^0)} + \chi(n,-n-1)^{(\nu^0)} \times \chi(n,-n+1)^{(\nu^0)}.$$

Multiplying the above relation with characters, one gets Proposition 4.1.

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