ON CHOOSING AN ESTIMATE OF THE SPECTRAL DENSITY FUNCTION OF A STATIONARY TIME SERIES[1]

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1. Introduction. The problem of estimating the spectral density function of a stationary time series has been extensively discussed recently (see references). The present period of research may be said to have commenced about 1945, when Bartlett and Daniell (see [1]) pointed out that the periodogram needs to be smoothed if it is to form a consistent estimate of the spectral density. About 1948–49, several consistent estimates were proposed by Bartlett [2] and Tukey [15]. Later, Grenander [6] and Rosenblatt [9] considered a general class of estimates of the spectral density. The present writer also considered in [11] and [12] a general class of estimates, treating continuous parameter, as well as discrete parameter stationary time series.

In the work of Grenander, Rosenblatt, and ourselves, the mean square error $E\,|\,f_T^*(\omega) - f(\omega)\,|^2$ is adopted as the figure of merit of an estimate $f_T^*(\omega)$ of the spectral density function $f(\omega)$. In our paper [12], the asymptotic bias, asymptotic variance, and asymptotic mean square error are computed for a certain general class of estimates. Certain general conclusions are stated as to (1) the highest order of consistency with which the spectral density function of a given stationary time series, whose covariance function satisfies certain conditions, could be estimated, using a suitable sequence of estimates of the form considered, and (2) the order of consistency which a given sequence of estimates could achieve for any stationary time series satisfying certain conditions. Conclusions of type (2) were stated also by Grenander and Rosenblatt [9] for certain estimates whose asymptotic bias, and consequently whose mean square error, they were able to evaluate.

In a more recent paper [14], we carry these results a good deal further, and show how to construct estimates of the spectral density function which achieve a maximum order of consistency and a minimum asymptotic mean square error.

Nevertheless, these results, as they stand now, cannot be said to constitute a practical solution to the problem of estimating the spectral density function of a stationary time series. For while many estimates have been proposed, little attention has been paid to the question of how to choose among them. Given any exponent α, $0 < \alpha < 1$, one may construct many estimates $f_T^*(\omega)$ having the property that they are consistent of order $T^{-\alpha}$, in the sense that their mean square

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errors are such that $\lim_{T\to\infty} T^{\alpha} E \mid f_T^*(\omega) - f(\omega) \mid^2$ is finite and non-zero. The question naturally arises how to form a most desirable (or optimum) estimate of order of consistency $T^{-\alpha}$. In this paper, especially in Sec. 5, we put forth certain considerations which indicate that the usual discussions of this question (such as in Grenander and Rosenblatt [9], pp. 154–155) are not adequate to settle the question. We then put forth certain notions on how to compare two estimates of the spectral density to determine which is more desirable. More importantly, we indicate a possible method of designing a spectral analysis.

It is to be emphasized that this paper is open to the criticism that it employs relations for samples of finite size which are true only in the limit. For this reason, it is to be regarded as an attempt to obtain, on somewhat heuristic grounds, a "practical" solution to the problem of estimating the spectral density function of a stationary time series. The paper contains no theorems.

This paper is to be read as a sequel to our paper [12], whose results form the base of the present paper. We employ the definitions, assumptions, and notations of [12], the most important of which will be explained here as they arise.

We discuss only the case of continuous parameter time series, since this seems to us to be the case of greatest physical interest. In our opinion, in considering a discrete parameter time series, one should always keep in mind the continuous parameter time series from which the discrete parameter one was obtained by means of sampling at discrete times. An interesting problem, which is briefly discussed in [13] and [14], is the relation which exists between the problems of estimating the spectral density of a continuous parameter stationary time series, and estimating the spectral density of a discrete parameter time series obtained by sampling the continuous parameter one at discrete times.

2. A class of estimates of the spectral density function. Consider a continuous parameter stationary time series $x(t)$, with mean $m = E[x(t)]$, and integrable covariance function

$$R(v) = E[y(t)y(t+v)] = \int_{-\infty}^{\infty} e^{i\omega v} f(\omega)\, d\omega, \tag{2.1}$$

where $y(t) = x(t) - m$. One calls $f(\omega)$ the spectral density function of $x(t)$.

Let $x(t)$ be observed for $0 \leqq t \leqq T$. Let $Y_T(t)$ denote the deviations from the sample mean defined by

$$\begin{aligned} Y_T(t) &= x(t) - \frac{1}{T}\int_0^T x(t)\, dt, && 0 < t \leqq T \\ &= 0, && \text{otherwise.} \end{aligned} \tag{2.2}$$

Let $R_T(v)$ denote the sample covariance function, defined by

$$\begin{aligned} R_T(v) &= \frac{1}{T}\int_0^{T-|v|} Y_T(t)Y_T(t + |v|)\, dt, && |v| \leqq T \\ &= 0, && \text{otherwise} \end{aligned} \tag{2.3}$$

Its Fourier transform

$$f_T(\omega) = \frac{1}{2\pi}\int_{-T}^{T} e^{-iv\omega}R_T(v)\,dt \tag{2.4}$$
$$= \frac{1}{2\pi T}\left|\int_0^T e^{it\omega}Y_T(t)\,dt\right|^2$$

may be called the sample spectral density function or periodogram.

We consider estimates of the spectral density function of the form

$$f_T^*(\omega) = \frac{1}{2\pi}\int_{-T}^{T} e^{-iv\omega}k(B_T v)R_T(v)\,dv, \tag{2.5}$$

where the function $k(u)$, called a covariance averaging kernel, is even, bounded, square integrable, $k(0) = 1$, and $|u|^{(1/2)+\epsilon}|k(u)|$ is bounded in u, for some $\epsilon > 0$. The constants B_T are assumed to tend to 0 as $T \to \infty$ in such a way that $TB_T \to \infty$.

In [12] it is shown that the properties of the estimates $f_T^*(\omega)$ depend on $k(u)$ and B_T in the following way. The variance $\sigma^2[f_T^*(\omega)] = E|f_T^*(\omega) - Ef_T^*(\omega)|^2$ satisfies

$$\lim_{T\to\infty} TB_T\sigma^2[f_T^*(\omega)] = \int_{-\infty}^{\infty} k^2(u)\,du\, f^2(\omega)\{1 + \delta(0,\omega)\}, \tag{2.6}$$

where $\delta(\omega_1, \omega_2) = 1$ or 0 according as $\omega_1 = \omega_2$ or $\omega_1 \neq \omega_2$. The bias $b[f_T^*(\omega)] = Ef_T^*(\omega) - f(\omega)$ satisfies

$$\lim_{T\to\infty} B_T^{-r}b[f_T^*(\omega)] = k^{(r)}f^{(r)}(\omega) \tag{2.7}$$

if $r > 0$ is such that

$$k^{(r)} = \lim_{u\to 0}\frac{1 - k(u)}{|u|^r} \tag{2.8}$$

is finite,

$$f^{(r)}(\omega) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} e^{-iv\omega}|v|^r R(v)\,dv \tag{2.9}$$

exists as an absolutely summable integral, and

$$0 < \liminf_T TB_T^{1+2r} \leqq \limsup_T TB_T^{1+2r} < \infty. \tag{2.10}$$

The function $f^{(r)}(\omega)$ is to be regarded as a generalized rth derivative of the spectral density function $f(\omega)$.

If a kernel $k(u)$ has the property that there is a unique positive number r such that $k^{(r)}$ exists and is non-zero, then r is defined to be the *characteristic exponent* of the kernel $k(u)$, and $k^{(r)}$ is defined to be the *characteristic coefficient.*

3. On achieving a figure of merit with minimum observation time. As a figure of merit of the estimate $f_T^*(\omega)$ one may use its mean square percentage error $\eta^2[f_T^*(\omega)]$, defined by (for $f(\omega) > 0$),

$$\eta^2[f_T^*(\omega)] = \frac{E\,|\,f_T^*(\omega) - f(\omega)\,|^2}{f^2(\omega)} = \frac{\sigma^2[f_T^*(\omega)]}{f^2(\omega)}\left\{1 + \frac{b^2[f_T^*(\omega)]}{\sigma^2[f_T^*(\omega)]}\right\}, \tag{3.1}$$

Another figure of merit one may use is the Gaussian range of percentage error, $\Delta[f_T^*(\omega)]$, defined by

$$\Delta[f_T^*(\omega)] = \gamma_p \frac{\sigma[f_T^*(\omega)]}{f(\omega)}\left\{1 + \frac{b[f_T^*(\omega)]}{\gamma_q\,\sigma[f_T^*(\omega)]}\right\}, \tag{3.2}$$

where γ_p is the p percentile of the normal distribution, defined by the relation

$$\int_{\gamma_p}^{\infty} e^{-(1/2)y^2}\,dy = \sqrt{2\pi}\,(p/2).$$

It was shown in [13] that the use of the Gaussian range of percentage error leads qualitatively to the same conclusions as does the mean square percentage error. In order not to overload the present paper, we merely mention the existence of the notion of Gaussian range of percentage error, but do not discuss its properties, or the motivation for its consideration.

We now make the crucial simplification on which the discussion of this paper is based. We suppose that the relations (2.6) and (2.7) which are valid in the limit as $T \to \infty$ may be written as equations valid for finite values of T. We then obtain the following expression for the mean square percentage error (which we write only for the case $\omega > 0$ in order to drop the term $1 + \delta(0, \omega)$):

$$\eta^2[f_T^*(\omega)] = \frac{1}{TB_T}\int_{-\infty}^{\infty} k^2(u)\,du \left\{1 + TB_T^{1+2r}\frac{|\,k^{(r)}f^{(r)}(\omega)\,|^2}{f^2(\omega)\int_{-\infty}^{\infty} k^2(u)\,du}\right\}. \tag{3.3}$$

Now for a given choice of the covariance averaging kernel $k(u)$, and length of observation time T, (3.3) defines η^2 as a function of B_T. Similarly, for fixed $k(u)$ and η^2, (3.3) defines T implicitly as a function of B_T. One may solve for T explicitly, and one obtains

$$T = \frac{\dfrac{1}{B_T}\dfrac{1}{\eta^2}\displaystyle\int_{-\infty}^{\infty} k^2(u)\,du\,(1 + \delta(0, \omega))}{\left\{1 - \left(\dfrac{B_T}{\lambda_r(\omega)}\right)^{2r}\left(\dfrac{k^{(r)}}{\eta}\right)^2\right\}}, \tag{3.4}$$

where we define the quantity $\lambda_r(\omega)$ by

$$\lambda_r(\omega) = \left|\frac{f(\omega)}{f^{(r)}(\omega)}\right|^{1/r}. \tag{3.5}$$

It is clear that $\lambda_r(\omega)$ has the dimensions of bandwidth (i.e., of the reciprocal of time). It may be shown, by a consideration of examples, that $\lambda_r(\omega)$ is related to the notion of bandwidth of a spectrum as it is usually defined in the physical literature. Consequently, $\lambda_r(\omega)$ may be interpreted as an extension of the notion of bandwidth, and we call $\lambda_r(\omega)$ the[2] *spectral bandwidth of order r at the frequency* ω.

Equation (3.4) gives the observation time required in order that the estimate $f_T^*(\omega)$, given by (2.5), with a specified value of the constant B_T, have a mean square percentage error equal to η^2. We now consider kernels $k(u)$, and $r > 0$ such that $k^{(r)} \neq 0$. One may determine the value $B_{\min}$ of B which minimizes the observation time T, and the value $T_{\min}$ of T at this minimum. One obtains

$$T_{\min} = \frac{1}{\lambda_r(\omega)} \frac{1}{\eta^{2+(1/r)}} T(k)C(r) \left(1 + \frac{1}{2r}\right) (1 + \delta(0, \omega)), \tag{3.6}$$

$$B_{\min} = \lambda_r(\omega)\eta^{1/r} \frac{1}{| k^{(r)} |^{1/r} C(r)}, \tag{3.7}$$

where we define, for a kernel $k(u)$ for which $k^{(r)} \neq 0$,

$$T(k) = | k^{(r)} |^{1/r} \int_{-\infty}^{\infty} k^2(u)\, du \tag{3.8}$$

and where, for $r > 0$,

$$C(r) = (1 + 2r)^{1/2r}. \tag{3.9}$$

One could also give a formula for the minimum mean square percentage error $\eta_{\min}^2$ obtainable with a fixed observation time T (compare Grenander and Rosenblatt [9], pp. 154–155). However, there is no need to write this formula explicitly, for a calculation of $\eta_{\min}$ shows that it may be obtained from (3.6) by replacing $T_{\min}$ by T, and η by $\eta_{\min}$.

Equation (3.6) may be used to make a comparison of the effect of using different kernels $k(u)$. Before making this comparison, we introduce some possible averaging kernels.

4. Some possible covariance averaging kernels. There are very large numbers of possible functions which one may consider as possible covariance averaging kernels $k(u)$ in the formula (2.5) for the estimated spectral density function $f_T^*(\omega)$.

To begin with, one may consider the following 3-parameter families of functions.

The algebraic family, defined for $r > 0$, $\gamma > 0$, and $0 < \mu \leqq 1/\gamma$:

$$\begin{aligned} k_A(u; \gamma, \mu, r) &= 1 - (\gamma | u |)^r \quad \text{for} \quad |u| \leqq \mu \\ &= 0 \qquad\qquad\qquad \text{otherwise.} \end{aligned} \tag{4.1}$$

The cosine family, defined for $r > 0$, $\gamma > 0$, and $0 < \mu \leqq \pi/2\gamma$:

[2] J. W. Tukey has suggested the alternative name of spectral bandscale, rather than spectral bandwidth.

$$(4.2)\qquad k_C(u; \gamma, \mu, r) = \frac{1 + \cos 2(\gamma \mid u \mid)^{r/2}}{2} \text{ for } \mid u \mid \leqq \mu$$
$$= 0 \qquad \text{otherwise.}$$

The exponential family, defined for $r > 0, \gamma > 0$, and $0 < \mu \leqq \infty$:

$$(4.3)\qquad k_E(u; \gamma, \mu, r) = e^{-(\gamma|u|)^r} \quad \text{for} \quad |u| \leqq \mu$$
$$= 0 \qquad \text{otherwise.}$$

The geometric family, defined for $r > 0, \gamma > 0$, and $0 < \mu \leqq \infty$:

$$(4.4)\qquad k_G(u; \gamma, \mu, r) = \frac{1}{1 + (\gamma \mid u \mid)^r} \quad \text{for} \quad \mid u \mid \leqq \mu$$
$$= 0 \qquad \text{otherwise.}$$

The name "the geometric family" is motivated by the fact that the expression in (4.4) is the sum of a geometric series.

By expanding these functions in power series, it is immediately clear that each or these kernels has characteristic exponent r, and characteristic coefficient $k^{(r)} = \gamma^r$.

Another kernel that should be considered is given by

$$(4.5)\qquad k_\infty(u) = 1 \quad \text{for} \quad |u| < 1$$
$$= 0 \quad \text{otherwise}$$

which can be regarded as the limit of $k_A(1, 1, r)$ as $r \to \infty$. This kernel give rise to the "truncated" estimate of the spectral density (see Grenander and Rosenblatt [9], p. 148).

The estimates for the spectral density which have been suggested by Bartlett (see [9], p. 146) and Tukey (see [15], or [10], or [9], p. 149, for similar estimates) can be obtained from (2.5) by using respectively the kernels

$$(4.6)\qquad k_B(u) = 1 - |u| \quad \text{for} \quad |u| \leqq 1$$
$$= 0 \qquad \text{otherwise}$$

and

$$(4.7)\qquad k_T(u) = \frac{1 + \cos \pi u}{2} \quad \text{for} \quad \mid u \mid \leqq 1$$
$$= 0 \qquad \text{otherwise.}$$

These kernels are the same as $k_A(u; 1, 1, 1)$ and $k_C(u; \pi/2, 1, 2)$, respectively.

Another estimate, which has been suggested by Daniell (see [1], or [3], or [9], p. 147, where it is called the rectangular estimate), corresponds to using the kernel

$$(4.8)\qquad k_D(u) = \frac{\sin u}{n}.$$

This kernel may be written as the Fourier transform of $k_\infty(u)$, defined by (4.5), in the following way:

$$k_D(u) = \frac{\int_{-\infty}^{\infty} e^{iu\omega} k_\infty(\omega)\, d\omega}{\int_{-\infty}^{\infty} k_\infty(\omega)\, d\omega}. \tag{4.9}$$

In a similar way, from the families of kernels k_A, k_C, k_E and k_G, one may obtain new families of kernels k'_A, k'_C, k'_E and k'_G. One gives only the definition of k'_A, since the others may be defined similarly:

$$k'_A(u; \gamma, \mu, r) = \frac{\int_{-\infty}^{\infty} e^{iu\omega} k_A(\omega; \gamma, \mu, r)\, d\omega}{\int_{-\infty}^{\infty} k_A(\omega; \gamma, \mu, r)\, d\omega}. \tag{4.10}$$

To determine the properties of these primed kernels, one considers a general kernel $h(u)$, defined by

$$h(u) = \frac{\int_{-\infty}^{\infty} e^{iu\omega} H(\omega)\, d\omega}{\int_{-\infty}^{\infty} H(\omega)\, d\omega}, \tag{4.11}$$

where $H(\omega)$ is a function satisfying the conditions

$$\int_{-\infty}^{\infty} |\, H(\omega)\, |\, d\omega < \infty, \qquad \int_{-\infty}^{\infty} H^2(\omega)\, d\omega < \infty. \tag{4.12}$$

Then

$$\int_{-\infty}^{\infty} h^2(u)\, du = \frac{2\pi \int_{-\infty}^{\infty} H^2(\omega)\, d\omega}{\left[\int_{-\infty}^{\infty} H(\omega)\, d\omega\right]^2}. \tag{4.13}$$

One determines the characteristic exponent of $h(u)$ under the assumptions that

$$\int_{-\infty}^{\infty} \omega H(\omega)\, d\omega = 0, \qquad \int_{-\infty}^{\infty} \omega^2 H(\omega)\, d\omega < \infty. \tag{4.14}$$

By expanding $e^{iu\omega}$ in Taylor series, it follows immediately that the characteristic exponent is $r = 2$, and the characteristic coefficient is given by

$$h^{(2)} = \left| \frac{\int_{-\infty}^{\infty} \omega^2 H(\omega)\, d\omega}{2\int_{-\infty}^{\infty} H(\omega)\, d\omega} \right|. \tag{4.15}$$

Now each of the kernels k_A, k_C, and k_E satisfy (4.12) and (4.14). Therefore, the primed kernels k'_A, k'_C, and k'_E have characteristic exponent 2, with characteristic coefficient given by (4.15). The kernel k_G does not satisfy (4.14) for $r \leqq 3$; therfore the kernels k_G', for $r \leqq 3$, will have a characteristic exponent less than 2.

One next evaluates the coefficient $T(k)$, defined by (3.8), for some of the kernels that have been introduced. One notes first that $T(k)$ remains unchanged under a change of scale; i.e., if two kernels $k_1(u)$ and $k_2(u)$ are related by the formula $k_1(u) = k_2(Bu)$ for some positive number B, then $T(k_1) = T(k_2)$.

It is noted next that for the kernel $h(u)$ defined by (4.11), if (4.14) holds,

$$T^2(h) = \frac{2\pi^2 \int_{-\infty}^{\infty} \omega^2 H(\omega)\, d\omega \left(\int_{-\infty}^{\infty} H^2(\omega)\, d\omega\right)^2}{\left(\int_{-\infty}^{\infty} H(\omega)\, d\omega\right)^5}. \tag{4.16}$$

Consequently, for the kernels $k_A'(u; 1, 1, r)$, one obtains

$$T^2[k_A'(u; 1, 1, r)] = \frac{\pi^2}{6}\frac{(r+1)^3}{r^4(r+3)}\left(r - 1 + \frac{r+1}{2r+1}\right)^2. \tag{4.17}$$

For the algebraic kernels, one obtains

$$T[k_A(u; \gamma, \mu, r)] = 2\left\{(\mu\gamma) - \frac{2}{r+1}(\mu\gamma)^{r+1} + \frac{1}{2r+1}(\mu\gamma)^{2r+1}\right\}. \tag{4.18}$$

For the exponential kernels, one obtains

$$T[k_E(u; \gamma, \mu, r)] = 2\left(\frac{1}{2r}\right)^{1/r}\int_0^{\mu\gamma(2r)^{1/r}} e^{-\frac{1}{r}t^r}\, dt. \tag{4.19}$$

For the cosine and geometric families of kernels, the results are given only for $r = 2$:

$$T[k_C(u; \gamma, \mu, 2)] = \tfrac{3}{4}\mu\gamma + \tfrac{1}{2}\sin 2\mu\gamma + \tfrac{1}{16}\sin 4\mu\gamma, \tag{4.20}$$

$$T[k_G(u; \gamma, \mu, 2)] = \frac{\mu\gamma}{1 + (\mu\gamma)^2} + \tan^{-1}(\mu\gamma). \tag{4.21}$$

5. On choosing a covariance averaging kernel. In this section, we argue the major proposition with which this paper is concerned, namely, that the notions of order of consistency and of asymptotic mean square error do not by themselves provide a basis for choosing between competing estimates of the spectral density. This is a negative statement. We also consider the positive question of how to make the choice.

In order to put this proposition in its clearest light, we shall consider the effect of using covariance averaging kernels of different functional forms but with the same characteristic exponent r. To compare two kernels of the same characteristic exponent r, we see from (3.6) that it is only necessary to compare $T(k)$, since the minimum observation time $T_{\min}$ is directly proportional to $T(k)$, for fixed r and η^2.

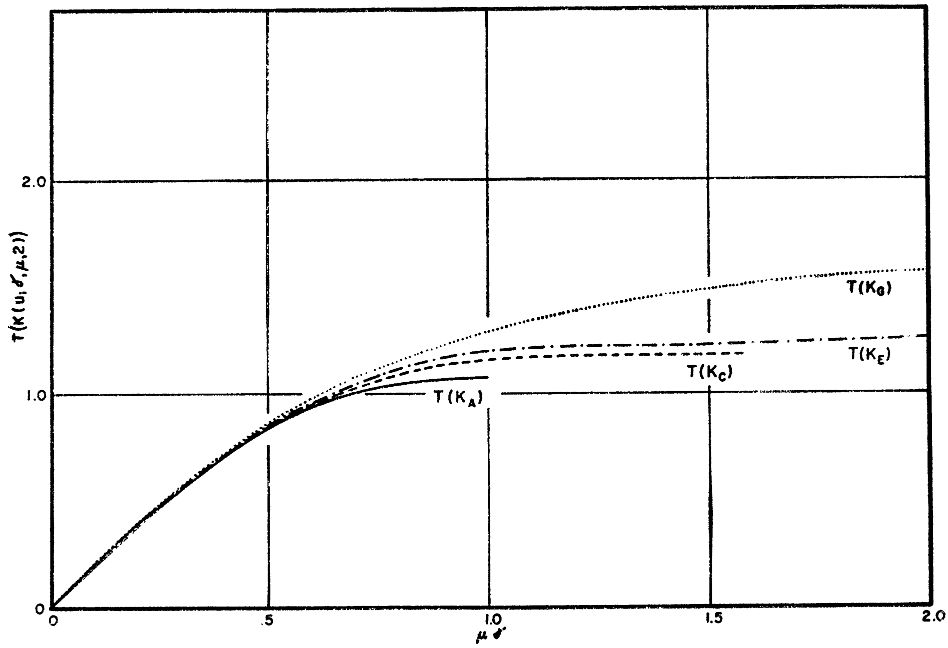

FIG. 1. *The coefficient* $T(k)$ *defined by* (3.8) *plotted, as a function of* $\mu\gamma$ *for the algebraic, cosine, exponential, and geometric families of kernels of characteristic exponent* $r = 2$, *denoted respectively by* $k_A(u; \gamma, \mu, 2)$, $k_C(u; \gamma, \mu, 2)$, $k_E(u; \gamma, \mu, 2)$, and $k_G(u; \gamma, \mu, 2)$.

In Fig. 1, we plot $T(k(u; \gamma, \mu, r))$ for the algebraic, cosine, exponential, and geometric families with characteristic exponent 2. It turns out that $T(k)$ is a function only of the product of the parameters μ and γ. Further, $T(k)$ may be made as small as we please by choosing $\mu\gamma$ sufficiently small. This fact cannot be correct, of course, and emphasizes that (3.6) is not valid for $\mu\gamma$ close to 0, which is not a case where T is large and B_T is small.

However, even if we confine our attention to $T(k)$ for values of $\mu\gamma > \frac{1}{2}$, say, the graphs in Fig. 1 are still disquieting. They seem to imply that the functional form of the kernel $k(u)$ is not too important, since if $k_0(u; \gamma_0, \mu_0, 2)$ is a kernel belonging to the family of kernels $k_0(u; \gamma, \mu, 2)$, then given any other family of kernels $k_1(u; \gamma, \mu, 2)$, there will exist a choice of parameter values γ_1 and μ_1 such that the coefficients $T(k)$ corresponding to the kernels $k_1(u; \gamma_1, \mu_1, 2)$ and $k_0(u; \gamma_0, \mu_0, 2)$ are equal. Thus there appears to be a need for a principle which would provide a means of choosing the parameters μ and γ. Such a principle may provide a means of choosing between kernels of different functional form.

One such principle may be obtained as follows. An estimate $f_T^*(\omega)$ of the form of (2.5) may achieve a small mean square error at the price of averaging over a large band of frequencies. This fact is important if we desire to estimate the difference $f(\omega_2) - f(\omega_1)$ between the values of the spectral density at two neighboring frequencies ω_1 and ω_2. As a measure of the ability of the estimate $f_T^*(\omega)$ to estimate such differences we compute the mean square percentage error

$$\eta^2[D_d f_T^*(\omega)] = \frac{E \mid \{f_T^*(\omega + dB_T) - f_T^*(\omega)\} - \{f(\omega + dB_T) - f(\omega)\} \mid^2}{f^2(\omega)} \tag{5.1}$$

of the increment $D_d f_T(\omega) = f_T(\omega + dB_T) - f_T(\omega)$, where $d > 0$ is fixed. One may show that

$$\lim_{T\to\infty} TB_T \eta^2[D_d f_T^*(\omega)] = 2 \int_{-\infty}^{\infty} (1 - \cos du) k^2(u)\, du \,\{1 + \delta(0, \omega)\}. \tag{5.2}$$

If one regards (5.2) as holding approximately for large values of T and small values of B_T one has, for $\omega \neq 0$,

$$\eta^2[D_d f_T^*(\omega)] = \frac{1}{TB_T} 2 \int_{-\infty}^{\infty} (1 - \cos du) k^2(u)\, du. \tag{5.3}$$

If one takes $T = T_{\min}$ and $B_T = B_{\min}$, given by (3.6) and (3.7), respectively, then

$$\eta^2[D_d f_{T_{\min}}^*(\omega)] = \frac{\eta^2}{\left(1 + \frac{1}{2r}\right)} 2 \frac{\int_{-\infty}^{\infty} (1 - \cos du) k^2(u)\, du}{\int_{-\infty}^{\infty} k^2(u)\, du}. \tag{5.4}$$

The mean square percentage error in (5.4) is of the estimate of the increment $f(\omega + dB_{\min}) - f(\omega)$. If one desires the mean square percentage error of the estimate $\Delta_\beta f_T^*(\omega) = f_T^*(\omega + \beta) - f_T^*(\omega)$ of the increment $\Delta_\beta f(\omega) = f(\omega + \beta) - f(\omega)$, one has, by setting $d = \beta / B_{\min}$, that

$$\eta^2[\Delta_\beta f_{T_{\min}}^*(\omega)] = \frac{2\eta^2}{\left(1 + \frac{1}{2r}\right)} S_{\beta'}(k) \tag{5.5}$$

where we define

$$S_{\beta'}(k) = \frac{\int_{-\infty}^{\infty} \{1 - \cos(\beta' \mid k^{(r)} \mid^{1/r} u)\} k^2(u)\, du}{\int_{-\infty}^{\infty} k^2(u)\, du} \tag{5.6}$$

and β' is defined as a function of β by

$$\beta' = \frac{\beta}{\lambda_r(\omega)} \frac{C(r)}{\eta^{1/r}}. \tag{5.7}$$

Next, consider a kernel $k(u)$ defined by

$$\begin{aligned} k(u) &= h(\gamma u), \qquad |u| \leq \mu \\ &= 0, \qquad \text{otherwise} \end{aligned} \tag{5.8}$$

in terms of a kernel $h(u)$ of characteristic exponent r and characteristic coefficient

$h^{(r)} = 1$. For $k(u)$ defined by (5.8) we obtain from (3.8), (2.8), and (5.4)

$$T(k) = 2 \int_0^{\mu\gamma} h^2(u)\, du, \tag{5.9}$$

$$| k^{(r)} |^{1/r} = \gamma \tag{5.10}$$

$$S_{\beta'}(k) = \frac{\int_0^{\mu\gamma} (1 - \cos \beta' u) h^2(u)\, du}{\int_0^{\mu\gamma} h^2(u)\, du}. \tag{5.11}$$

We see from (5.9) and (5.11) that the properties of the kernel $k(u)$ depend only on the product $\mu\gamma$. We consequently take $\gamma = 1$. Next, let us consider how to choose μ. One sees that, as $\mu \to 0$, $T(k) \to 0$ in (5.9) and $S_{\beta'}(k) \to 1$ in (5.11). Now $T(k)$ is proportional to the minimum observation time which can be attained using the kernel corresponding to μ, while $S_{\beta'}(k)$ is proportional to the mean square error of the increment $\Delta_\beta f(\omega)$, using the kernel corresponding to μ. Thus μ must be chosen so as to strike a balance among these quantities.

Our criterion for choosing μ will follow from the following *assumption on how to design a spectral analysis.* To our mind, in order to design a spectral analysis, one must first specify a quantity η^2, which one desires the mean square percentage error of one's estimate of the spectral density not to exceed. One next specifies quantities β and η_1^2, such that one desires to estimate the increment $f(\omega + \beta) - f(\omega)$ with a mean square percentage error less than or equal to η_1^2. One finally assumes that the spectral bandwidth, of order r at the frequency ω, of the spectral density function being estimated, is greater than or equal to a known quantity $\lambda_r(\omega)$. Given a kernel $k(u)$ of functional form (5.8), with $\gamma = 1$, one chooses μ so that $\eta^2[\Delta_\beta f^*_{T_{\min}}(\omega)]$, given by (5.5), is $\leqq \eta_1^2$. One then lets $T = T_{\min}$, given by (3.6), and $B_T = B_{\min}$, given by (3.7). The estimate $f_T^*(\omega)$, given by (2.5), is then completely defined. It will have the desired properties.

Finally, we may compare the properties of two families of estimates $h_1(\mu u)$ and $h_2(\mu u)$. To each family $h_i(\mu u)$ by the foregoing procedure one obtains a minimum observation time $T^i_{\min}$. One chooses that family of kernels which leads to the smaller minimum observation time.

The foregoing procedure can be made routine if suitable tables and graphs are constructed. However, we have not made such computations. Before such a computational effort is made, it seems to us that the methods proposed for choosing an estimate of the spectral density function and for designing a spectral analysis, which after all are somewhat heuristic, should receive some public acceptance.

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