RELATÓRIO TÉCNICO

ON CLIQUE CONVERGENT GRAPHS

Claudson Ferreira Bornstein
NCE/UFRJ

Jayme Luiz Szwarcfiter
NCE/UFRJ

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Junho

Núcleo de Computação Eletrônica
Universidade Federal do Rio de Janeiro

Tel.: 598-3212 - Fax.: (021) 270-3842
Caixa Postal 2324 - CEP 20001
Rio de Janeiro - RJ

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On Clique Convergent Graphs

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Jayme Luiz Szwarcfiter

Resumo

Um grafo G é **convergente** quando existe um inteiro finito $n \geq 0$, tal que o n–ésimo grafo clique iterado $K^n(G)$ possui um único vértice. O menor n que satisfaz esta condição é o **índice** de G. A **deficiência Helly** de um grafo convergente é o menor inteiro h tal que $K^h(G)$ é clique Helly, ou seja, suas cliques maximais satisfazem a propriedade Helly. Bandelt e Prisner provaram que a deficiência Helly de um grafo cordal é no máximo um e indagaram acerca da existência de algum grafo cuja deficiência Helly excede a diferença entre seu índice e diâmetro por mais de um. Neste artigo é fornecida uma resposta afirmativa a esta questão. Para um inteiro arbitrário n, é exibido um grafo cuja deficiência Helly excede em n unidades a diferença entre seu índice e diâmetro.

Abstract

A graph G is **convergent** when there is some finite integer $n \geq 0$, such that the n-th iterated clique graph $K^n(G)$ has only one vertex. The smallest such n is the **index** of G. The **Helly defect** of a convergent graph is the smallest h such that $K^h(G)$ is clique Helly, that is, its maximal cliques satisfy the Helly property. Bandelt and Prisner proved that the Helly defect of a chordal graph is at most one and asked whether there is a graph whose Helly defect exceeds the difference of its index and diameter by more than one. In the present paper an affirmative answer to the question is given. For any arbitrary finite integer n, it is exhibited a graph, the Helly defect of which exceeds by n the difference of its index and diameter.

1 Introduction

G denotes a simple finite undirected graph, $V(G)$ and $E(G)$ its vertex and edge sets, respectively. A **clique** is a subset of vertices of G which induces a complete subgraph. G is **clique Helly** if its maximal cliques satisfy the **Helly property**, that is, every family of pairwise intersecting maximal cliques of G has a nonempty intersection. The **clique graph** $K(G)$ of G is the intersection graph of the maximal cliques of G. Denote $K^0(G) = G$. The **iterated clique graph** $K^i(G)$ is defined as $K(K^{i-1}(G))$, $i \geq 1$. G is **(clique) convergent** if there is some finite integer $n \geq 0$, such that $K^n(G)$ has only one vertex v. In this case we say that G is n**–convergent to** v. The **index** of a convergent graph G is the smallest n such that G is n–convergent, while its **Helly defect** [1] is the smallest n such that $K^n(G)$ is clique Helly.

Bandelt and Prisner [1] proved that the Helly defect of a chordal graph is at most one and asked whether there is a graph whose Helly defect exceeds the difference of the index and diameter by more than one.

In the present paper, an affirmative answer to the above question is given. For any arbitrary finite integer $n \geq 0$ it is exhibited a graph in which the Helly defect exceeds by n the difference of its index and diameter. The following section contains lemmas about convergent graphs. They are applied to the study of a special class of graphs in Section 3. In the last section, we conclude that this class provides an answer to the considered problem.

2 Basic Lemmas

First we introduce some notation.

The vertex set of $K^n(G)$ is denoted by V^n_G. When possible, we simply write V^n. Let C be a maximal clique of $K^{n-1}(G)$ and v be the vertex of $K^n(G)$ corresponding to C. Then write $K^{-1}_{G,n}(v) = C$. For a subset of vertices $V' \subset V^n_G$, let $K^{-1}_{G,n}(V')$ represent the union of all the sets $K^{-1}_{G,n}(v)$, for $v \in V'$. For $v \in V^n$, $K^{-i}_{G,n}(v)$ denotes the set $K^{-1}_{G,n-(i-1)}(K^{-(i-1)}_{G,n}(v))$, that is, $K^{-i}_{G,n}(v)$ corresponds to the vertices of V^{n-i} which led to v by applying the iterated clique graph operation. Call $K^{-i}_{G,n}(v)$ the i**–th inverse image** of v. Similarly, if $V' \subset V^n_G$ then $K^{-i}_{G,n}(V')$ is the union of the i–th inverse images of all $v \in V'$. Whenever possible, we ommit G, n or both in the notations. Finally, let H be a subgraph of G. Observe that $V^n_H \not\subset V^n_G$ in general, but $K^{-n}(V^n_H) \subset K^{-n}(V^n_G)$. $\mathcal{P}(V)$ denotes the power set of V.

The following lemma is a slight variation of Hedman's theorem [2].

Lemma 1 *Let G' be a connected subgraph of $K^i(G)$, with vertex set V'. Let H be the* *subgraph induced in G by $K^{-i}(V')$. Then $diam(H) \leq diam(G') + i$.*

Proof.: The lemma follows immediately if we can prove it for $i = 1$. by repeatedly applying the same argument.

Let v, w be vertices in $K^{-1}(V')$, and c_v,c_w be vertices in V' such that $v \in K^{-1}(c_v)$ and $w \in K^{-1}(c_w)$. Since G' is connected there is a path $c_v = c_0, c_1, c_2, \ldots c_l = c_w$ in G' of length $l \leq$diam(G'). Each pair of consecutive vertices in this path corresponds to cliques in $K^{i-1}(G)$, which have at least a one vertex intersection. Let $v_j \in (K^{-1}(c_{j-1}) \cap K^{-1}(c_j))$, $1 \leq j \leq l$. Also let $v_0 = v$. and $v_{l+1} = w$. Each pair of vertices v_j, v_{j+1} is adjacent since they are in the same clique $K^{-1}(c_j)$, for $0 \leq j \leq l$. Therefore the vertices v_j induce a path between v and w of length $l+1$, and thus dist$(v, w) \leq$ diam$(V')+1$ for any $v, w \in K^{-1}(V')$, concluding the proof. ■

The idea of the following lemma is simple. By applying the clique graph operation on a graph G we are somehow also applying it to graphs isomorphic to the subgraphs of G. There are in $K(G)$ vertices corresponding to the cliques of any subgraph H, although these cliques may or may not be maximal in G. In the latter case, these vertices correspond to maximal cliques of G containing those of H. We can apply the same ideas to as many iterations of the clique graph operator as we want and if H is n-convergent the vertex of $K^n(G)$ which corresponds to the unique vertex of $k^n(H)$ will satisfy Lemma 2.

Lemma 2 *Let H and G be graphs such that H is n–convergent and a subgraph of G. Then there is at least one vertex $v \in V_G^n$ such that $V(H) \subset K^{-n}(v)$.*

Proof.: It will sufice to show that there exist functions $f^i : \mathcal{P}(V_H^i) \rightarrow \mathcal{P}(V_G^i)$. such that $K^{-i}(f^i(V_H^i))$ covers $V(H)$.

First we show that there are functions $f^i : \mathcal{P}(V_H^i) \rightarrow \mathcal{P}(V_G^i)$ satisfying the following conditions. for $V \subset V_H^i$.

If $i = 0$, then $f^0(V) = V$.

If $i > 0$, then $f^{i-1}(K^{-1}(V)) \subset K^{-1}(f^i(V))$.

The case $i = 0$ is trivial. Assume that we can define functions f^i, $i < k$, satisfying the conditions above, and such that if v, $w \in V_H^i$ are adjacent, then $f^i(\{v\})$ and $f^i(\{w\})$ are also single vertices, which either coincide or are adjacent in $K^i(G)$. Then we show how to construct f^k. Start from one vertex subsets and define the image of the larger subsets as the union of the images of its subsets of size one. The empty set is its own image.

For each $v \in V_H^k$, there is a maximal clique $C = K_{H,k}^{-1}$, and the above conditions imply that the set $\{f^{k-1}(\{u\})/u \in C\}$ induces in $K^{k-1}(G)$ a complete subgraph. Taking a maximal clique C' in $K^{k-1}(G)$ covering this set, we define $f^k(\{v\})$ to be the unique vertex v' in V_G^k such that $K^{-1}(v') = C'$. Given a set V, f^k is defined as the union of the sets $f^k(\{v\})$, $v \in V$.

Clearly f^k satisfies the required conditions. In addition, every pair of singleton sets $\{v\}$ and $\{w\}$ where v is adjacent to w are mapped to singleton sets $\{v'\}$ and $\{w'\}$ where v' and w' are either adjacent or coincide. A pair of vertices is adjacent in $K^k(H)(k > 0)$ when its corresponding cliques in $K^{k-1}(H)$ intersect. So the corresponding sets in $K^{k-1}(G)$ also intersect. Therefore the maximal cliques taken in $K^{k-1}(G)$ also intersect.

The next step is to show that the fuctions f^i above defined satisfy the lemma.

For $i = 0$, obviously $K^{-i}(f^i(V_H^i)) = f^0(V(H))$ covers (coincides with) $V(H)$.

Assume as induction hypothesis that f^i satisfies the lemma for $i = p$, and we show that it does so for $p + 1$.

By the definition of the K^{-1} operation, $K^{-1}(V_H^{p+1}) = V_H^p$, and since

$$f^p(K^{-1}(V_H^{p+1})) \subset K^{-1}(f^{p+1}(V_H^{p+1})),$$

$$f^p(V_H^p) \subset K^{-1}(f^{p+1}(V_H^{p+1})),$$

applying K^{-p}, $K^{-p}(f^p(V_H^p)) \subset K^{-p-1}(f^{p+1}(V_H^{p+1}))$.

By hypothesis $K^{-p}(f^p(V_H^p))$ covers $V(H)$. Therefore, the set in which it is contained also covers $V(H)$, thus completing the proof. ∎

Lemma 3 *Let $H' \subset H \subset G$ be graphs and $v', v \in V_G^n$ distinct vertices, such that H' and H are the graphs induced in G respectively by $K_G^{-n}(v')$ and $K_G^{-n}(v)$. Then there are two distinct vertices w and w' in V_H^n such that H' and H are induced in H by $K_H^{-n}(w')$ and $K_H^{-n}(w)$.*

Proof.: By definition of K^{-n} all the vertices of G that give rise to v and v' through the construction of $K^n(G)$ originate from the subgraph H. Therefore there are distinct vertices w and w' in V_H^n whose inverse images by K^{-n} must induce H and H'. ∎

Corollary 1 *Let $H \subset G$ be a n-convergent graph with $H = K^{-n}(v)$ for some $v \in K^n(G)$. Then v is the unique vertex in V_G^n for which $K^{-n}(v)$ is entirely contained in $V(H)$.*

Lemma 4 *Let H be a maximal subgraph with diameter n of G, such that H is n-convergent. Then $H = K^{-n}(v)$ for a unique v in V_G^n.*

Proof.: By Lemma 2 there is a vertex $v \in V_G^n$ with $H \subset K^{-n}(v)$. By lemma 1 this set must induce a graph with diameter at most n, and thus $H = K^{-n}(v)$. The uniqueness of v follows from corollary 1. ■

Lemma 5 *Let H. H_1 and H_2 be distinct induced subgraphs of G. satisfying the following conditions :*

- *H_1 and H_2 are n–convergent respectively to u_1 and u_2*
- *H is $(n-1)$–convergent to u and $H \subset (V(H_1) \cap V(H_2))$.*
- *H. H_1 and H_2 are maximal subgraphs of G with diameters $n-1$. n and n respectively.*

Then there are in V_G^n two vertices v_1 and v_2 whose inverse images $K^{-n}(v_i)$ are exactly the vertex sets $V(H_i)$. In addition v_1 and v_2 are adjacent.

Proof.: The first part follows from Lemma 4 that assures the existence and uniqueness of v_1 and v_2 in V_G^n.

We show that v_1 and v_2 are adjacent. The argument below implies that $K^{-1}(v_i)$, $i = 1, 2$ have at least one common vertex, which completes the proof.

Let f^i. g^i and h^i be functions

$$f^i : \mathcal{P}(V_{H_1}^i) \rightarrow \mathcal{P}(V_G^i)$$

$$g^i : \mathcal{P}(V_H^i) \rightarrow \mathcal{P}(V_{H_1}^i)$$

$$h^i : \mathcal{P}(V_H^i) \rightarrow \mathcal{P}(V_G^i)$$

as those defined in the proof of Lemma 2. It follows that h^i can be obtained as a composition of f^i and g^i.

Clearly. this is true for $i = 0$.

The functions satisfy $f^{i-1}(K^{-1}(v)) \subset K^{-1}(f^i(\{v\}))$ and thus

$$f^{i-1}(g^{i-1}(K^{-1}(v))) \subset f^{i-1}(K^{-1}(g^i(\{v\}))) \subset K^{-1}(f^i(g^i(\{v\}))).$$

Therefore the composition of the funtions f^i and g^i is a function h^i.

By Lemma 4 there is a unique value for $h^{n-1}(\{u\})$.

On the other hand, by Lemma 4 again, $v_1 = f^n(\{u_1\})$.

By the definition of the functions g^i there is a vertex $v \in V_{H_1}^{n-1}$ with $g^{n-1}(\{u\}) = \{v\}$. Clearly $v \in K^{-1}(u_1)$, and thus $f^{n-1}(\{v\}) \subset K^{-1}(v_1)$. Since $h^{n-1}(\{u\})$ is unique, $f^{n-1}(g^{n-1}(\{u\})) = f^{n-1}(\{v\}) = h^{n-1}(\{v\})$. Therefore $h^{n-1}(\{u\}) \subset K^{-1}(v_1)$.

Applying the same argument to H_2, leads to $h^{n-1}(\{u\}) \subset K^{-1}(v_2)$, what concludes the proof. ∎

3 The F_n and H_n^i graphs

Let G be a graph with minimum degree δ. An **extreme vertex** of G is one with degree δ. An **extreme path** is a shortest path between two extreme vertices such that all internal vertices of it have degree $\delta + 1$ or $\delta + 2$.

Next we define the class of F_n graphs:

F_1 is the graph K_3. For $i > 1$, F_i is obtained as follows: Let $v_0, v_1, \ldots, v_{i-1}$ be an extreme path of F_{i-1}, and $w_0, w_1, \ldots, w_i \notin V(F_{i-1})$. F_i has vertex set $V(F_{i-1}) \cup \{w_0, w_1, \ldots, w_i\}$, and edge set $E(F_{i-1}) \cup \{(v_j, w_j), (w_j, w_{j+1}), (v_j, w_{j+1}) / 0 \le j < i\}$.

Figure 1 illustrates the graphs F_1, F_2 and F_3.

The graphs H_n^i are defined as follows.

H_n^1 is the graph obtained from F_n by removing its extreme vertices, and in general H_n^i is obtained from F_n by removing the vertices of F_n that are at a distance less than i from its extreme vertices. We deal only with those H_n^i satisfying $i \le n/2$. F_n is also denoted by H_n^0. Figure 2 shows the graph H_7^2 and its six extreme paths.

Theorem 1 *F_n and H_n^i are both convergent graphs. Moreover, each of them has equal index and diameter.*

Proof.: With regard to F_1, F_2 and F_3 the theorem can be checked by inspection. The same for H_2^1 and H_3^1.

Clearly $\text{diam}(F_n) = n$ and $\text{diam}(H_n^i) = n - i$.

The proof is by induction on n. Let $n > 3$.

Note that when n is even, $H_n^{n/2}$ is $F_{n/2}$. When n is odd, $H_n^{(n-1)/2}$ coincides with

$H^1_{(n+3)/2}$, and that $(n+3)/2$ is less than n when $n > 3$. The induction hypothesis is that the theorem holds for F_k and H^j_k, $k < n$, and for H^j_n, $n/2 \geq j > i$;

The outline of the proof for H^i_n is as follows. We find the maximal subgraphs H' of H^i_n with diameter $n - i - 1$. These graphs will be among those which converge in $n - i - 1$ iterations by hypothesis.

Any subgraph of H^i_n with diameter less than $n - i - 1$ is contained in some H'. Hence, by Corollary 1 there is a one to one correspondence between the vertices of $K^{n-i-1}(H^i_n)$ and the subgraphs H'.

Finally we verify that each pair of distinct subgraphs H' satisfies the conditions of Lemma 5. Therefore the corresponding vertices in $K^{n-i-1}(H^i_n)$ are adjacent. Thus the graph $K^{n-i-1}(H^i_n)$ is complete, what proves the theorem.

Now we proceed to find all subgraphs H'.

Consider H^i_n, $0 < i < n/2$. H^i_n has exactly six extreme paths. Label these paths clockwise from 1 to 6, so that the paths 2, 4 and 6 have length $n - 2i$, while 1, 3 and 5 have length i. An example is given in Figure 2.

Let S be a subgraph of H^i_n with diameter less than $n - i$.

If S has some vertex from the extreme path 1, it cannot have any vertex from path 4, since these vertices would be at a distance $n - i$. If S does not have vertices from paths 3 and 5, then S is contained in a graph H^i_{n-1} (class 1). See Figure 3.

Assume that S has vertices from path 3 and not from 5. Then S cannot have vertices from path 6, since its diameter must be at most $n - i - 1$. Hence S is contained in the subgraph obtained by removing the vertices from the extreme paths 4, 5 and 6, that is, a graph H^{i-1}_{n-2} (class 2).

Otherwise if S has vertices from both paths 3 and 5, it cannot have any vertex from paths 2, 4 and 6. In this case, it is contained in a graph H^{i-2}_{n-3} (class 3), $i > 1$. If $i = 1$ we would have a graph F_{n-3}, which is contained in all other classes.

Finally, if S does not have vertices from paths 1, 3 and 5 it is contained in a graph H^{i+1}_n (class 4).

The remaining cases are all similar and lead to subgraphs belonging to the four classes above.

Now, the last step of the proof:

All the maximal subgraphs with diameter $n - i - 1$ above have a common subgraph obtained by removing from H_n^i its extreme paths, that is, a graph H_{n-3}^{i-1}. By hypothesis it is $(n-i-2)$–convergent.

By applying Lemma 5 we conclude that H_n^i is $(n-i)$–convergent.

We apply a similar strategy to prove the theorem for F_n. Let S be a subgraph of F_n with diameter less than n. Then if S has any extreme vertex, it cannot have any vertex belonging to the extreme path opposite to this vertex. Therefore S is contained in a subgraph F_{n-1}.

On the other hand, if S does not have any extreme vertex it must be contained in H_n^1.

Every maximal subgraph with diameter $n-1$ of F_n is either a F_{n-1} or a H_n^1 graph. They have pairwise intersections which contain subgraphs F_{n-2} or H_{n-1}^1. These graphs are $(n-2)$–convergent by hypothesis. By Lemma 5, $K^{n-1}(F_n)$ is complete and the theorem is proved. ∎

In addition we have shown that, for $n > 3$, $K^{n-1}(F_n)$ is the graph K_4.

Next we determine the Helly defect of F_n. The following notation is used. Given a graph F_n, $n > 3$, let its extreme vertices be called v_1, v_2 and v_3 arbitrarily. $S_{i,j}$ $(1 \leq i \leq j \leq 3)$ is the subgraph induced in F_n by the vertices of F_n such that:

if $i \neq j$, their distances to v_i and v_j is less than n;

if $i = j$, their distances to v_i is less than $n-1$.

F_n contains six such subgraphs, which are examined in the following lemma.

Lemma 6 *F_2 is an induced subgraph of $K^{n-2}(F_n)$, $n > 3$.*

Proof.: The six subgraphs $S_{i,j}$ are F_{n-2} subgraphs induced in F_n. We show that these graphs converge to six distinct vertices of $K^{n-2}(F_n)$, forming an induced F_2 in $K^{n-2}(F_n)$.

By Theorem 1, $S_{i,j}$ is $(n-2)$–convergent. By Lemma 4, each maximal subgraph of F_n with diameter $n-2$ corresponds to a unique vertex of $K^{n-2}(F_n)$. $S_{1,1}$, $S_{1,2}$ and $S_{1,3}$ pairwise intersect in subgraphs F_{n-3}, and their corresponding vertices in $K^{n-2}(F_n)$ are therefore adjacent. The same happens with $S_{1,2}$, $S_{2,2}$ and $S_{2,3}$, as well as with $S_{1,3}$, $S_{2,3}$ and $S_{3,3}$.

Except those above, no other edges exist among the vertices of $K^{n-2}(F_n)$ whose inverse images are the $S_{i,j}$'s. Otherwise $K^{n-1}(F_n)$ would contain vertices whose corresponding inverse images in G induce a subgraph with diameter n, contradicting Lemma 1. ∎

Theorem 2 *$K^{n-2}(F_n)$ is not clique Helly ($n \geq 2$).*

Proof.: For a graph containing an induced F_2 to be clique Helly, the three cliques that contain respectively the extreme vertices of the F_2 must have at least one vertex in common. Hence there must be a vertex v adjacent to the extreme vertices of the induced F_2.

It follows that the inverse image of v in F_n is a subgraph F' of the induced F_{n-3}, obtained from F_n by removing its extreme paths. Otherwise, Lemma 1 is contradicted.

But F' is properly contained in the three subgraphs $S_{i,j}$, $i \neq j$ that give rise to the non–extreme vertices of the induced F_2. By Corollary 1 there is only one vertex in $K^{n-2}(F_n)$ with its inverse image contained in each of these subgraphs $S_{i,j}$. The latter correspond to the non–extreme vertices of the F_2. This is inconsistent with the existence of a vertex v adjacent to the three extreme vertices of the F_2.

Therefore $K^{n-2}(F_n)$ is not clique Helly. ∎

4 Conclusion

The clique graph of a clique Helly graph is also clique Helly [3]. Therefore Theorem 2 implies that the Helly defect of F_n is exactly $n-1$, since $K^{n-1}(F_n)$ is a complete graph. From Theorem 1, F_n has its diameter and index equal to n. Hence the Helly defect of F_n exceeds the difference between its index and diameter by $n-1$.

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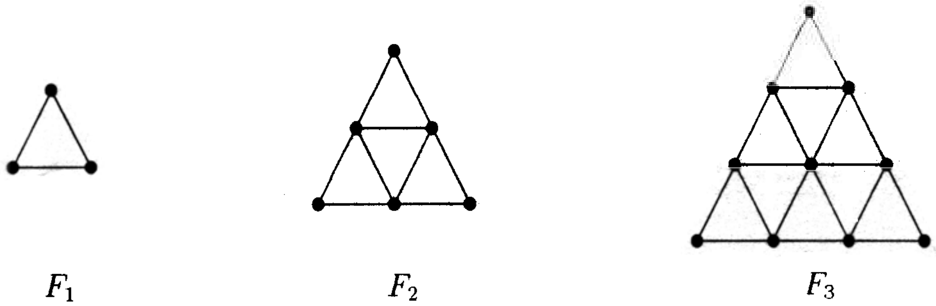


Figure 1 – Some F_i graphs

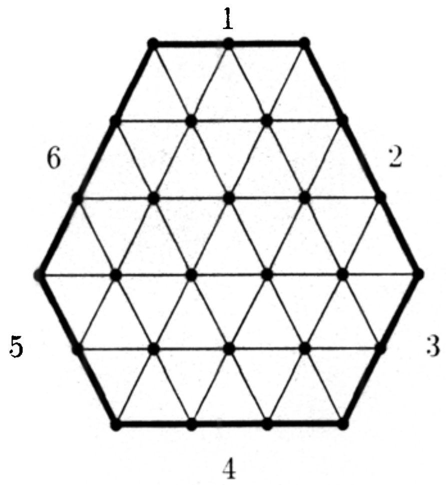
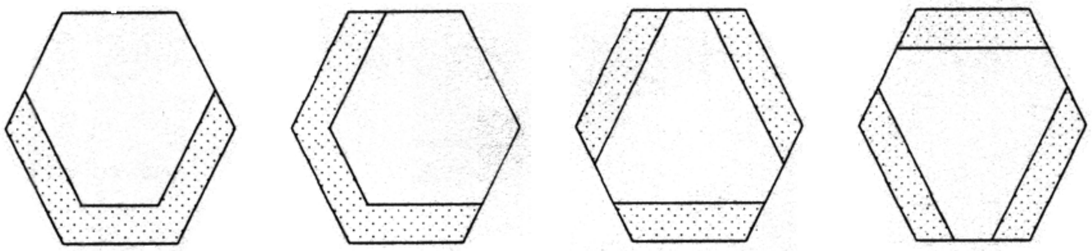


Figure 2 – H_7^2 and the labelling of its six extreme paths



Class 1 Class 2 Class 3 Class 4

Figure 3 – H_n^i maximal subgraph classes with diameter $n-1$.